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OSCILLATION AND ASYMPTOTIC PROPERTIES OF n-TH ORDER DIFFERENTIAL EQUATIONS

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We consider the equation

$$L_n u(t) + p(t) f\big(u(g(t))\big) = 0, \tag{1}$$

where $n \geqslant 2$ and L_n denotes the disconjugate differential operator

$$L_n = \frac{\mathrm{d}}{\mathrm{d}t} r_{n-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} r_{n-2}(t) \frac{\mathrm{d}}{\mathrm{d}t} \cdots \frac{\mathrm{d}}{\mathrm{d}t} r_1(t) \frac{\mathrm{d}}{\mathrm{d}t}.$$

We always assume $f, p, r_k, g \in C\big([t_0, \infty)\big)$, $r_k(t), p(t) > 0$ for $1 \leqslant k \leqslant n-1$, $g(t) \to \infty$ as $t \to \infty$ and $xf(x) > 0$ for $x \neq 0$.

In the sequel we will restrict our attention to nontrivial solutions of the equations considered. Such a solution is called oscillatory if the set of its zeros is unbounded. Otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

We introduce the notation:

$$L_0 u(t) = u(t),$$
$$L_k u(t) = r_k(t)\big[L_{k-1} u(t)\big]', \quad 1 \leqslant k \leqslant n-1.$$

We say that the operator L_n is in canonical form if

$$R_k(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r_k(s)} \to \infty \text{ as } t \to \infty, \ 1 \leqslant k \leqslant n-1.$$

For $k \in \{1, 2, \ldots, n-1\}$ and $t \geqslant t_0$ we define

$$I_0 = 1,$$
$$I_k(t) = \int_{t_0}^{t} \frac{I_{k-1}(s)}{r_k(s)} \mathrm{d}s.$$

Definition 1. Equation (1) is said to *have property* (A) if for n even equation (1) is oscillatory, and for n odd every nonoscillatory solution $u(t)$ of (1) satisfies $L_k u(t) \to 0$ as $t \to \infty$, $0 \leqslant k \leqslant n-1$.

Definiton 2. Equation (1) is said to *have property* (C) if for n even equation (1) is oscillatory, and for n odd every nonoscillatory solution $u(t)$ of (1) satisfies $I_k(t)L_k u(t) \to 0$ as $t \to \infty$, $0 \leqslant k \leqslant n-1$.

Recently W. E. Mahfoud [2] has shown that if the equation

$$y^{(n)}(t) + p(t)f\big(y(g(t))\big) = 0 \tag{2}$$

has property (A), that is every nonoscillatory solution y of (1) satisfies $y^{(k)}(t) \to 0$ as $t \to \infty$ for $0 \leqslant k \leqslant n-1$, then equation (2) has property (C) as well, that is $t^k y^{(k)}(t) \to 0$ as $t \to \infty$ for $0 \leqslant k \leqslant n-1$. J. Ohriska [3] defined property (C) for the equation

$$\big(r(t)\dots\big(r(t)u'(t)\big)'\dots\big)' + p(t)f\big(y(g(t))\big) = 0 \tag{3}$$

and extended some Mahfoud's results from equation (2) to equation (3). The aim of this paper is to show that properties (A) and (C) are equivalent for equation (1), and to extend some results known for equation (3) to (1). In the whole paper we will deal only with equation (1) whose operator L_n is in canonical form.

Theorem 1. *Let u be a nonoscillatory solution of (1) such that $u(t) \to 0$ as $t \to \infty$. Then*

(i) $\left|\int^{\infty} \frac{L_k u(t)}{r_k(t)}\,\mathrm{d}t\right| < \infty$ *for* $k = 1, 2, \dots, n-1$.

(ii) $R_k(t)L_k u(t) \to 0$ *as* $t \to \infty$ *for* $k = 1, 2, \dots, n-1$.

(iii) $\left|\int^{\infty} \frac{I_{k-1}(t)L_k u(t)}{r_k(t)}\,\mathrm{d}t\right| < \infty$ *for* $k = 1, 2, \dots, n-1$.

(iv) $I_k(t)L_k u(t) \to 0$ *as* $t \to \infty$ *for* $k = 1, 2, \dots, n-1$.

Proof. Assume $u(t) > 0$ for $t \geqslant t_0$. Then a modification of the well-known lemma of Kiguradze [1] guarantees that $|L_k u(t)|$ are decreasing for all large t, say $t \geqslant t_1$, and since $u(t) \to 0$ as $t \to \infty$ we have $L_k u(t) \to 0$ as $t \to \infty$ for $k = 0, 1, \dots, n-1$. Hence for all $t \geqslant t_1$ and $1 \leqslant k \leqslant n-1$ we have

$$\left|\int_{t_1}^{t} \frac{L_k u(s)}{r_k(s)}\,\mathrm{d}s\right| = \left|L_{k-1}u(t) - L_{k-1}u(t_1)\right| \leqslant 2\left|L_{k-1}u(t_1)\right|.$$

