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Equivalent definitions of regular generalized Perron integral

EQUIVALENT DEFINITIONS OF REGULAR GENERALIZED PERRON INTEGRAL

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Dedicated to the memory of Professor Earl A. Coddington

0. INTRODUCTION

The aim of the present paper is to compare some notions of multidimensional generalized Riemann integrals whose definitions involve the notion of regularity of interval (i.e. the minimum of the ratios of its edges), among them those introduced by J. Mawhin [5], W. F. Pfeffer [6] and the authors [3].

Let I be a compact interval in $\mathbb{R}^n$, $f: I \to \mathbb{R}$. In connection with his investigation of the Gauss-Green theorem, J. Mawhin in [5] introduced a generalized Riemann integral which we will call the M-integral (see Definitions 5 and 1 in Section 2). Let us assume that f is M-integrable. It is known that $F(J) = M \int_J f$ exists for every interval $J \subset I$ and that F is additive in the following sense:

let J, K be nonoverlapping intervals such that $J \cup K$ is an interval, $J \cup K \subset I$; then

$$F(J \cup K) = F(J) + F(K).$$

On the other hand, $g: J \cup K \to \mathbb{R}$ need not be M-integrable even if both its restrictions $g|_J$, $g|_K$ are M-integrable (see [2], Example 1). This drawback was removed by W. F. Pfeffer [6] and, in a different way that will not be followed here, by the authors [2]. Pfeffer's integral, here called the Pf-integral (see Definitions 5 and 3 in Section 2) has the following properties:

(i) if f is Pf-integrable, then it is M-integrable and both integrals are equal to each other;

(ii) if J and K have the same properties as above, if $g\colon J \cup K \to \mathbf{R}$ and both its restrictions $g|_J, g|_K$ are Pf-integrable, then g is Pf-integrable and

$$\mathrm{Pf}\int_{J\cup K} g = \mathrm{Pf}\int_J g + \mathrm{Pf}\int_K g.$$

Let L be a compact interval, $I \subset \mathrm{Int}\, L$, $f\colon I \to \mathbf{R}$. Extend f to L by zero, i.e. define $f_{\mathrm{ex}}\colon L \to \mathbf{R}$ by

$$f_{\mathrm{ex}}(t) = \begin{cases} f(t) & \text{for } t \in I, \\ 0 & \text{for } I \in L\setminus I. \end{cases}$$

We will prove in the paper that f is Pf-integrable iff f_{ex} is M-integrable (and, of course, the integrals are equal to each other). Moreover, a simplified version of the Pf-integral (see Definitions 5 and 4 in Section 2) is shown to be equivalent to the original one. (This result was announced in [3], Example 7.)

Since a descriptive definition of the M-integral is available, see [3] and Section 4 of this paper, we thus solve also one of the problems posed by W. F. Pfeffer in [6].

1. Preliminaries

Throughout the paper, $n > 1$ is an integer, $\mathbf{R}^n$ is the n-dimensional Euclidean space with the norm

$$\|x\| = \max\{|x_i|\,;\, i = 1, 2, \ldots, n\}.$$

For $x \in \mathbf{R}^n, r > 0$ we denote

$$V(x, r) = \{y \in \mathbf{R}^n\,;\, \|y - x\| \leqslant r\}.$$

(Using these sets instead of balls $B(x, r)$ does not affect our considerations but often simplifies them technically.) The symbols $\mathrm{Int}\, M$, $\mathrm{Cl}\, M$, ∂M, $\mathrm{dist}(x, M)$ are used with the standard meaning provided $x \in \mathbf{R}^n$, $M \subset \mathbf{R}^n$, $m(M)$ is the n-dimensional Lebesgue measure, $|M|$ the number of elements of a finite set.

If $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$, $a_i < b_i$ for $i = 1, 2, \ldots, n$, we write

$$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] = [a, b],$$
$$[a_1, b_1) \times [a_2, b_2) \times \ldots \times [a_n, b_n) = [a, b).$$

Given $k \in \{0, 1, 2, \ldots, n-1\}$ then any k-dimensional linear manifold E in $\mathbf{R}^n$ which is **parallel to k distinct coordinate axes** will be called a k-*plane* (plane, if its dimension is not specified). (Note that the definition does not include the case $k = n$, i.e. $E = \mathbf{R}^n$.)

