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OSCILLATION OF ODD ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Summary. A necessary and sufficient condition for the oscillation of all solutions of

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big[x(t) - cx(t-\tau)\Big] + p(t)x(\sigma(t)) = 0,$$

where n is odd integer is obtained. A new sufficient condition for the oscillation of all solutions is derived along with some comparison results.

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1. INTRODUCTION

Oscillation of higher order neutral differential equations of the type

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[x(t) - cx(t-\tau)\right] + p(t)x(\sigma(t)) = 0, \quad t \geqslant t_0 \tag{1.1}$$

have been recently considered by several authors (Ladas and Sficas [5,6]], Wang [8], Zahariev and Bainov [10] and Zhang and Gopalsamy [12]). The purpose of this article is to discuss the asymptotic behavior of (1.1) when n is an odd positive integer. We recall that a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$ and (1.1) is said to be oscillatory if every solution his equation is oscillatory.

First we derive necessary and sufficient conditions in section 2 for the oscillation of all solutions of (1.1) and discuss certain comparison results in section 3. In section 4, we establish new results for the oscillations of all solutions of (1.1) We note that all inequalities are assumed to hold for all sufficiently large t.

2. OSCILLATIONS

We begin with the following Lemma which describes the asymptotic behavior of nonoscillatory solutions of (1.1).

Lemma 2.1. *Assume the following:*
(i) c *and* τ *are constants with* $0 \leqslant c < 1$ *and* $\tau > 0$;
(ii) $\sigma \in C(R_+, R)$, $\quad R_+ = [0, \infty)$, $\quad \lim\limits_{t\to\infty} \sigma(t) = \infty$;
(iii) $p \in C(R_+, R_+)$ *and* $\int\limits_T^\infty p(s)\,\mathrm{d}s = \infty$, $\quad T \geqslant t_0$.
If $x(t)$ *is an eventually positive solution of* (1.1), *then*

$$\text{(a)} \qquad \lim_{t\to\infty} x(t) = 0$$

$$\text{(b)} \qquad (-1)^i Z^{(i)} > 0 \quad \textit{eventually and}$$
$$\lim_{t\to\infty} Z^{(i)}(t) = 0, \quad i = 0, 1, 2, \dots, n-1,$$

where

$$Z(t) = x(t) - cx(t-\tau). \tag{2.1}$$

Proof. Let $x(t)$ be an eventually positive solution of (1.1); then $Z^{(n)} \leqslant 0$. Since $p(t) \not\equiv 0$ we must have either

$$\lim_{t\to\infty} Z^{(n-1)}(t) = -\infty \tag{2.2}$$

or

$$\lim_{t\to\infty} Z^{(n-1)}(t) = \ell. \tag{2.3}$$

It is easy to see that in the case of (2.2) we have

$$\lim_{t\to\infty} Z(t) = -\infty. \tag{2.4}$$

However, when $c = 0$, $Z(t) = x(t)$ and so (2.4) is not possible. When $0 < c < 1$, (2.4) implies that $\lim\limits_{t\to\infty} x(t) = 0$; consequently $\lim\limits_{t\to\infty} Z(t) = 0$ and thus (2.4) is again impossible. Let us then consider the only possible case namely (2.3). If $\ell \neq 0$, then (2.3) implies that either

$$\lim_{t\to\infty} Z(t) = +\infty \tag{2.5}$$

or

$$\lim_{t\to\infty} Z(t) = -\infty. \tag{2.6}$$

If (2.5) holds, then we obtain

$$\lim_{t\to\infty} x(t) = +\infty. \tag{2.7}$$

Now integrating both sides of (1.1) from T to ∞ and using (2.3) we get

$$\int_T^\infty p(s)x(\sigma(s))\,\mathrm{d}s < \infty, \tag{2.8}$$

which together with (2.7) leads to

$$\int_T^\infty p(s)\,\mathrm{d}s < \infty \tag{2.9}$$

providing a contradiction to the assumption (iii) of the Lemma. Therefore $\ell = 0$ which implies that $Z^{(n-1)} > 0$ and $\lim_{t\to\infty} Z^{(n-1)}(t) = 0$. It will follow from this that

$$(-1)^i Z^{(i)}(t) > 0, \quad i = 1, 2, \dots, (n-1),$$

and

$$\lim_{t\to\infty} Z^{(i)}(t) = 0, \quad i = 1, 2, \dots, (n-1).$$

In particular $Z'(t) < 0$. Hence

$$\lim_{t\to\infty} Z(t) = \ell_1.$$

As before we can show that $\ell_1 < 0$ is impossible. If $\ell_1 > 0$ for some $T \geqslant t_0$ we will have

$$0 < \ell_1 < Z(t) < x(t), \quad \text{for } t \geqslant T.$$

This together with (2.8) leads to (2.9). Thus we must have $\ell_1 = 0$, and hence

