Danica Jakubíková-Studenovská
On ideal extensions of partial monounary algebras

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 331–344

Persistent URL: `http://dml.cz/dmlcz/128261`

Czechoslovak Mathematical Journal, 58 (133) (2008), 331–344

ON IDEAL EXTENSIONS OF PARTIAL MONOUNARY ALGEBRAS

Danica Jakubíková-Studenovská, Košice

(Received August 10, 2005)

Abstract. In the present paper we introduce the notion of an ideal of a partial monounary algebra. Further, for an ideal (I, f_I) of a partial monounary algebra (A, f_A) we define the quotient partial monounary algebra $(A, f_A)/(I, f_I)$. Let (X, f_X), (Y, f_Y) be partial monounary algebras. We describe all partial monounary algebras (P, f_P) such that (X, f_X) is an ideal of (P, f_P) and $(P, f_P)/(X, f_X)$ is isomorphic to (Y, f_Y).

Keywords: partial monounary algebra, ideal, congruence, quotient algebra, ideal extension

MSC 2000: 08A60

0. Introduction

In the present paper we introduce the notion of an ideal of a partial monounary algebra. Further, for an ideal (I, f_I) of a partial monounary algebra (A, f_A) we define the quotient partial monounary algebra $(A, f_A)/(I, f_I)$.

The aim of the paper is as follows. Let (X, f_X), (Y, f_Y) be partial monounary algebras. Find all partial monounary algebras (P, f_P) such that (X, f_X) is an ideal of (P, f_P) and $(P, f_P)/(X, f_X)$ is isomorphic to (Y, f_Y). In particular, we consider the question under which assumptions on (X, f_X), (Y, f_Y) such a (P, f_P) exists. If for given (X, f_X), (Y, f_Y) such a (P, f_P) exists, then we describe constructively all (P, f_P) with this property; (P, f_P) is called an *ideal extension* of (Y, f_Y) by (X, f_X).

This paper is a continuation of [3] where subalgebra extensions of partial monounary algebras were dealt with; the investigation was motivated by extension problems for various structures, cf., e.g., the extension problem for groups: Given

This work was supported by the Slovak VEGA Grant No. 1/3003/06 and by the Science and Technology Assistance Agency under the contract No. APVT-20-004104.

two groups H and K, construct all groups G which have a normal subgroup N such that N is isomorphic to H and the quotient G/N of G by N is isomorphic to K. G is the well known Schreier's extension of H by K. Following the extension of groups, the ideal extension of semigroups has been considered by A. H. Clifford [1]. Related investigations dealing with extensions by ideals were performed for lattice ordered groups (in connection with the product of torsion classes, cf. Martinez [7]), for ordered and totally ordered semigroups (Kehayopulu, Tsingelis [6], Hulin [2]) and for lattices (Kehayopulu, Kiriakuli [5]).

1. Preliminaries

Monounary and partial monounary algebras play a significant role in the study of algebraic structures (cf., e.g., Jónsson [4], M. Novotný [8]).

A *partial monounary algebra* is a pair (A, f_A), where A is a nonempty set and f_A is a partial unary operation on A. If $\operatorname{dom} f_A = A$, then (A, f_A) is called *complete*; if $\operatorname{dom} f_A \neq A$, then (A, f_A) is said to be *incomplete*. The class of all partial monounary algebras will be denoted by $\mathscr{U}$.

A partial monounary algebra (A, f_A) is said to be *trivial* if $|A| = 1$.

We will denote by $\mathbb{N}$ the set of all positive integers and if $n \in \mathbb{N}$, then we put $\mathbb{N}_n = \{0, 1, \ldots, n\}$. Let $(A, f_A) \in \mathscr{U}$, $x, y \in A$. Put $f_A^0(x) = x$ and $f_A^{-1}(x) = \{z \in \operatorname{dom} f_A \colon f_A(z) = x\}$. If $n \in \mathbb{N}$, $f_A^{n-1}(x)$ is defined and $f_A^{n-1}(x) \in \operatorname{dom} f_A$, then we put $f_A^n(x) = f_A(f_A^{n-1}(x))$. Next we put $x \sim y$ if there are $m, n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(x)$, $f_A^m(y)$ are defined and $f_A^n(x) = f_A^m(y)$. Then $\sim$ is an equivalence relation on the set A and the elements of $A/\sim$ are called *connected components* of (A, f_A). Further, (A, f_A) is said to be *connected* if it has only one connected component. An element $c \in A$ is called *cyclic* if $f_A^k(c) = c$ for some $k \in \mathbb{N}$. The set of all cyclic elements of a connected component of (A, f_A) is called a *cycle* of (A, f_A). An element $c \in A$ is called a *top* of (A, f_A) if (A, f_A) is connected and either $c \notin \operatorname{dom} f_A$ or $\{c\}$ is a cycle.

