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TWO SIDED NORM ESTIMATE OF THE BERGMAN PROJECTION ON L^p SPACES

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Abstract. We give some explicit values of the constants C_1 and C_2 in the inequality $C_1/\sin(\pi/p) \leqslant |P|_p \leqslant C_2/\sin(\pi/p)$ where $|P|_p$ denotes the norm of the Bergman projection on the L^p space.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and let $\mathrm{d}A(z)$ be the Lebesgue measure on $\mathbb{D}$. For $0 < p < \infty$, let $L^p(\mathbb{D})$ denote the space of complex-valued measurable functions f on $\mathbb{D}$ such that

$$|f|_p = \left(\int_{\mathbb{D}} |f|^p \, \mathrm{d}A \right)^{\frac{1}{p}} < \infty.$$

We denote by P the integral operator on $L^p(\mathbb{D})$ defined by

$$Pf(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{(1 - z\overline{\xi})^2} \, \mathrm{d}A(\xi) \quad \text{(the Bergman projection).}$$

We denote by $|P|_p$ the norm of P on $L^p(\mathbb{D})$. It is well known (see [2], for example) that P is a bounded operator on $L^p(\mathbb{D})$ $(1 < p < \infty)$. In [3] the interesting fact is proved that the norm of the Bergman projection on $L^p(B)$ (B is the unit ball in $\mathbb{C}^n$) is comparable to $1/\sin \pi/p$ for $1 < p < \infty$.

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In this note we give new concrete values of the constants C_1 and C_2 in the inequality

$$C_1 \frac{1}{\sin \pi/p} \leqslant |P|_p \leqslant C_2 \frac{1}{\sin \pi/p}.$$

2. Result

Let

$$K_p = \max_{\alpha > -1} \frac{\Gamma\big(1+\frac{\alpha}{p}\big)\Gamma\big(\frac{2}{p}\big)}{\Gamma\big(\frac{\alpha+2}{p}\big)} \sqrt[p]{\frac{\Gamma^2\left(1+\frac{\alpha}{2}\right)}{\Gamma\left(1+\alpha\right)}}$$

(Γ is the Euler gamma function).

Theorem 1. *If* $2 \leqslant p \leqslant +\infty$, *then*

$$K_p \leqslant |P|_p \leqslant \frac{\pi}{\sin \pi/p},$$

and if $1 < p \leqslant 2$, *we have*

$$K_{\frac{p}{p-1}} \leqslant |P|_p \leqslant \frac{\pi}{\sin \pi/p}.$$

Proof. Let q: $1/p + 1/q = 1$ and $h(\xi) = (1 - |\xi|^2)^{-1/pq}$. Then, after simple calculations, we get

$$\frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{\left|1 - z\overline{\xi}\right|^2} h(\xi)^q \, \mathrm{d}A(\xi) = \sum_{n=0}^{\infty} |z|^{2n} \, B\Big(1 - \frac{1}{p}, n+1\Big) \tag{1}$$

($B(\cdot,\cdot)$ is the Euler beta function). Since

$$B\Big(1 - \frac{1}{p}, n+1\Big) = \frac{\Gamma\big(1 - \frac{1}{p}\big)\Gamma\left(n+1\right)}{\Gamma\big(n + 2 - \frac{1}{p}\big)}$$

and

$$\binom{s}{n} = (-1)^n \frac{\Gamma(n-s)}{\Gamma(-s)\,\Gamma(n+1)}$$

we obtain from (1)

$$\frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{\left|1 - z\overline{\xi}\right|^2} h(\xi)^q \, \mathrm{d}A(\xi) \tag{2}$$

$$= \Gamma\Big(1 - \frac{1}{p}\Big)\Gamma\Big(\frac{1}{p}\Big) \sum_{n=0}^{\infty} (-1)^n \binom{-1/p}{n} |z|^{2n} \frac{\Gamma^2\left(n+1\right)}{\Gamma\big(n + 2 - \frac{1}{p}\big)\Gamma\big(n + \frac{1}{p}\big)}.$$