Consequently, (i) holds and for any given $\varepsilon > 0$, there exists $t_2 \geqslant t_1$ such that $\left|\int_{t_2}^{\infty} \frac{L_k u(s)}{r_k(s)}\mathrm{d}s\right| < \varepsilon$. As $|L_k u(t)|$ are decreasing for all $t \geqslant t_1$, we have

$$\varepsilon > \left|\int_{t_2}^{t} \frac{L_k u(s)}{r_k(s)}\,\mathrm{d}s\right| \geqslant \left|L_k u(t)\right|\big(R_k(t) - R_k(t_2)\big), \text{ for all } t \geqslant t_2.$$

Since $L_k u(t) \to 0$ as $t \to \infty$, from the above inequality we necessarily have $2\varepsilon > |R_k(t) L_k u(t)|$ for all large t. Part (ii) is proved.

To verify (iii) and (iv) we use induction on k. Note that for $k = 1$ conditions (i) and (ii) are equivalent to (iii) and (iv), respectively. We assume that (iii) and (iv) hold for an arbitrary $k \in \{1, 2, \ldots, n-2\}$ and show that

$$\left| \int_{t_1}^{\infty} \frac{I_k(t) L_{k+1} u(t)}{r_{k+1}(t)} \mathrm{d}t \right| < \infty \tag{4}$$

and

$$I_{k+1}(t) L_{k+1} u(t) \to 0 \text{ as } t \to \infty. \tag{5}$$

Integration by parts yields

$$\left| \int_{t_1}^{t} \frac{I_k(s) L_{k+1} u(s)}{r_{k+1}(s)} \mathrm{d}s \right| = \left| I_k(t) L_k u(t) - I_k(t_1) L_k u(t_1) - \int_{t_1}^{t} \frac{I_{k-1}(s) L_k u(s)}{r_k(s)} \mathrm{d}s \right|,$$

and so (4) holds by the induction hypothesis. Hence, for any given $\varepsilon > 0$ there exists $t_3 > t_1$ such that

$$\left| \int_{t_3}^{\infty} \frac{I_k(t) L_{k+1} u(t)}{r_{k+1}(t)} \mathrm{d}t \right| < \varepsilon.$$

Proceeding exactly as above we see that

$$\varepsilon > |L_{k+1} u(t)| \int_{t_3}^{t} \frac{I_k(s)}{r_{k+1}(s)} \mathrm{d}s = |L_{k+1} u(t)| \big(I_{k+1}(t) - I_{k+1}(t_3) \big),$$

which completes our proof. □

The above theorem shows that if u is a nonoscillatory solution of (1) with the property $u(t) \to 0$ as $t \to \infty$ then a stronger result concerning the asymptotic behavior of the solution holds, namely $I_k(t) L_k u(t) \to 0$ as $t \to \infty$ for $k = 0, 1, \ldots, n-1$.

Remark 1. By virtue of Theorem 1, all results holding for equation (1) with property (A) hold also for (1) with property (C).

Now we prepared to introduce several examples of extending the results known for equation (3) to equation (1).

In what follows we assume that r is a positive and continuous function satisfying

$$r(t) \geqslant \max_{1 \leqslant k \leqslant n-1} r_k(t) \quad \text{and} \quad R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} \to \infty \text{ as } t \to \infty. \tag{6}$$

Theorem 2. *Assume that (6) is satisfied and* $g(t) \leqslant t$ *for* $t \geqslant t_0$. *Let* f *be a nondecreasing function on* $(-\infty, -\alpha] \cup [\alpha, \infty)$ *for some* $\alpha > 0$.

(i) Equation (1) has property (C) *if*

$$\int^{\infty} p(t) R^k(t) f\left[\mp c\left(R(g(t))\right)^{n-k-2}\right] \mathrm{d}t = \mp\infty \tag{7}$$

for every $c > 0$ *and every* $k \in \{0, 1, \ldots, n-2\}$.

Let, moreover, f *be bounded above or below.*

(ii) Equation (1) has property (C) *if*

$$\int^{\infty} p(t) f\left[\mp c\left(R(g(t))\right)^{n-1}\right] \mathrm{d}t = \mp\infty \tag{8}$$

for every $c > 0$.

Proof. Suppose that f and g satisfy the assumptions. Let us consider equation (3) with the function r given in (6). Theorem 3.6 in [3] shows that (3) has property (C) if (7) holds or f is bounded above or below and (8) is satisfied. Applying the comparison theorem for property (A) (e.g. Theorem 1 in [1]) to equations (3) and (1) and taking Remark 1 into account we obtain the assertions of the theorem. □

References

[1] *T. Kusano, M. Naito*: Comparison theorems for functional differential equations with deviating argument, J. Math. Soc. Japan *33* (1981), 509–532.

[2] *W. E. Mahfoud*: Oscillation and asymptotic behavior of solutions of n-th order nonlinear delay differential equations, J. Diff. Equations *24* (1977), 75–98.

[3] *J. Ohriska*: Oscillation of differential equations and v-derivatives, Czechoslovak Math. J. *39(114)* (1989), 24–44.

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