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} [x(t) - cx(t-\tau)] = 0$$

from which one can derive using $c \in [0, 1)$ that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 2.2. *Let the hypotheses of Lemma 2.1 hold; furthermore assume that*

$$\sigma(t) \leqslant t \quad \textit{when} \quad 0 < c < 1$$

and

$$\sigma(t) < t, \ p(t) > 0 \quad \textit{when} \quad c = 0.$$

Then (1.1) *is oscillatory if and only if the differential inequality*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t) - cx(t-\tau)\big] + p(t)x(\sigma(t)) \leqslant 0 \tag{2.10}$$

has no eventually positive solution.

Proof. The sufficiency is obvious. To prove the necessity we let $x(t)$ be an eventually positive solution of (2.10). We shall show that (1.1) has a nonoscillatory solution. As in the proof of Lemma 2.1, we have

$$\lim_{t\to\infty} Z^{(i)} = 0, \quad i = 0, 1, 2, \dots, (n-1) \quad \text{and} \quad \lim_{t\to\infty} x(t) = 0.$$

If $x(t) > 0$ for some $T \geqslant t_0$ we let

$$T_0 = \inf_{t\geqslant T} \sigma(t) \leqslant T, \quad \text{and} \quad T_1 = T + \max(T - T_0, \tau).$$

There exists a T_2 such that

$$x(T_2) = \min_{t\in[T,T_2]} x(t).$$

Integrating (2.10) n-times from t to ∞ we have

$$x(t) \geqslant cx(t-\tau) + \int_t^\infty \frac{(s-t)^{(n-1)}}{(n-1)!} p(s)\, x(\sigma(s))\,\mathrm{d}s, \quad t \geqslant T.$$

If $0 < c < 1$ then we define

$$y_0(t) = x(t) \quad \text{for } t \geqslant T,$$

$$y_1(t) = \begin{cases} cy_0(t-\tau) + \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s)\, y_0(\sigma(s))\,\mathrm{d}s, & t \geqslant T_2, \\ y_1(T_2) + x(T) - x(T_2), & t \in [T, T_2]. \end{cases}$$

It follows that

$$0 < y_1(t) \leqslant y_0(t), \quad t \geqslant T.$$

In general we define

$$(2.11)\qquad y_{m+1}(t)=\begin{cases} cy_m(t-\tau)+\int\limits_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}p(s)\,y_m(\sigma(s))\,\mathrm{d}s, & t\geqslant T_2,\\ y_{m+1}(T_2)+x(t)-x(T_2), & t\in[T,T_2]. \end{cases}$$

By induction it can be found that

$$0<y_n(t)\leqslant y_{n-1}(t)\leqslant\ldots\leqslant y_0(t),\quad t\geqslant T.$$

From the fact that $x(t)\geqslant cx(t-\tau)$ for $t\geqslant T_1$ we can derive that

$$x(t)\geqslant\alpha\,\mathrm{e}^{-\mu t},$$

where $\alpha=x(T_1)\exp(\mu T_1)>0$, $\mu=(-\frac{1}{\tau})\ln[c]>0$, and that

$$\alpha\,\mathrm{e}^{-\mu t}\leqslant y_n(t)\leqslant y_{n-1}(t)\leqslant\ldots\leqslant y_0(t),\quad t\geqslant T_1.$$