Since the function $x \longmapsto \ln \Gamma(x)$ is convex, we have

$$\frac{\Gamma^2(n+1)}{\Gamma\big(n+2-\frac{1}{p}\big)\Gamma\big(n+\frac{1}{p}\big)} \leqslant 1$$

and from (2) we conclude that

$$(3) \qquad \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{1}{\left|1-z\overline{\xi}\right|^2} h(\xi)^q \, \mathrm{d}A(\xi) \leqslant \frac{\pi}{\sin \pi/p}(1-|z|^2)^{-1/p} = \frac{\pi}{\sin \pi/p} h(z)^q .$$

Similarly,

$$(4) \qquad \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{1}{\left|1-z\overline{\xi}\right|^2} h(z)^p \, \mathrm{d}A(z) \leqslant \frac{\pi}{\sin \pi/q} h(\xi)^p .$$

From (3) and (4), by Schur's theorem ([2], p. 42) we obtain

$$|P|_p \leqslant \frac{\pi}{\sin \pi/p}.$$

In order to estimate $|P|_p$ from below, it is enough to suppose that $p > 2$; then the case $1 < p < 2$ follows by duality. Let $0 < \lambda < 1$, $\alpha > -1$ and

$$f_\lambda(z) = (1-|z|^2)^{\alpha/p}(1-\lambda z)^{-(\alpha+2)/p}, \qquad z \in \mathbb{D}.$$

Then we can easily conclude that

$$(5) \qquad |f_\lambda|_p^p = \frac{\pi\,\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)} \sum_{n=0}^{\infty} \lambda^{2n} \frac{\Gamma^2\left(n+1+\frac{\alpha}{2}\right)}{\Gamma(n+1)\,\Gamma(n+\alpha+2)}.$$

Since

$$\frac{\Gamma^2\big(n+1+\frac{\alpha}{2}\big)}{\Gamma(n+1)\,\Gamma(n+\alpha+2)} = \frac{1}{n+1} + O\Big(\frac{1}{(n+1)^2}\Big),$$

we obtain from (5)

$$(6) \qquad |f_\lambda|_p^p = \frac{\pi\Gamma(1+\alpha)}{\Gamma^2\big(1+\frac{\alpha}{2}\big)}\Big(-\frac{\ln(1-\lambda^2)}{\lambda^2}\Big) + g_1(\lambda)$$

where g_1 is a bounded function on $[0,1]$. By direct calculation, we get

$$(7) \qquad Pf_\lambda(z) = \frac{\Gamma\big(1+\frac{\alpha}{p}\big)}{\Gamma\big(\frac{\alpha+2}{p}\big)} \sum_{n=0}^{\infty} (\lambda z)^n (n+1) \frac{\Gamma\big(n+\frac{\alpha+2}{p}\big)}{\Gamma\big(n+2+\frac{\alpha}{p}\big)}$$
$$= \frac{\Gamma\big(1+\frac{\alpha}{p}\big)\Gamma\big(\frac{2}{p}\big)}{\Gamma\big(\frac{\alpha+2}{p}\big)} \sum_{n=0}^{\infty} (\lambda z)^n (-1)^n \binom{\frac{-2}{p}}{n} \frac{\Gamma(n+2)\Gamma\big(n+\frac{\alpha+2}{p}\big)}{\Gamma\big(n+\frac{2}{p}\big)\Gamma\big(n+2+\frac{\alpha}{p}\big)}.$$

Since

$$\frac{\Gamma(n+2)\Gamma\left(n+\frac{\alpha+2}{p}\right)}{\Gamma\left(n+\frac{2}{p}\right)\Gamma\left(n+2+\frac{\alpha}{p}\right)}=1+O\left(\frac{1}{n+1}\right)$$

and $\left|\binom{-2/p}{n}\right| \leqslant \operatorname{const} n^{-(1-2/p)}$, from (7) it follows (if $p>2$) that

$$Pf_\lambda(z)=\frac{\Gamma\left(1+\frac{\alpha}{p}\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{\alpha+2}{p}\right)}(1-\lambda z)^{-2/p}+g_2(\lambda,z) \tag{8}$$

where $|g_2(\lambda,z)| \leqslant M<+\infty$, $z\in\mathbb{D}$, $\lambda\in[0,1]$. Having in mind that

$$|(1-\lambda z)^{-\frac{2}{p}}|_p=\left(\frac{\pi}{\lambda^2}\right)^{1/p}\left(-\ln(1-\lambda^2)\right)^{1/p},$$

from (8) we obtain

$$|Pf_\lambda|_p \geqslant \frac{\Gamma\left(1+\frac{\alpha}{p}\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{\alpha+2}{p}\right)}\left(\frac{\pi}{\lambda^2}\right)^{1/p}\left(-\ln(1-\lambda^2)\right)^{1/p}-|g_2|_p. \tag{9}$$