Let $(A, f_A), (B, f_B) \in \mathscr{U}$. Let $B \subseteq A$, $\operatorname{dom} f_B \subseteq \operatorname{dom} f_A$ and if $x \in B \cap \operatorname{dom} f_A$ then $x \in \operatorname{dom} f_B$, $f_B(x) = f_A(x)$. Then (B, f_B) is called a *subalgebra* of (A, f_A).

Let $(A, f_A) \in \mathscr{U}$, $\emptyset \neq X \subseteq A$. We will denote by $f_A \restriction X$ the partial operation on X defined as follows: $\operatorname{dom}(f_A \restriction X) = \{x \in X \cap \operatorname{dom} f_A \colon f_A(x) \in X\}$ and if $x \in \operatorname{dom}(f_A \restriction X)$ then $(f_A \restriction X)(x) = f_A(x)$. The partial algebra $(X, f_A \restriction X)$ is called the *relative subalgebra* of (A, f_A) with the support X.

Let $(A, f_A) \in \mathscr{U}$. If $x, y \in A$, then we set $x \leqslant y$ if $f_A^k(x) = y$ for some $k \in \mathbb{N} \cup \{0\}$. Notice that the relation $\leqslant$ is a quasi-order on the set A. For a quasiordered set $(A, \leqslant)$ and an element $a \in A$ we put $(a\rangle = \{x \in A \colon x \leqslant a\}$, $\langle a) = \{x \in A \colon a \leqslant x\}$. The notion of an ideal of a lattice is well known. Let us modify the definition for lattices to the following definition for quasi-ordered sets: Let $(Q, \leqslant)$ be a quasi-ordered set, $\emptyset \neq X \subseteq Q$. Then $(X, \leqslant)$ is called an *ideal* in $(Q, \leqslant)$ if the following conditions are satisfied:

(1) if $a \in X$, $b \leqslant a$, then $b \in X$,

(2) if $a, b \in X$ and $c \in Q$ is a minimal upper bound of $\{a, b\}$, then $c \in X$.

1.1 Definition. Let $(A, f_A) \in \mathscr{U}$, $\emptyset \neq X \subseteq A$. If $(X, \leqslant)$ is an ideal of $(A, \leqslant)$, then the relative subalgebra $(X, f_A \restriction X)$ of (A, f_A) with the support X is called an *ideal* of (A, f_A).

Clearly, (A, f_A) is an ideal of (A, f_A); it will be called *improper*. If (X, f_X) is an ideal of (A, f_A) such that $X \neq A$, it will be said to be *proper*.

Let $(A, f_A) \in \mathscr{U}$. An equivalence relation θ on A is said to be a *congruence* of (A, f_A) if $\{x, y\} \subseteq \operatorname{dom} f_A$, $(x, y) \in \theta$ implies $(f_A(x), f_A(y)) \in \theta$. For $x \in A$, the block (equivalence class) of θ containing x is denoted by $[x]_\theta$ or simply $[x]$. The *quotient algebra* $(A, f_A)/\theta = (A/\theta, f_{A/\theta})$ is such that $\operatorname{dom} f_{A/\theta} = \{[x] \in A/\theta \colon [x] \subseteq \operatorname{dom} f_A\}$ and if $[x] \in \operatorname{dom} f_{A/\theta}$, then $f_{A/\theta}([x]) = [f_A(x)]$.

1.2 Notation. Let $(A, f_A) \in \mathscr{U}$, $\emptyset \neq X \subseteq A$. We denote by θ_X the smallest congruence of (A, f_A) such that if $x, y \in X$ belong to the same connected component of (A, f_A), then x, y belong to the same equivalence class of the congruence θ_X.

Furthermore, we denote by $G(X)$ the support of the subalgebra generated by the set X.

1.3 Lemma. *Let (A, f_A) be a connected monounary algebra, $\emptyset \neq X \subseteq A$ a set such that there exists $x_0 \in X$ with $f_A(x_0) \in X$. Then $A/\theta_X = \{G(X)\} \cup \{\{x\} \colon x \in A - G(X)\}$.*

Proof. (1) Consider the system of sets $\{G(X)\} \cup \{\{x\} \colon x \in A - G(X)\}$. It is easy to see that this is a system of blocks of a congruence ψ on (A, f_A) and that X is a subset of the block $G(X)$. Thus $\theta_X \subseteq \psi$.

(2) Clearly, $X \subseteq [x_0]_{\theta_X}$. Suppose $y \in [x_0]_{\theta_X}$. Then $(x_0, y) \in \theta_X$. If $f_A(y)$ exists, we obtain $(f_A(x_0), f_A(y)) \in \theta_X$, $f_A(y) = [x_0]_{\theta_X}$ because $f_A(x_0) \in [x_0]_{\theta_X}$. Hence $[x_0]_{\theta_X}$ is the support of a subalgebra of (A, f_A). It follows that $G(X) \subseteq [x_0]_{\theta_X}$, hence the block $G(X)$ of ψ is a subset of the block $[x_0]_{\theta_X}$ of θ_X. Thus $\psi \subseteq \theta_X$. $\square$

1.4 Notation. Let $(A, f_A) \in \mathscr{U}$ and let (X, f_X) be an ideal of (A, f_A). By the *quotient partial monounary algebra* $(A, f_A)/(X, f_X) = (A/X, f_{A/X})$ we understand the partial algebra $(A, f_A)/\theta_X$.