By Lebesgue's convergence theorem it follows that the pointwise limit of $\{y_n(t)\}$ exists as $n\longrightarrow\infty$. Thus there exists a y^* such that

$$\lim_{t\longrightarrow\infty}y_n(t)=y^*(t),\quad t\geqslant T_1.$$

From (2.11) we have

$$\alpha\,\mathrm{e}^{-\mu t}\leqslant y^*(t)=cy^*(t-\tau)+\int\limits_t^{\infty}\frac{s-t)^{n-1}}{(n-1)!}p(s)\,y^*(\sigma(s))\,\mathrm{d}s,\quad T\geqslant T_2,$$

which implies that y^* is a positive solution of (1.1). If $c=0$, we define a sequence $\{y_n\}$ as follows;

$$y_0(t)=x(t),\quad t\geqslant T,$$

$$y_{m+1}(t)=\begin{cases} \int\limits_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}p(s)\,y_m(\sigma(s))\,\mathrm{d}s, & t\geqslant T_1,\\ y_{m+1}(T_1)+x(t)-x(T_1), & t\in[T,T_1]. \end{cases}$$

Proceeding as before we can prove that there exists a function y^* such that

$$\lim_{n\longrightarrow\infty}y_n(t)=y^*(t),\quad t\geqslant T_1,$$

and

$$y^*(t) = \begin{cases} \int\limits_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s)\, y^*(\sigma(s))\,\mathrm{d}s, & t \geqslant T_1 \\ y^*(T_1) + x(t) - x(T_1), & t \in [T, T_1]. \end{cases}$$

From Lemma 2.1, $x'(t) < 0$ (in case $c = 0$) and hence

$$y^*(t) \geqslant x(t) - x(T_1) > 0, \quad \text{for } t \in [T, T_1]$$

and it follows that $y^*(t) > 0$ for all $t \geqslant T_1$. Therefore y^* is a positive solution of (1.1).

We wish to remark that the conclusion of Theorem 2.2 can be extended to equations with several delays of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[x(t) - cx(t-\tau)\right] + \sum_{i=1}^{m} p_i(t) x(\sigma_i(t)) = 0, \quad t \geqslant t_0. \tag{2.12}$$

☐

3. Comparison results

It is sometimes possible to conclude the oscillatory nature of one equation by comparing it with another suitable equation. We derive results of this type here.

Theorem 3.1. *Assume that the hypotheses of Theorem 2.2 hold and that*

$$0 < c \leqslant \bar{c} < 1, \quad q(t) \geqslant p(t) \geqslant 0.$$

Then the oscillation of (1.1) *implies the oscillation of*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[x(t) - \bar{c}x(t-\tau)\right] + q(t)x(\sigma(t)) = 0. \tag{3.1}$$

Proof. Suppose that (1.1) is oscillatory and that (3.1) is not oscillatory; let $x(t)$ be an eventually positive solution of (3.1). We set

$$Z(t) = x(t) - \bar{c}x(t-\tau)$$

and obtain by Lemma 2.1

$$\begin{aligned} x(t) &= \bar{c}x(t-\tau) + \int\limits_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s) x(\sigma(s))\,\mathrm{d}s \\ &\geqslant cx(t-\tau) + \int\limits_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s) x(\sigma(s))\,\mathrm{d}s. \end{aligned} \tag{3.2}$$

By Theorem 2.2, it now follows that (1.1) has a nonoscillatory solution which contradicts the assumption that (1.1) is oscillatory. This completes the proof ☐

Example 3.1. Consider the odd order neutral equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t) - cx(t-\tau)\big] + (p^n + q(t))x(t - n\sigma) = 0, \quad t \geqslant t_0, \tag{3.3}$$

where c, τ, σ and p are positive constants with $0 < c < 1$ and $q \in C(R_+, R_+)$. If

$$p e \sigma > (1-c)^{\frac{1}{n}}, \tag{3.4}$$

then every solution of (3.3) is oscillatory. In fact, from a result to be proved below (see Theorem 4.2) and condition (3.4) it follows that (3.3) is oscillatory for the case $q(t) \equiv 0$. Thus the assertion regarding (3.3) follows from the comparison Theorem 3.1.

We proceed to establish a comparison result for delay differential equations of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) + p(t)x(\tau(t)) = 0, \tag{3.5}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t) + q(t)x(\sigma(t)) = 0. \tag{3.6}$$

Theorem 3.2. *Let the assumptions of Theorem 2.2 hold for* (3.5). *Furthermore, suppose that*

$$\sigma(t) \leqslant \tau(t) < t, \quad p(t) \leqslant q(t), \quad \text{for } t \geqslant T^* \geqslant t_0, \tag{3.7}$$

where T^* is possibly sufficiently large. Then the oscillation of (3.5) *implies the oscillation of* (3.6).

