Czechoslovak Mathematical Journal

Jan Hamhalter

Completeness and modular cross-symmetry in normed linear spaces

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 1, 1–5

Persistent URL: http://dml.cz/dmlcz/128311

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

COMPLETENESS AND MODULAR CROSS-SYMMETRY IN NORMED LINEAR SPACES

JAN HAMHALTER, Praha

(Received April 29, 1986)

1. PRELIMINARIES

Let us first recall the basic notions and facts (see [3, 4]). In what follows let the symbol $L_C(X)$ mean the lattice of all closed subspaces of a normed linear space X. The partial ordering in $L_C(X)$ is given by set inclusion. We shall be mainly interested in the modular properties of $L_C(X)$.

1.1. **Definition.** Two elements a, b of a lattice are said to form a *modular pair* (in symbol $(a,b)\,\mathrm{Mod}$) when $(x \vee a) \wedge b = x \vee (a \wedge b)$ holds for all $x \leqslant b$, and they are said to form a *dual-modular pair* (in symbol $(a,b)\,\mathrm{Mod}^*$) when $(x \wedge a) \vee b = x \wedge (a \vee b)$ holds for all $x \geqslant b$.

In the papers [1, 2] G. W. Mackey and S. S. Holland, Jr. have obtained the following description of modular (dual modular, respectively) pairs in $L_C(X)$.

1.2. Theorem. *Let us suppose that $A, B \in L_C(X)$. Then*
(i) $(A,B)\,\mathrm{Mod}^*$ *if and only if* $A + B \in L_C(X)$.
(ii) *Let $A \cap B = \{0\}$. If we define projections P_1, P_2 such that $P_1(a+b) = a$ and $P_2(a+b) = b$ ($a \in A$, $b \in B$), then we have $(A,B)\,\mathrm{Mod}$ if and only if P_1, P_2 are bounded as maps on the normed linear space $A + B$.*

Outline of the proof (for details see [1]). The proof of the statement (i) is not difficult and is based only upon easy algebraic computations (see [1]).

The proof of the statement (ii) is more complicated, in contrast to the simple characterization of dual-modularity. The idea of this proof is to transfer the problem to the conjugate space, which is a Banach space so that the closed graph theorem holds, and in which the original modular pair is transferred to a dual modular pair so that we can use (i). This transformation is a dual-isomorphic mapping of the lattice $L_C(X)$ to the lattice $\mathcal{L}(X^*)$ of all weakly* closed linear subspaces of X^* and has the form $A \to A^0$, where A^0 is the anihilator of A. By (i) $(A,B)\,\mathrm{Mod}^*$ in $L_C(X)$ if and only if $(A^0, B^0)\,\mathrm{Mod}^*$ in $\mathcal{L}(X^*)$, which is equivalent to $A^0 + B^0 = X^*$. Without loss of generality we can suppose that $X = \overline{A+B}$. The adjoints P_1 and P_2 are easily computed. The domains of P_1^*, P_2^* are $A^0 + B^0$, and $P_1^*(a' \dot{+} b') = b'$, $P_2^*(a' + b') = a'$ for every $a' \in A^0$ and $b' \in B^0$.

If $(A, B)\,\mathrm{Mod}$ in $L_C(X)$, then $A^0 + B^0 = X^*$. Thus P_1^* and P_2^* are closed, everywhere defined linear operators on the Banach space X^* and, by the closed graph theorem, they are bounded. Therefore P_1^{**} and P_2^{**} are also bounded, and with them P_1 and P_2.

Conversely, if P_1 and P_2 are bounded then P_1^* and P_2^* are bounded operators defined everywhere on X^* and $X^* = \mathrm{dom}(P_1^*) = \mathrm{dom}(P_2^*) = A^0 + B^0$. By (i) we have $(A, B)\,\mathrm{Mod}$, which concludes the proof. □

We shall be mainly concerned with spaces possessing an *unconditional basis.*

1.3. **Definition.** A subset $(e_n)_{n=1}^{\infty}$ of a normed linear space X is called an unconditional basis if the following conditions are satisfied:

(i) $\overline{\mathrm{sp}}(e_n)_{n=1}^{\infty} = X$,

(ii) there is a positive number c with the property: $\|\sum_{i\in F} \alpha_i e_i\| \leqslant \|\sum_{i\in F'} \alpha_i e_i\|$ for all $F \subset F'$, where F, F' are finite subsets of N, and for any sequence $(\alpha_i)_{i=1}^{\infty}$ of real numbers.

