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AN INFINITE COLLECTION OF ABSOLUTELY CONVEX SUBGROUPS

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A subgroup H of an orderable group G is said to be ***absolutely convex*** if H is convex in every ordering of G. The literature contains very few examples of such groups. We show that a well known infinite collection has this property.

Torsion free nilpotent groups are known to be orderable [3]. The following theorem provides a sufficient condition for the center of a nilpotent o-group to be convex.

Theorem. *If the center $Z(G)$ of a nilpotent o-group G is Archimedean, then it is convex.*

Proof. Let $b \in Z(G)$, $a \in G$, with $e < a < b$.

Suppose $a \notin Z(G)$. Then there exists a largest non-trivial commutator w starting with a.

$$w = [a, x_1, x_2, \ldots, x_n].$$

By the maximality of lenght, $w \in Z(G)$.

Let $y = [a, x_1, \ldots, x_{n-1}]$. (It may be that $y = a$.)

Since G is nilpotent, it is weakly abelian [4]. Therefore, $|y| \leqslant a < b$, and, without loss of generality, $y < b$. Again, because G is weakly abelian,

$$[y, x_n] \ll a.$$

Thus

$$[y, x_n] \ll b \quad \text{i.e.} \quad w \ll b.$$

This contradicts the Archimedean property of $Z(G)$. Thus no such non-trivial commutator exists and $a \in Z(G)$. □

Corollary. *If the center of a torsion free nilpotent group is of rank* 1, *then it is absolutely convex.*

Proof. Every order of a rank 1 group is Archimedean.

An infinite collection of absolutely convex subgroups:

Let S be a unitary subring of the rational numbers, e.g. the integers or finite decimals. Let n be a positive integer greater than 1 and let G be the group of all $n \times n$ lower triangular matrices with 1's on the diagonal and entries from S. G is known to be nilpotent of class $n - 1$ and to be orderable.

The center $Z(G)$ consists of all such matrices whose only possible non-zero non-diagonal entry is in the corner. This is isomorphic to the additive structure of S and, therefore, has rank 1. By the corollary, $Z(G)$ is absolutely convex. □

References

[1] *L. Fuchs*: Partially ordered algebraic systems, Pergamom Press, 1963.

[2] *H. A. Hollister*: Nilpotent ℓ-groups are representable, Algebra Universalis *8* (1978), 65–71.

[3] *A. I. Malcev*: On the full ordering of groups, Trudy Mat. Inst. Steklov *38* (1951), 173–175.

[4] *N. R. Reilly*: Nilpotent, weakly abelian, and Hamiltonian lattice-ordered groups, Czechoslovak Math. J. *33(108)* (1983), 348–353.

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