So

$$|P|_p \geqslant \frac{|Pf_\lambda|_p}{|f_\lambda|_p} \geqslant \frac{\frac{\Gamma\left(1+\frac{\alpha}{p}\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{\alpha+2}{p}\right)}\left(\frac{\pi}{\lambda^2}\right)^{1/p}\left(-\ln(1-\lambda^2)\right)^{1/p}-|g_2|_p}{\left(\frac{\pi\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)}\left(-\frac{\ln(1-\lambda^2)}{\lambda^2}\right)+g_1(\lambda)\right)^{1/p}}.$$

From that, when $\lambda\to 1-$, we get

$$|P|_p \geqslant \frac{\Gamma\left(1+\frac{\alpha}{p}\right)\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{\alpha+2}{p}\right)}\sqrt[p]{\frac{\Gamma^2\left(1+\frac{\alpha}{2}\right)}{\Gamma(1+\alpha)}}.$$

Since the previous inequality holds for every $\alpha>-1$, we have

$$|P|_p \geqslant K_p \quad (p>2)\,.$$

□

Remark 1. It is clear that (putting $\alpha=p-2$)

$$K_p \geqslant \Gamma\left(2-\frac{2}{p}\right)\Gamma\left(\frac{2}{p}\right)\sqrt[p]{\frac{\Gamma^2\left(\frac{p}{2}\right)}{\Gamma(p-1)}}$$

i.e.

$$K_p \geqslant \frac{\pi\left(\frac{1}{2}-\frac{1}{p}\right)}{\sin\pi\left(\frac{1}{2}-\frac{1}{p}\right)}\frac{1}{\sin\frac{\pi}{p}}\sqrt[p]{\frac{\Gamma^2\left(\frac{p}{2}\right)}{\Gamma(p-1)}}.$$

We observe that $\sqrt[p]{\Gamma^2(\frac{1}{2}p)/\Gamma(p-1)} \geqslant \frac{1}{2}$ for $p \geqslant 2$ Indeed, previous inequality is equivalent to the inequality

$$\frac{\Gamma^2\left(\frac{p}{2}\right)}{\sqrt{\pi}\,\Gamma(p)} \geqslant \frac{1}{2^p(p-1)\sqrt{\pi}}$$

and, according to Legandre duplication formula, we obtain

$$\frac{\Gamma^2\left(\frac{p}{2}\right)}{2^{p-1}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p+1}{2}\right)} \geqslant \frac{1}{2^p(p-1)\sqrt{\pi}}$$

i.e.

$$\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} \geqslant \frac{1}{2(p-1)\sqrt{\pi}}.$$

From that, it follows

$$\frac{\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} \geqslant \frac{p}{p-1}\cdot\frac{1}{4\sqrt{\pi}}.$$

If $p \geqslant 2$, then $p(p-1)^{-1}\cdot 1/\left(4\sqrt{\pi}\right) < 1$ and $\Gamma\left(1+\frac{p}{2}\right)/\Gamma\left(\frac{p+1}{2}\right) > 1$ because $\Gamma(x)$ is the increasing function if $x \geqslant x_1 \approx 1.4616\ldots$ So,

$$K_p \geqslant \frac{\pi\left(\frac{1}{2}-\frac{1}{p}\right)}{\sin\pi\left(\frac{1}{2}-\frac{1}{p}\right)}\frac{1}{\sin\frac{\pi}{p}}\cdot\frac{1}{2} \quad (p>2) \tag{10}$$

or, more roughly (because $\pi\left(\frac{1}{2}-\frac{1}{p}\right)/\sin\pi\left(\frac{1}{2}-\frac{1}{p}\right) \geqslant 1$),

$$K_p \geqslant \frac{1}{2}\cdot\frac{1}{\sin\pi/p}.$$

So,

$$\frac{1}{2}\cdot\frac{1}{\sin\pi/p} \leqslant |P|_p \leqslant \frac{\pi}{\sin\pi/p}, \quad 1<p<\infty.$$