It is easy to see that if X has an unconditional basis $(e_n)_{n=1}^{\infty}$ then $(e_n)_{n=1}^{\infty}$ is an unconditional basis of the completion $\widetilde{X}$ of X and, moreover, if $x \in \widetilde{X}$, then we can write $x = \sum_{n=1}^{\infty} \alpha_n e_n$ with uniquely given coefficients α_n $(n \in N)$. For every nonempty subset $F \subset N$, let us denote by P_F the projection on $\widetilde{X}$ determined by the formula

$$P_F\Big(\sum_{n=1}^{\infty} \alpha_n e_n\Big) = \sum_{n\in F} \alpha_n e_n \ .$$

The correctness of this definition verifies easily.

1.4. **Definition** (see [3]). An unconditional basis $(e_n)_{n=1}^{\infty}$ of a normed linear space is called a *subsymmetric basis* if for every subsequence $(e_{n_i})_{i=1}^{\infty}$ there is an isomorphism T between X and $\overline{\mathrm{sp}}(e_{n_i})_{i=1}^{\infty}$ such that $T(e_i) = e_{n_i}$ for all $i \in N$.

2. Modularity and dual modularity in $L_C(X)$ for X with an unconditional basis

2.1. **Definition.** A lattice L is called *cross-symmetric* if $(a, b)\,\mathrm{Mod}$ implies $(b, a)\,\mathrm{Mod}^*$ for every $a, b \in L$ (see [4]).

Let X be a normed linear space. A lattice $L_C(X)$ is called *stable cross-symmetric* if the lattice $L_C(\mathrm{sp}(X \cup \{a\}))$ is cross-symmetric for every a from the completion $\widetilde{X}$ of X.

2.2. Proposition. *Let X be a normed linear space and let $(e_n)_{n=1}^{\infty}$ be its unconditional basis. If $F_1, F_2 \subset N$ and $F_1 \cap F_2 = \emptyset$, then the spaces $A_1 = \overline{\mathrm{sp}}(e_n)_{n\in F_1}^{\infty}$, $A_2 = \overline{\mathrm{sp}}(e_n)_{n\in F_2}^{\infty}$ form a modular pair in $L_C(X)$.*

The proof follows from the continuity of P_{F_2}, P_{F_2} and Theorem 1.2. P_{F_1} and P_{F_2} are bounded as projections on $\widetilde{X}$ and they also have to be bounded when understood as projections on $A_1 + A_2$.

Using this observation we can prove the following theorem.

2.3. Theorem. *Let us suppose that X is a normed linear space and let $(e_n)_{n=1}^{\infty}$ be its unconditional basis. If there is an $x \in X$ such that $P_F(x) \notin X$ for a set $F \subset N$, then $L_C(X)$ is not cross-symmetric.*

Proof. Let us consider all closure operations as if they were in $\widetilde{X}$. Since $(e_n)_{n=1}^{\infty}$ is an unconditional basis of $\widetilde{X}$, we have $x = P_F(x) + P_{n\setminus F}(x)$ for every $x \in X$. Let us suppose that $P_F(x) \notin X$ and let $L_C(X)$ be cross-symmetric. We have $P_F(x) = \lim_{n\to\infty} x_n$ and $P_{N\setminus F}(x) = \lim_{n\to\infty} y_n$, where $(x_n)_{n=1}^{\infty} \subset \overline{\mathrm{sp}}(e_n)_{n\in F} \bigcap X$ and $(y_n)_{n=1}^{\infty} \subset \overline{\mathrm{sp}}(e_n)_{n\in N\setminus F} \bigcap X$. Put $A_1 = \overline{\mathrm{sp}}(e_n)_{n\in F} \bigcap X$ and $A_2 = \overline{\mathrm{sp}}(e_n)_{n\in N\setminus F} \bigcap X$. By Proposition 2.1, we see that $(A_1, A_2)\,\mathrm{Mod}$ in $L_C(X)$. If $L_C(X)$ were cross-symmetric, then $(A_1, A_2)\,\mathrm{Mod}^*$. Obviously, $A_1 + A_2 \in L_C(X)$. We therefore have $x = P_F(x) + P_{N-F}(x) \in \overline{A_1 + A_2} \bigcap X = A_1 + A_2$. Further, $P_F(x) + P_{N\setminus F}(x) = a + b$, where $a \in A_1$, $b \in B_2$. This implies that $a - P_F(x) = P_{N\setminus F}(x) - b \in \overline{A_1} \cap \overline{A_2} = \{0\}$ which is a contradiction. □