With help of the notion just introduced we will generalize the well known concept of regularity of an interval. Recall that the *regularity* of a nondegenerate interval $J = [a, b]$ in $\mathbf{R}^n$ is the number

$$\operatorname{reg} J = \min\{b_i - a_i, i = 1, 2, \ldots, n\} / \max\{b_i - a_i, i = 1, 2, \ldots, n\}.$$

Let E be a k-plane parallel to the x_{j_i}-axes, $j_i \neq j_\ell$ for $i \neq \ell$, $i, \ell = 1, 2, \ldots, k$. If $J \cap E = \emptyset$, we define $\operatorname{reg}_E J = \operatorname{reg} J$; if $J \cap E \neq \emptyset$ we set

(1) $\quad \operatorname{reg}_E J = \min\{b_{j_i} - a_{j_i}; i = 1, 2, \ldots, k\} / \max\{b_j - a_j; j = 1, 2, \ldots, n\}$

provided $k \in \{1, 2, \ldots, n - 1\}$;

$$\operatorname{reg}_E J = 1 \quad \text{provided} \quad k = 0.$$

Let $\mathscr{E} = \{E_1, E_2, \ldots, E_\ell\}$ be a finite family of planes (generally of various dimensions). We define

(2) $$\operatorname{reg}_{\mathscr{E}} J = \max\{\operatorname{reg}_E J; E \in \mathscr{E} \cup \{\mathbf{R}^n\}\},$$

where we put $\operatorname{reg}_{\mathbf{R}^n} J = \operatorname{reg} J$ (which is formally justified by the formulas defining $\operatorname{reg}_E J$, $\operatorname{reg} J$).

R e m a r k . Modifying our definitions, we can introduce

$$\operatorname{rg} J = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)/(\max\{b_i - a_i; i = 1, 2, \ldots, n\})^n$$

and

$$\operatorname{rg}_E J = (b_{j_1} - a_{j_1})(b_{j_2} - a_{j_2})\ldots(b_{j_k} - a_{j_k})/(\max\{b_i - a_i; i = 1, 2, \ldots, n\})^k,$$

if E is as above, $E \cap J \neq \emptyset$, $k > 0$;

$$\operatorname{rg}_E J = 1 \qquad \text{if } E \cap J \neq \emptyset, k = 0;$$
$$\operatorname{rg}_E J = \operatorname{rg} J \quad \text{if } E \cap J = \emptyset.$$

Finally, for a finite system $\mathscr{E}$ of planes we set

$$\operatorname{rg}_{\mathscr{E}} J = \max\{\operatorname{rg}_E J; E \in \mathscr{E} \cup \{\mathbf{R}^n\}\},$$

putting $\operatorname{rg}_{\mathbf{R}^n} J = \operatorname{rg} J$ analogously as above.

Let us note that $\mathrm{rg}_{\mathscr{E}}\, J$ is the quantity used by W. F. Pfeffer to define $(\varepsilon, \mathscr{E})$-intervals in his paper on generalized Riemann integrals [6] (except for the inessential difference in taking the diameter of J instead of the maximal edge). Since the inequalities

$$(\mathrm{reg}_E\, J)^k \leqslant \mathrm{rg}_E\, J \leqslant \mathrm{reg}_E\, J$$

evidently hold, the difference between reg and rg is immaterial.

Let $I \subset \mathbb{R}^n$ be a nondegenerate compact interval. Any finite set of pairs (t, J), where J is a nondegenerate compact interval, $t \in J \subset I$, and all intervals J from the set are (pairwise) nonoverlapping, is called *a system in I*. If, moreover, the union of all J's is the whole I, the system is called *a partition of I*.

Let $0 < \alpha \leqslant 1$. We say that an interval J is *α-regular* (*$E\alpha$-regular, $\mathscr{E}\alpha$-regular*) if $\mathrm{reg}\, J \geqslant \alpha$ ($\mathrm{reg}_E\, J \geqslant \alpha$, $\mathrm{reg}_{\mathscr{E}}\, J \geqslant \alpha$). A system Δ is called *α-regular* (*$E\alpha$-regular, $\mathscr{E}\alpha$-regular*) if J is *α-regular* (*$E\alpha$-regular, $\mathscr{E}\alpha$-regular*) for every J for which there is t such that $(t, J) \in \Delta$.

Let $\delta \colon I \to (0, \infty)$ (δ is called *a gauge on I*). A pair (t, J), $t \in J \subset I$ is said to be *δ-fine* if $J \subset V(t, \delta(t))$. A system Δ in I is said to be *δ-fine* if each of its elements is δ-fine.

If $f \colon I \to \mathbb{R}$ and $\Delta = \{(t, J)\}$ is a system in I, we will write

$$S(f, \Delta) = \sum_{(t, J) \in \Delta} f(t) m(J).$$

2. Definitions and main results

In the present section we will define several types of integral and formulate the main results (Theorems 1, 2) describing the relations between them. Throughout the section we assume that $I \subset \mathbb{R}^n$ is a compact nondegenerate interval, $f \colon I \to \mathbb{R}$, $0 < \alpha < 1$.