Remark 2. In a similar way, we can give two sided norm estimate for the Bergman projection on the weighted space $L^p(B, \mathrm{d}v_\alpha)$ where B is the open unit ball in $\mathbb{C}^n$ and $\mathrm{d}v_\alpha = (\alpha+1)\left(1-|z|^2\right)^\alpha \mathrm{d}v(z)$ where $\mathrm{d}v$ is the normalized volume measure on B.

Remark 3. Let Ω be bounded, simply connected domain in $\mathbb{C}$ with $C^{1+\varepsilon}$ $(\varepsilon>0)$ boundary. By F we denote a conformal mapping of Ω onto $\mathbb{D}$. Let $\varphi = F^{-1}$. It is well known that $\varphi' \in C\left(\overline{\mathbb{D}}\right)$ and $\varphi'(z) \neq 0$ on $\overline{\mathbb{D}}$. The Bergman projection on $L^p(\Omega)$ is defined by

$$P_\Omega f(z) = \frac{1}{\pi}\int_\Omega \frac{F'(z)\overline{F'(\xi)}}{(1-F(z)\overline{F(\xi)})^2} f(\xi)\,\mathrm{d}A(\xi) \quad \text{(see [1], p. 184)}.$$

If we define the operators V and M by

$$\begin{aligned} &V\colon\ L^p(\Omega) \longrightarrow L^p(\mathbb{D}) \\ &Vf(z) = f(\varphi(z))\cdot \varphi'(z)^{2/p}, \\ &M\colon\ V\colon\ L^p(\mathbb{D}) \longrightarrow L^p(\mathbb{D}) \\ &Mf(z) = \varphi'(z)^{1-2/p} f(z), \end{aligned}$$

we have

$$P_\Omega = V^{-1}M^{-1}P\,M\,V.$$

Since V is an isometry, we obtain

$$|P_\Omega|_p \leqslant |M^{-1}|_p \cdot |M|_p \cdot |P|_p$$

and

$$|P|_p \leqslant |M|_p \cdot |M^{-1}|_p \cdot |P_\Omega|_p$$

i.e.

$$\frac{|P|_p}{|M|_p \cdot |M^{-1}|_p} \leqslant |P_\Omega|_p \leqslant |P|_p \cdot |M|_p \cdot |M^{-1}|_p.$$

Let

$$C(\Omega) = \frac{\max\limits_{\overline{\mathbb{D}}} |\varphi'(z)|}{\min\limits_{\overline{\mathbb{D}}} |\varphi'(z)|},$$

then

$$|M|_p \cdot |M^{-1}|_p \leqslant \begin{cases} (C(\Omega))^{1-2/p}; & 2 \leqslant p < \infty \\ (C(\Omega))^{2/p-1}; & 1 < p \leqslant 2 \end{cases}$$

and we have

$$\begin{aligned} \frac{1}{2}(C(\Omega))^{2/p-1}\, 1/\sin \pi/p \leqslant |P_\Omega|_p \leqslant \frac{\pi}{\sin \pi/p}(C(\Omega))^{1-2/p}; \quad 2 \leqslant p < \infty, \\ \frac{1}{2}(C(\Omega))^{1-2/p}\frac{1}{\sin \pi/p} \leqslant |P_\Omega|_p \leqslant \frac{\pi}{\sin \pi/p}(C(\Omega))^{2/p-1}; \quad 1 < p \leqslant 2. \end{aligned}$$

Here $|P_\Omega|_p$, $|M|_p$, $|M^{-1}|_p$ denote the norms of the operators P_Ω, M, M^{-1} on the space $L^p(\Omega)$ and $L^p(\mathbb{D})$, respectively.

Question. From (10) it follows that for large p we have $K_p \geqslant c(\sin \pi/p)^{-1}$ where the constant c is near $\frac{1}{4}\pi$. Having in mind that $|P|_2 = 1$ it is natural to ask whether

$$|P|_p = \frac{1}{\sin \pi/p}.$$

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