Proof. Suppose the contrary and let $x(t)$ be an eventually positive solution of (3.6). Then by Lemma 2.1 we have $x'(t) < 0$ since $c = 0$. Hence

$$x(\tau(t)) \leqslant x(\sigma(t)) \quad \text{for } t \geqslant T \geqslant T^*.$$

Now

$$\begin{aligned} \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + p(t)x(\tau(t)) &= \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + q(t)x(\sigma(t)) + [p(t)x(\tau(t)) - q(t)x(\sigma(t))] \\ &= [p(t)x(\tau(t)) - q(t)x(\sigma(t))] \leqslant 0 \end{aligned} \tag{3.8}$$

which in view of Theorem 2.2 implies that (3.5) has a nonoscillatory solution. This contradiction proves the assertion of the Theorem. □

For odd order neutral equations the type

$$(3.9)\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t)-cx(t-\tau)]+p(t)x(\sigma(t))+F\big(t,x(t),x(g_1(t)),\dots,x(g_m(t))\big)=0,\quad t\geqslant t_0$$

the following is an immediate consequence of Theorem 2.2.

Theorem 3.3. *Let the hypotheses of Theorem 2.2 hold. Furthermore let*

(i) $F\in C(R_+\times R^{m+1},R)$, *and* $F(t,y_0,y_1,\dots,y_m)y_0>0$ *whenever* $y_0y_i>0$, $i=1,2,\dots,m$;

(ii) $g_i\in C(R_+,R)$, $\lim\limits_{t\to\infty}g_i(t)=\infty$, $i=1,2,\dots,m$.

Then the oscillation of (1.1) *implies that of* (3.9).

Remark 3.1. In (3.9) the arguments g_i can be of delay type, advanced type or of mixed type. For example consider

$$(3.10)\qquad \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t)-px(t-=\tau)\big]+\sum_{i=1}^{K}p_ix(t-\tau_i)+\sum_{j=1}^{L}q_jx(t-\sigma_j)=0,\quad t\geqslant t_0,$$

where $0<p<1$, τ, p_i, τ_i, σ_j and q_j are positive constants, $i=1,2,\dots,K$; $j=1,2,\dots,L$ and n is odd. Using Theorem 3.3 we can conclude that the oscillation of

$$(3.11)\qquad \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t)-px(t-\tau)\big]+\sum_{i=1}^{K}p_ix(t-\tau_i)=0,$$

implies that of (3.10).

Theorem 3.4. *Assume that the hypotheses of Theorem 2.2 hold. Further assume that* $p(t)>0$ *and* $\sigma(t)<t$. *Then the oscillation of (1.1) with* $c=0$ *implies that of (1.1) with* $0<c<1$. *The converse is false.*

Proof. Suppose the contrary and let (1.1) with $c=0$ be oscillatory and when $0<c<1$, there is an eventually positive solution $x(t)$ of (1.1). By Lemma 2.1 we know that $Z(t)<x(t)$. Hence

$$(3.12)\qquad \frac{\mathrm{d}^n}{\mathrm{d}t^n}Z(t)+p(t)Z(\sigma(t))\leqslant 0,$$

which by Theorem 2.2 implies that

$$(3.13)\qquad \frac{\mathrm{d}^n}{\mathrm{d}t^n}y(t)+p(t)y(\sigma(t))=0$$

has a nonoscillatory solution, and this contradicts our assumption that (1.1) with $c=0$ is oscillatory.

To establish the second part of our proof we consider

$$(3.14)\qquad \frac{d}{\mathrm{d}t}\big[x(t)-cx(t-\tau)\big]+x\left(t-\frac{1}{e}\right)=0.$$

It is known that when $c=0$, (3.14) has a nonoscillatory solution; however (3.14) is oscillatory for $0<c<1$ (for details see [11]). The proof is complete. □

4. OSCILLATIONS OF (1.1)

In the following we are concerned with the investigation of oscillations of a special case of (1.1); that is we shall assume that $\sigma(t) = t - \sigma^*$ where σ^* is a positive constant.