Definition 1. The function f is said to be M α-*integrable* (M for Mawhin) if there is $c \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a gauge δ on I such that

$$(3) \qquad\qquad |S(f, \Delta) - c| \leqslant \varepsilon$$

for every δ-fine α-regular partition Δ of I. We write $c = \mathrm{M}\alpha \int_I f$; c is called the Mα-*integral* (of f over I).

Definition 2. Let $I \subset \mathrm{Int}\, L \subset \mathbb{R}^n$, L a compact interval, let $f_{\mathrm{ex}} \colon L \to \mathbb{R}$ be defined by

$$f_{\mathrm{ex}}(x) = \begin{cases} f(x) & \text{for } x \in I, \\ 0 & \text{for } x \in L \setminus I. \end{cases}$$

The function $f: I \rightarrow \mathbb{R}$ is called *extensively α-integrable* (ex α-*integrable*) if there exists L, $I \subset \operatorname{Int} L$, such that $f_{\mathrm{ex}}: L \rightarrow \mathbb{R}$ is M α-integrable (on L). We write ex $\alpha \int_I f = \mathrm{M} \, \alpha \int_L f_{\mathrm{ex}}$ and call this number *the extensive α-integral* (of f over I).

R e m a r k . Neither the extensive α-integrability nor the value of the extensive α-integral depends on the choice of L. The proof is straightforward.

Definition 3. The function f is said to be Pf α-*integrable* (Pf for Pfeffer) if there is $c \in \mathbb{R}$ such that for every $\varepsilon > 0$ and every system of planes $\mathscr{E}$ there is a gauge δ on I such that (3) holds for every δ-fine $\mathscr{E}\alpha$-regular partition Δ of I. We write $c = \mathrm{Pf} \, \alpha \int_I b$; c is called the Pf α-*integral* (of f over I).

Definition 4. The function f is said to be *weakly* Pf α-*integrable* if there is $c \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a gauge δ on I such that (3) holds for every δ-fine $\mathscr{F}\alpha$-regular partition Δ of I, where $\mathscr{F}$ is the family of all k-planes which include a k-dimensional face of I, $k = 0, 1, 2, \ldots, n - 1$. We write $c = w \, \mathrm{Pf} \, \alpha \int_I f$; c is called the *weak* Pf α-*integral* (of f over I).

Definition 5. The function f is M-*integrable* (*extensively integrable*, Pf-*integrable*, *weakly* Pf-*integrable*) if it is M α-integrable (extensively α-integrable, Pf α-integrable, weakly Pf α-integrable) for every α, $0 < \alpha < 1$. We then write

$$\mathrm{M} \int_I f = \mathrm{M} \, \alpha \int_I f \ (\textit{the} \ \mathrm{M}\textit{-integral}),$$

$$\mathrm{ex} \int_I f = \mathrm{ex} \, \alpha \int_I f \ (\textit{the extensive integral}),$$

$$\mathrm{Pf} \int_I f = \mathrm{Pf} \, \alpha \int_I f \ (\textit{the} \ \mathrm{Pf}\textit{-integral}),$$

$$w \, \mathrm{Pf} \int_I f = w \, \mathrm{Pf} \, \alpha \int_I f \ (\textit{the weak} \ \mathrm{Pf}\textit{-integral}),$$

where $\alpha \in (0, 1)$ is chosen arbitrarily. (The correctness of the definition is obvious.)

For each of the above notions of integral we can define the respective primitive function F by $F: K \mapsto F(K) = \star \int_K f$, where $K \subset I$ is an interval and $\star$ stands for any one of the symbols from Definition 5. By the elementary properties of the above integrals F exists and is an additive function of interval (on I). By additivity we can extend its definition to finite unions of (nonoverlapping) intervals.

The following two theorems represent the main result of the present paper. Their proof is obtained by simply combining Propositions 1–4, which are formulated and proved in the next section.

Theorem 1. *The following conditions are equivalent:*

(i_α) f is extensively α-integrable,

(ii_α) f is Pf α-integrable,

(iii_α) f is weakly Pf α-integrable.

If one of the above conditions is fulfilled, then also

(iv_α) f is M α-integrable

and

$$\mathrm{ex}\,\alpha \int_I f = \mathrm{Pf}\,\alpha \int_I f = w\,\mathrm{Pf}\,\alpha \int_I f = \mathrm{M}\,\alpha \int_I f.$$

Theorem 2. *The following conditions are equivalent:*

(i) f *is extensively integrable,*

(ii) f *is Pf-integrable,*

(iii) f *is w Pf-integrable.*

If one of the above conditions is fulfilled, then also

(iv) f *is M-integrable*

and

$$\mathrm{ex} \int_I f = \mathrm{Pf} \int_I f = w\,\mathrm{Pf} \int_I f = \mathrm{M} \int_I f.$$

R e m a r k . The condition (iv) is not equivalent to the other conditions in Theorem 2, neither is (iv_α) equivalent to the other conditions in Theorem 1. This is demonstrated (after a minor routine modification) by [2], Example 1.