The last theorem yields the following two corollaries. By Theorem 2.3, we can prove with technical innovations the result of S. S. Holland ([1, Theorem 1]).

2.4. Corollary (Holland). *Let X be an inner product space and let $L_C(X)$ be cross-symmetric. Then X is a Hilbert space.*

Proof. Let us suppose that $\widetilde{X} \setminus X \neq \emptyset$. Take an element $a \in \widetilde{X} \setminus X$. Since X is dense in $\widetilde{X}$, there is $x \in X$ such that $\langle a, x\rangle \neq 0$. We put $b = a - (\langle a, x\rangle^{-1} \|a\|^2)x$. It is obvious that $a - b \in X$ and $\langle a, b\rangle = 0$. Using the standard Hilbert space technique we can now find two sequences $(a_n)_{n=1}^{\infty} \subset X$ and $(b_n)_{n=1}^{\infty} \subset X$ with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $a_j \perp b_i$ for all $i, j \in N$. The Gramm-Schmidt process allows us to construct an orthonormal sequence $(e_n)_{n=1}^{\infty} \subset X$ such that $\mathrm{sp}(e_{2n})_{n=1}^{\infty} = \mathrm{sp}(a_n)_{n=1}^{\infty}$ and $\mathrm{sp}(e_{2n-1})_{n=1}^{\infty} = \mathrm{sp}(b_n)_{n=1}^{\infty}$. The set $(e_n)_{n=1}^{\infty}$ is an unconditional basis for the space $\overline{\mathrm{sp}}(e_n)_{n=1}^{\infty} \bigcap X$. Put $y = a - b$ and set $F = \{2n \mid n \in N\}$. It follows that $P_F(y) \notin X$, which is a contradiction. □

2.5. Corollary. *Let X be a normed space and let $(e_n)_{n=1}^{\infty}$ be its unconditional basis. Then for every $a \in X \setminus \mathrm{sp}(e_n)_{n=1}^{\infty}$ the lattice $L_C(\mathrm{sp}((e_n)_{n=1}^{\infty} \cup \{a\}))$ is not cross-symmetric.*

Proof. Obviously, the sequence $(e_n)_{n=1}^{\infty}$ is an unconditional basis of the space $A = \mathrm{sp}\big((e_n)_{n=1}^{\infty} \cup \{a\}\big)$. We can write $a = \sum_{n=1}^{\infty} a_n e_n$ and choose such infinite disjoint sets $F_1, F_2 \subset N$ that $a_n \neq 0$ for all $n \in F_1 \cup F_2$. It is easy to see that $P_{F_1}(a) \notin X$, and we can apply Theorem 2.3. □

As a by-product, observe that the foregoing corollary makes it possible to construct various incomplete spaces whose lattices of all closed subspaces are not cross-symmetric and which are not, generally, inner product spaces.

Example. Let X be either the space c_0 or l_p $(1 \leqslant p < \infty)$. Let $(e_n)_{n=1}^{\infty}$ be the standard unconditional basis $e_1 = (1, 0, \ldots, 0, \ldots)$; $e_2 = (0, 1, 0, \ldots, 0, \ldots)$; .... Then $L_C\big(\mathrm{sp}((e_n)_{n=1}^{\infty} \cup \{a\})\big)$ is not cross-symmetric for any $a \in X$ with infinitely many nonzero coordinates.

A similar technique allows us to derive the following result.