Theorem 4.1. *In addition to the assumptions of Theorem 2.2, if*

$$\liminf_{t\to\infty} \int_{t-\sigma^*}^{t} p(s)(s-t)^{n-1}\mathrm{d}s > (1-c)(n-1)!, \tag{4.1}$$

then (1.1) *is oscillatory.*

Proof. Suppose that there exists an eventually positive solution $x(t)$ of (1.1). Then

$$\begin{aligned} Z^{(n)}(t) &= -p(t)x(t-\sigma^*) \\ &= -p(t)Z(t-\sigma^*) - cp(t)x(t-\sigma^*-\tau) \\ &= -p(t)Z(t-\sigma^*) - cp(t)Z(t-\sigma^*-\tau) \\ &\quad - c^2p(t)x(t-\sigma^*-2\tau), \end{aligned} \tag{4.2}$$

and so on. By Lemma 2.1, $Z(t) < x(t)$, and $Z'(t) < 0$. Hence (4.2) implies that

$$Z^{(n)}(t) \leqslant -p(t)Z(t-\sigma^*)[1+c+c^2+\ldots+c^m] \tag{4.3}$$

where m is an arbitrary large positive integer. From (4.1) it follows that we can choose an arbitrarily large positive integer m such that

$$\limsup_{t\to\infty} \int_{t-\sigma^*}^{t} p(s)(s-t)^{n-1}\mathrm{d}s > \left(\frac{1-c}{1-c^m}\right)[(n-1)!]. \tag{4.4}$$

In view of Lemma 2.1, for $t > s$ we have

$$\begin{aligned} Z(s-\sigma^*) &= \frac{Z(t-\sigma^*) + Z'(t-\sigma^*)(s-t) + \ldots + Z^{(n-1)}(t-\sigma^*)(s-t)^{n-1}}{(n-1)!} \\ &\quad + \frac{Z^{(n)}(\xi-\sigma^*)}{n!}(s-t)^n \\ &\leqslant \frac{Z^{(n-1)}(t-\sigma^*)}{(n-1)!}(s-t)^{n-1}, \end{aligned} \tag{4.5}$$

where $\xi \in (s,t)$. Substituting (4.5) in (4.3) where we first change t to s, one can derive that

$$Z^{(n)}(s) \leqslant -\frac{1-c^m}{(1-c)[(n-1)!]}p(s)\,(s-t)^{n-1}Z^{(n-1)}(t-\sigma^*). \tag{4.6}$$

Integrating (4.6) from $t-\sigma^*$ to t we get

$$Z^{(n-1)}(t)-Z^{(n-1)}(t-\sigma^* \leqslant -\left(\frac{1-c^m}{(1-c)\,(n-1)!}\right)Z^{(n-1)}(t-\sigma^*)\int_{t-\sigma^*.}^{t} p(s)\,(s-t)^{n-1}\mathrm{d}s.$$

That is

$$Z^{(n-1)}(t)+Z^{(n-1)}(t-\sigma^*)\Big[\frac{1-c^m}{(1-c)\,(n-1)!}\int_{t-\sigma^*}^{t} p(s)\,(s-t)^{n-1}\mathrm{d}s-1\Big] \leqslant 0. \tag{4.7}$$

By Lemma 2.1, $Z^{(n-1)}(t)$ is eventually positive and therefore in view of (4.4), inequality (4.7) provides a contradiction. The proof is now complete. ☐

For (1.1) with constant parameters we have the following result:

Theorem 4.2. *If* $p(t) \equiv p > 0$, $\sigma(t) = t-\sigma^*$, $\sigma^* > 0$, $0 < c < 1$ *and*

$$\left(\frac{p}{1-c}\right)^{\frac{1}{n}}\frac{\sigma^*}{n} > \frac{1}{\mathrm{e}}, \tag{4.8}$$

then (1.1) *is oscillatory.*

Proof. Since $0 < c < 1$, it follows from (4.8) tht there exists a sufficintly large integer m such that

$$\left[\frac{p(1-c^m)}{1-c}\right]^{\frac{1}{n}}\frac{\sigma^*}{n} > \frac{1}{\mathrm{e}}. \tag{4.9}$$

Since (4.3) becomes

$$Z^{(n)}(t)+p\frac{1-c^m}{1-c}Z(t-\sigma^*) \leqslant 0, \tag{4.10}$$

which implies that (4.10) has an eventually positive solution, we arrive at a contradiction to a known result (see [6], Lemma 3(ii)). This completes the proof. ☐

Remark 4.1. The condition (4.8) improves the condition of Theorem 3 in [5] since the parameter of the neutral term appears in (4.8) whereas such parametes do not appear in the condition used in Theorem 3 in [5].

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