3. Proofs

Proposition 1. *If f is Pf α-integrable, then it is weakly Pf α-integrable, and the two integrals coincide.*

P r o o f follows directly from Definitions 3, 4. □

Proposition 2. *If f is weakly Pf α-integrable, then it is extensively α-integrable, and the two integrals coincide.*

We need the following lemma, proved in a more general setting as Lemma 2.11 in [3].

Lemma 1. *Let $\varepsilon > 0$, $N \subset I$, $m(N) = 0$, $f \colon I \to \mathbb{R}$. Then there exists a gauge δ_0 on N such that*

$$S(|f|, \Theta) \leqslant \varepsilon$$

for every δ_0-fine system Θ in I such that $s \in N$ for all $(s, K) \in \Theta$.

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P r o o f . Denote $D_1 = \{x \in N \,;\, |f(x)| \leqslant 2\}$, $D_k = \{x \in N \,;\, 2^{k-1} < |f(x)| \leqslant 2^k\}$ for $k = 2, 3, \ldots$. There exist open sets $G_k \supset D_k$ with $m(G_k) \leqslant \varepsilon 2^{-2k}$, $k = 1, 2, \ldots$. Choose δ_0 so that $V(x, \delta_0(x)) \subset G_k$ for $x \in D_k$. The proof is now completed in a straightforward way. □

P r o o f of Proposition 2. Let $\varepsilon > 0$. Set $\varepsilon_1 = \frac{1}{3}\varepsilon$ and find the corresponding gauge δ_1 from the definition of the weak Pf α-integral. Choose an interval L such that $\operatorname{Int} L \supset I$. Without loss of generality we will assume that

$$V(t, \delta_1(t)) \subset \operatorname{Int} I \text{ for } t \in \operatorname{Int} I,$$

$$V(t, \delta_1(t)) \subset L \text{ for } t \in \partial I.$$

Find the gauge δ_0 from Lemma 1 corresponding to ε_1 and $N = \partial I$. Define a gauge δ on L as follows:

$$\delta(t) = \begin{cases} \delta_1(t) & \text{for } t \in \operatorname{Int} I, \\ \min(\delta_1(t), \delta_0(t)) & \text{for } t \in \partial I, \\ \frac{1}{2}\operatorname{dist}(t, I) & \text{for } t \in L\backslash I. \end{cases}$$

Let Δ be a δ-fine α-regular partition of L. Define

$$\Delta_1 = \{(t, K \cap I) \,;\, (t, K) \in \Delta, K \cap I \text{ nondegenerate}\}.$$

Evidently, Δ_1 is a δ_1-fine partition of I; moreover, we will prove that it is $\mathcal{F}\alpha$-regular.

Indeed, let $E \in \mathcal{F}$; then $\operatorname{reg}_E(K \cap I) \geqslant \operatorname{reg}_E K$ since in the ratio defining the E-regularity the numerator does not change (the lengths of only those edges are included which are parallel to E and hence not cut by E), while the terms in the denominator either do not change or become smaller. (The case $k = 0$ is trivial.) Moreover, it is evident from (1) that $\operatorname{reg} K \leqslant \operatorname{reg}_E K$ for any E, hence $\operatorname{reg}_{\mathcal{F}}(K \cap I) \geqslant \operatorname{reg}_{\mathcal{F}} K \geqslant \operatorname{reg} K \geqslant \alpha$ since Δ was assumed α-regular.

Consequently,

$$|S(f, \Delta_1) - w\operatorname{Pf}\alpha\int_I f| \leqslant \varepsilon_1.$$

Further, we have

$$|S(f, \Delta) - S(f, \Delta_1)| \leqslant 2\varepsilon_1$$

since in virtue of the definition of the gauge δ all summands $f(t)m(J)$ cancel except those with $t \in \partial I$, and for them we have the estimate following from Lemma 1. Hence

$$|S(f, \Delta) - w\operatorname{Pf}\alpha\int_I f| \leqslant$$

$$\leqslant |S(f, \Delta) - S(f, \Delta_1)| + |S(f, \Delta_1) - w\operatorname{Pf}\alpha\int_I f| \leqslant 3\varepsilon_1 = \varepsilon,$$

which completes the proof of Proposition 2. □

Proposition 3. *If f is ex α-integrable, then it is M α-integrable, and the two integrals coincide.*

P r o o f.   follows immediately from the fact that the M α-integral, and hence also ex α-integral, is additive in the sense mentioned in Introduction.   □

Proposition 4. *If f is ex α-integrable, then it is Pf α-integrable, and the two integrals coincide.*

P r o o f.   We shall prove: for every $\varepsilon > 0$ and any system of planes $\mathscr{E} = \{E_1, E_2, \ldots, E_\ell\}$ there is a gauge δ on I such that (3) holds for every δ-fine $\mathscr{E}\alpha$-regular partition Δ of I. Note that (3) with $c = \mathrm{ex}\,\alpha \int_I f$ is equivalent to

$$\left| \sum_{(t,J)\in\Delta} (F(J) - f(t)m(J)) \right| \leqslant \varepsilon,$$

where F is the primitive of f (in the sense of the ex α-integral).