2.6. Lemma. *Let X be a normed linear space and let $L_C(X)$ be stable cross-symmetric. If subspaces $A_1, A_2 \in L_C(X)$ with completions $\widetilde{A_1} \cap \widetilde{A_2} = \{0\}$ form a modular pair in $L_C(X)$, then at least one of A_i, $(i = 1, 2)$ is complete.*

Proof. Let us assume the opposite case and try to reach a contradiction. Let both A_1 and A_2 be incomplete. Then there are elements $a_1, a_2 \in \widetilde{X}$ with $a_1 \in \widetilde{A_1} - A_1$, $a_2 \in \widetilde{A_2} - A_2$. Let $a = a_1 + a_2$. Consider the space $\mathrm{sp}(X \cup \{a\})$. Observe first that the spaces A_1, A_2 are closed in $\mathrm{sp}(X \cup \{a\})$. Indeed, suppose e.g. that $x_n \to x$ for $(x_n)_{n=1}^{\infty} \subset A_1$ and $x \in \mathrm{sp}(\widetilde{X} \cup \{a\})$. We can assume that $x = \lambda a + y$, where $\lambda \in R$, $y \in X$. The spaces $\widetilde{A_1}$, $\widetilde{A_2}$ form a modular pair in $(e_n)_{n=1}^{\infty}$. The projections $\widetilde{P_1}$, $\widetilde{P_2}$ defined on $\widetilde{A_1} + \widetilde{A_2}$ and corresponding to $\widetilde{A_1}, \widetilde{A_2}$ are bounded and therefore $\widetilde{P_2}(x) = \lambda \widetilde{P_2}(a) = \lambda a_1 + P_2(y) = 0$. By the assumption $L_C(X)$ is cross-symmetric and we see (by computations as in the proof of Theorem 2.3) that $\widetilde{P_2}(y) \in X$. Thus $\lambda a_1 \in X$ and obviously $\lambda = 0$.

This means that A_1, A_2 form a modular pair in $L_C(\mathrm{sp}(X \cup \{a\}))$, which is a cross-symmetric space and therefore $a \in \overline{A_1 + A_2} \cap \mathrm{sp}(X \cup \{a\}) = A_1 + A_2$. This contradiction concludes the proof. □

If we apply this lemma in spaces with unconditional basis, we immediately obtain the following result.

2.7. Corollary. *Let X be a normed linear space and let $(e_n)_{n=1}^{\infty}$ be its unconditional basis. Let us take sets $F_1, F_2 \subset N$, $F_1 \cap F_2 = \emptyset$, and put $A_1 = \overline{\mathrm{sp}}(e_n)_{n \in F_1}$, $A_2 = \overline{\mathrm{sp}}(e_n)_{n \in F_2}$. If $L_C(X)$ is stable cross-symmetric, then at least one of A_i $(i = 1, 2)$ is complete.*

This corollary affirms that if we require the condition of stable cross-symmetry, then we can express X in the form of a direct sum of two spaces with one of the summands complete.

2.8. Theorem. *Let X be a normed linear space. If $L_C(X)$ is stable cross symmetric and if the product $X \times X$ is isomorphic to X, then X is complete.*

The assertion of this theorem follows easily from Lemma 2.6. From Corollary 2.5, we have the following criterion of completeness.

2.9. Theorem. *Let X be a normed linear space with a subsymmetric basis. If $L_C(X)$ is stable cross-symmetric, then X is complete.*

The above results show that some extensions of Mackey's conjecture hold. The next step in pursuing Mackey's problem seems to be a thorough analysis of the relation between the stable cross-symmetry and the equivalence of modularity and dual modularity.

References

[1] *S. S. Holland, Jr.*: Partial solution to Mackey's problem about modular pairs and completeness, Canad. J. Math. *21* (1969), 1518–1525.

[2] *G. W. Mackey*: On infinite dimensional linear spaces, Trans. Amer. Math. Soc. *57* (1945), 155–207.

[3] *T. J. Marti*: Introduction to the Theory of Bases, Springer Tracts in Natural Philosophy, 1969.

[4] *F. Maeda, S. Maeda*: Theory of Symmetric Lattices, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

Author's address: Czech Technical University, Electrical Engineering, Department of Mathematics, Technická 2, 166 27 Prague 6, Czechoslovakia.