Let $\varepsilon > 0$ and $\mathscr{E}$ be fixed. Without loss of generality we may and will assume

(4) $\mathscr{F} \subset \mathscr{E}$, i.e. $\mathscr{E}$ includes all planes which contain a (k-dimensional, $k = 0, 1, \ldots, n-1$) face of I and have the minimal dimension possible;

(5) if $E_j \cap E_k \neq \emptyset$ then $E_j \cap E_k \in \mathscr{E}$ $(j, k = 1, 2, \ldots, \ell)$.

Indeed, these assumptions are justified by the evident implication $\mathscr{E} \subset \mathscr{H} \Rightarrow \mathrm{reg}_{\mathscr{E}} J \leqslant \mathrm{reg}_{\mathscr{H}} J$ where $\mathscr{E}, \mathscr{H}$ are any two finite systems of planes; hence the gauge δ corresponding to $\mathscr{H}$ and $\varepsilon > 0$ in the definition of the Pf α-integral can be used for any system $\mathscr{E} \subset \mathscr{H}$ (and the same ε).

Since f is ex α-integrable, we choose an interval L, $\mathrm{Int}\, L \supset I$, and find a gauge δ_1 on L corresponding to ε_1 in the definition of the ex α-integral (ε_1 will be specified later). Without loss of generality we will assume

(6) $$V(t, \delta_1(t)) \cap E_j = \emptyset \text{ for } t \in L \backslash E_j, j = 1, 2, \ldots, \ell$$

(note that, in particular, $V(t, \delta_1(t)) \cap I = \emptyset$ for $t \in L \backslash I$ and $V(t, \delta_1(t)) \subset \mathrm{Int}\, I$ for $t \in \mathrm{Int}\, I$ by virtue of (4)). Further, find a gauge δ_0 from Lemma 1 corresponding to ε_1 and $N = \bigcup_{j=1}^{\ell} E_j$. Set

$$\delta(t) = \begin{cases} \delta_1(t) & \text{for } t \in L \backslash \bigcup_{j=1}^{\ell} E_j, \\ \min(\delta_0(t), \delta_1(t)) & \text{for } t \in L \cap \bigcup_{j=1}^{\ell} E_j. \end{cases}$$

Let Δ be a δ-fine $\mathscr{E}\alpha$-regular partition of I. Denote

$$\Delta_1 = \{(t, J) \in \Delta \,;\, t \notin \bigcup_{j=1}^{\ell} E_j\}, \quad \Theta = \Delta \backslash \Delta_1.$$

Then reg $J \geqslant \alpha$ for $(t, J) \in \Delta_1$ since in this case $J \cap \bigcup_{j=1}^{\ell} E_j = \emptyset$ (cf. (6)). The Saks-Henstock lemma [3, Lemma 2.4] yields

$$(7) \qquad \sum_{(t,J) \in \Delta_1} |F(J) - f(t)m(J)| \leqslant 2\varepsilon_1,$$

where F is the primitive of f (in the sense of the ex α-integral).

Further, since $\delta(t) \leqslant \delta_0(t)$ for $t \in \bigcup_{j=1}^{\ell} E_j \cap I$, we have (cf. Lemma 1)

$$(8) \qquad S(|f|, \Theta) \leqslant \varepsilon_1.$$

Hence to estimate the integral sum

$$\sum_{(t,J) \in \Delta} |F(J) - f(t)m(J)|$$

it suffices to establish an estimate for

$$\sum_{(t,J) \in \Theta} |F(J)| = \sum_{j=1}^{\ell} \sum_{(t,J) \in \Theta_j} |F(J)|$$

where $\Theta_j = \{(t, J) \in \Theta; \ t \in E_j \backslash \bigcup \{E_k; \ \dim E_k < \dim E_j\}\}$. (The identity $\Theta = \bigcup_{j=1}^{\ell} \Theta_j$ holds due to (5).)

Let us fix $j \in \{1, 2, \ldots, \ell\}$ and order the coordinates so that

$$E_j = \{x \in \mathbb{R}^n; \ x_1 = c_1, x_2 = c_2, \ldots, x_m = c_m\},$$

where $1 \leqslant m \leqslant n$ and $c_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \Lambda = \{0, 1\}^m$ let us denote

$$Q_j^\lambda = \{x \in \mathbb{R}^n; \ (-1)^{\lambda_i}(x_i - c_i) \geqslant 0, i = 1, 2, \ldots, m\}.$$

We will first estimate $\sum_{(t,J) \in \Theta_j} |F(J \cap Q_j^\lambda)|$ for a fixed λ, say for $\lambda = 0 \in \mathbb{R}^m$. (The other cases are quite analogous.)

Let $(t, J) \in \Theta_j$. Then $t_1 = c_1$, $t_2 = c_2$, ..., $t_m = c_m$ and writing

$$J = [a, b] = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

we have

$$J \cap Q_j^0 = [c_1, b_1] \times \ldots \times [c_m, b_m] \times [a_{m+1}, b_{m+1}] \times \ldots \times [a_n, b_n]$$

and, obviously, $c_i \in [a_i, b_i)$ for $i = 1, 2, \ldots, m$. (The case $c_i = b_i$ can be omitted since then $J \cap Q_j^0$ is a degenerate interval.)

Let us denote

(9)
$$h = \max\{b_i - a_i \,;\, i = 1, 2, \ldots, n\},$$

and let K be an interval such that

(10)
$$K = [d_1, e_1] \times \ldots \times [d_m, e_m] \times [a_{m+1}, b_{m+1}] \times \ldots \times [a_n, b_n],$$
$$K \subset J \cap Q_j^0,$$
$$0 < e_i - d_i \leqslant (1 - \alpha)h \text{ for } i = 1, 2, \ldots, m$$

(hence $a_i \leqslant c_i \leqslant d_i < e_i \leqslant b_i$ for $i = 1, 2, \ldots, m$). For $\nu = (\nu_1, \nu_2, \ldots, \nu_m) \in \Lambda$ let us set

$$P(K, \nu) = [e_1 - h, d_1 + \nu_1(e_1 - d_1)] \times \ldots \times [e_m - h, d_m + \nu_m(e_m - d_m)] \times$$

$$\times [a_{m+1}, b_{m+1}] \times \ldots \times [a_n, b_n].$$

Evidently, $\chi_{[d_i, e_i)} = \chi_{[e_i - h, e_i)} - \chi_{[e_i - h, d_i)}$ where χ_M is the characteristic function of M. Consequently,

$$F(K) = \sum_{\nu \in \Lambda} \omega(\nu) F(P(K, \nu))$$

provided $\omega(\nu) \in \{-1, 1\}$ are properly chosen. Note that this is the point where we essentially exploit the fact that the ex α-integral (and not only M α-integral) of f exists, since the intervals $P(K, \nu)$ go beyond the interval I if E_j is a face of I.

Let us prove that

(11)
$$\operatorname{reg} P(K, \nu) \geqslant \alpha.$$

We will prove that all edges of $P(K, \nu)$ have lengths in the interval $[\alpha h, h]$, which obviously yields the desired inequality. If $i \in \{1, 2, \ldots, m\}$ then the length of the i-th edge is either $e_i - (e_i - h) = h$ (if $\nu_i = 1$), or $d_i - (e_i - h) = h - (e_i - d_i) \geqslant h - (1 - \alpha)h = \alpha h$ (if $\nu_i = 0$). Now recall that J is $\mathscr{E}\alpha$-regular. We have $\operatorname{reg}_{E_k} J = \operatorname{reg} J$ if $t \notin E_k$ (cf. (6)); since $(t, J) \in \Theta j$, this occurs if $E_k \cap E_j = \emptyset$ or if $E_k \cap E_j \neq \emptyset$ and $\dim E_k < \dim E_j$ (see the definition of Θ_j). If $E_k \supset E_j$ then $\operatorname{reg}_{E_k} J \leqslant \operatorname{reg}_{E_j} J$ (this implication holds generally, as is easily seen from (1)). Consequently, $\operatorname{reg}_{E_k} J \leqslant \operatorname{reg}_{E_j} J$ for $k = 1, 2, \ldots, \ell$, which implies

$$\operatorname{reg}_{\mathscr{E}} J = \operatorname{reg}_{E_j} J = \frac{1}{h} \min\{b_i - a_i \,;\, i = m + 1, m + 2, \ldots, n\}$$

and, in view of the $\mathscr{E}\alpha$-regularity of J, we have $b_i - a_i \geqslant \alpha h$ for $i = m + 1, m + 2, \ldots, n$, which completes the proof of (11).

There exist nonoverlapping intervals K_1, K_2, ..., K_ϱ such that $\bigcup_{\sigma=1}^{\varrho} K_\sigma = J \cap Q_j^0$ and each of the intervals has the form (10), i.e. its edges $[d_i, e_i]$, $i = 1, 2, \ldots, m$ are not longer than $(1 - \alpha)h$. Since $b_i - c_i \leqslant b_i - a_i \leqslant h$ for $i = 1, 2, \ldots, m$ (cf. (9)), we can choose the intervals K_σ so that their number ϱ satisfies the inequality

$$\text{(12)} \qquad \qquad \varrho < [(1 - \alpha)^{-1} + 1]^m.$$

Now, let us construct the intervals K_1, K_2, ..., K_ϱ just described for every interval $J \cap Q_j^0$ where $(t, J) \in \Theta_j$; let us denote them by $K_\sigma(J)$ (evidently, they can be chosen so that ϱ is the same for all J's). Hence we have

$$\sum_{(t,J)\in\Theta_j} |F(J \cap Q_j^0)| \leqslant \sum_{\sigma=1}^{\varrho} \sum_{(t,J)\in\Theta_j} |F(K_\sigma(J))| \leqslant$$

$$\text{(13)} \qquad \qquad \leqslant \sum_{\sigma=1}^{\varrho} \sum_{\nu\in\Lambda} \sum_{(t,J)\in\Theta_j} |F(P(K_\sigma(J), \nu))|.$$

Let $(t, J), (t', J') \in \Theta_j, J \neq J'$ and let both $J \cap Q_j^0, J' \cap Q_j^0$ be nondegenerate intervals. Analogously to $J = [a, b]$ let us write

$$J' = [a', b'] = [a'_1, b'_1] \times [a'_2, b'_2] \times \ldots \times [a'_n, b'_n].$$

Since the intervals J, J' do not overlap, the same holds a fortiori for intervals $K_\sigma(J) \subset J$, $K_\sigma(J') \subset J'$, $\sigma \in \{1, 2, \ldots, \varrho\}$. Moreover, since $J \cap Q_j^0$, $J' \cap Q_j^0$ are nondegenerate n-dimensional intervals by assumption, the inclusion

$$[c_1, \min(b_1, b'_1)] \times \ldots \times [c_m, \min(b_m, b'_m)] \times$$
$$\times [\max(a_{m+1}, a'_{m+1}), \min(b_{m+1}, b'_{m+1})] \times \ldots \times [\max(a_n, a'_n), \min(b_n, b'_n)]$$
$$\subset J \cap J' \cap Q_j^0$$

implies that the intervals $[a_{m+1}, b_{m+1}] \times \ldots \times [a_n, b_n]$, $[a'_{m+1}, b'_{m+1}] \times \ldots \times [a'_n, b'_n]$ do not overlap, either (in $\mathbf{R}^{n-m}$). Combining the two results, we conclude that any two intervals $P(K_\sigma(J), \nu)$, $P(K_\sigma(J'), \nu)$ with $\sigma \in \{1, 2, \ldots, \varrho\}$, $\nu \in \Lambda$ are nonoverlapping (in $\mathbf{R}^n$). (If $m = n$ then $E_j = \{s\}$ is a one-point set, Q_j^0 is an n-dimensional orthant, and evidently there is only one element (s, J) such that $J \cap Q_j^0$ is nondegenerate.)

For σ, ν fixed let us estimate as follows:

$$\text{(14)} \qquad \sum_{(t,J)\in\Theta_j} |F(P(K_\sigma(J), \nu))| \leqslant$$

$$\leqslant \sum_{(t,J)\in\Theta_j} |F(P(K_\sigma(J), \nu)) - f(t)m(P(K_\sigma(J), \nu))|$$

$$+ \sum_{(t,J)\in\Theta_j} |f(t)|m(P(K_\sigma(J), \nu)).$$

The gauge δ has been chosen so small that by Lemma 1 the second sum on the right-hand side is not greater than ε_1. Further, we have (since $(t, J) \in \Theta_j$ implies $t \in E_j \cap J$)

$$t \in P(K_\sigma(J), \nu) \subset V(t, h) \subset V(t, \delta(t)),$$
$$\operatorname{reg} P(K_\sigma(J), \nu) \geqslant \alpha \quad (\text{see } (11)),$$

and the intervals $P(K_\sigma(J), \nu)$ do not overlap as proved above. Hence they form a δ-fine α-regular system, which justifies the application of the Saks-Henstock lemma to the first sum on the right-hand side of the inequality (14), hence

$$\sum_{(t,J)\in\Theta_j} |F(P(K_\sigma(J), \nu))| \leqslant 3\varepsilon_1.$$

Taking into account the estimate (12) for ϱ and the fact that $\operatorname{Card}\Lambda = 2^m$ we find from (13):

$$\sum_{(t,J)\in\Theta_j} |F(J \cap Q_j^0)| \leqslant 2^m[(1 - \alpha)^{-1} + 1]^m 3\varepsilon_1.$$

Here Q_j^0 can be replaced per analogiam by any Q_j^λ with $\lambda \in \Lambda$, hence

$$\sum_{(t,J)\in\Theta_j} |F(J)| \leqslant \sum_{\lambda\in\Lambda} \sum_{(t,J)\in\Theta_j} |F(J \cap Q_j^\lambda)| \leqslant 2^{2m}[(1 - \alpha)^{-1} + 1]^m 3\varepsilon_1$$

and, finally summing over all planes E_j we obtain (using the obvious inequality $m = m(j) \leqslant n$)

$$\sum_{(t,J)\in\Theta} |F(J)| \leqslant \sum_{j=1}^{\ell} \sum_{(t,J)\in\Theta_j} |F(J)| \leqslant 2^{2n}\ell[(1 - \alpha)^{-1} + 1]^n 3\varepsilon_1.$$

Combining this estimate with (7), (8) we conclude

$$\sum_{(t,J)\in\Delta} |F(J) - f(t)m(J)| \leqslant 3(1 + 2^{2n}\ell[(1 - \alpha)^{-1} + 1]^n)\varepsilon_1 = q\varepsilon_1.$$

If we start the proof with $\varepsilon_1 = q^{-1}\varepsilon$, we arrive at an estimate proving that the Pf α-integral of f over I exists and coincides with the ex α-integral. $\qquad\square$

4. Concluding Remarks

1. In [3], Theorem 2.5 it was proved (in a more general axiomatic setting) that the M α-primitive function is continuous in the interior of the interval I of the M α-integrability. (That is, if $L \subset \mathrm{Int}\, I$ and $\varepsilon > 0$ then there is $\eta > 0$ such that $|F(L) - F(K)| \leqslant \varepsilon$ for every interval $K \subset I$ with $m(L \div K) \leqslant \eta$, where $\div$ denotes the symmetric difference.) This evidently implies that the ex α-primitive function is continuous everywhere in I (including the boundary), and consequently, the Pf α-primitive has the same property.

2. Note that for every $\alpha, 0 < \alpha < 1$, there is a function $f \equiv f_\alpha$ which is M α_1-integrable for every $\alpha_1, \alpha < \alpha_1 < 1$, and is not M α_2-integrable for every $\alpha_2, 0 < \alpha_2 < \alpha$, see [4]. It is even true that for every $0 < \alpha < 1$ there are functions g, h such that g is M α_1-integrable iff $\alpha < \alpha_1 < 1$ and h is M α_2-integrable iff $\alpha \leqslant \alpha_2 < 1$.

3. Specifying Theorem 4.2 of [3] to comply with our setting of the problem, we obtain

Theorem. *A function $f \colon I \to \mathbb{R}$ is* M *α-integrable with a primitive F iff*
(i) *F is additive,*
(ii) *F is α-regularly differentiable to $f(t)$ at almost every $t \in I$,*
(iii) *F is α-variationally normal on I.*

The last property is also called "good behavior on sets of zero measure", meaning the following: for every $N \subset I$ with $m(N) = 0$ and every $\varepsilon > 0$ there is $\delta \colon I \to (0, \infty)$ such that $(\Delta) \sum |F(J)| \leqslant \varepsilon$ for every δ-fine α-regular system Δ such that $t \in N$ for every $(t, J) \in \Delta$. (The sum is taken over all J such that $(t, J) \in \Delta$ for some t.)

The above theorem represents a descriptive definition of the M α-integral, and by easy modification also of the ex α-integral and ex-integral. Since the latter two integrals coincide with the Pf α and Pf-integral, respectively, we have solved the first part of Problem 6.6 posed by W. F. Pfeffer in [6], namely to give a descriptive definition of his concept of integral.

4. Let us mention here T.S. Chew's paper [1] in which the author claims to have solved the same problem. However, the relevant part of Chew's definition of the (allegedly Pfeffer's) integral reads "for every $\varepsilon > 0$ there is a gauge δ and a regularity $\varrho \dots$", while Pfeffer's definition starts "for every $\varepsilon > 0$ and every regularity ϱ there is a gauge $\delta \dots$". The order of the quantifiers certainly affects the notion of the integral substantially. Indeed, the function $f \equiv f_\alpha$ mentioned in point 2 of this section provides an example of a function that is not Pf-integrable but is Pf α_1-integrable for $\alpha < \alpha_1 < 1$, thus being integrable in the sense of T. S. Chew. (In [4] only M α_1-integrability is claimed, but it is immediately seen from the construction of

f that it is also ex α_1-integrable and hence Pf α_1-integrable as well.) Thus, Theorem 3 in [1] solves a problem different from Problem 6.6 in [6].

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