1.5 Remark. Let (A, f_A) be a connected partial monounary algebra, (X, f_X) its improper ideal. Then the quotient partial algebra $(A, f_A)/(X, f_X)$ is trivial.

Let (A, f_A) be a connected partial monounary algebra and $c \in A$. Then c will be said to be *inconvenient* if it is either a top or a cyclic element. An element of A is called *convenient* if it is not inconvenient. A convenient element $a \in A$ will be said to be *very convenient* if $f^{-1}(a) \neq \emptyset$.

1.6 Lemma. *Let (A, f_A) be a connected partial monounary algebra, (X, f_X) its proper ideal. Then there exists a convenient element $a \in A$ such that $X = (a\rangle$.*

Proof. Let $c \in X$. If $f_A^n(c) \in X$ for any $n \in \mathbb{N} \cup \{0\}$, then, clearly, $A = X$, which is a contradiction. Thus, there exists the least $n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(c) \in X$ and either $f_A^{n+1}(c)$ does not exist or $f_A^{n+1}(c) \notin X$. Put $a = f_A^n(c)$. If $f_A^{n+1}(c) = f_A(a)$ does not exist, then a is the top of (A, f_A) and $A = X$, which is a contradiction. Hence, the only possibility is $f_A(a) = f_A^{n+1}(c) \notin X$, which entails that a is a convenient element of (A, f_A).

Since $a \in X$ and X is an ideal of (A, f_A), we obtain $(a\rangle \subseteq X$. Suppose $x \in X$. There exist $m \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$ such that $f_A^k(a) = f_A^m(x)$ where k has the least possible value. Then $f_A^k(a)$ is a minimal upper bound of $\{a, x\}$, which yields $f_A^k(a) \in X$. This implies $k = 0$ and, therefore, $x \leqslant a$. Thus, $X \subseteq (a\rangle$. □

1.7 Lemma. *If (A, f_A) is a connected partial monounary algebra and $a \in A$ a convenient element, then $A - (a\rangle$ is the support of a subalgebra of (A, f_A).*

Proof. Indeed, if $x \in A$ and $f_A(x) \in (a\rangle$, then $x \in (a\rangle$. Thus, $x \in A - (a\rangle$ implies $f_A(x) \in A - (a\rangle$ if $f_A(x)$ exists. Hence, $A - (a\rangle$ is the support of a subalgebra of (A, f_A). □

1.8 Lemma. *Let (A, f_A) be a connected partial monounary algebra, $a \in A$ a convenient element. Then $(a\rangle \cup \langle f_A(a))$ is the support of the subalgebra of (A, f_A) generated by the set $(a\rangle$.*

Proof. (1) We prove that the set $(a\rangle \cup \langle f_A(a))$ is the support of a subalgebra of (A, f_A). Suppose $x \in (a\rangle \cup \langle f_A(a))$. Then either $x \leqslant a$ or $f_A(a) \leqslant x$.

In the former case there exists $n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(x) = a$. If $n = 0$, we obtain $x = a$, $f_A(x) = f_A(a) \in \langle f_A(a))$. The case $n > 0$ entails $f_A^{n-1}(f_A(x)) = a$, hence $f_A(x) \in (a\rangle$.

If $f_A(a) \leqslant x$ and $f_A(x)$ exists, we obtain $f_A(a) \leqslant f_A(x)$ and, therefore, $f_A(x) \in \langle f_A(a))$.

Hence, $(a\rangle \cup \langle f_A(a))$ is the support of a subalgebra of (A, f_A).

(2) Let Y be the support of a subalgebra of (A, f_A) such that $(a\rangle \subseteq Y$. If $x \in \langle f_A(a))$, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(f_A(a)) = x$, which yields $x = f_A^{n+1}(a) \in Y$ because $a \in Y$. It follows that $(a\rangle \cup \langle f_A(a)) \subseteq Y$ and, hence, $(a\rangle \cup \langle f_A(a))$ is the support of the subalgebra generated by the set $(a\rangle$. □

Let (A, f_A) be a connected partial monounary algebra, $a \in A$ a convenient element. Put $X = (a\rangle$, $Y = \langle f_A(a))$, $Z = X \cup Y$. Then the sets $A - X$, Y, Z are supports of subalgebras of (A, f_A) by 1.7, 1.8.

1.9 Proposition. *Let (A, f_A) be a connected partial monounary algebra, $a \in A$ a convenient element. Put $X = (a\rangle$, $Y = \langle f_A(a))$, $Z = X \cup Y$. Then the factor partial algebras $(A, f_A)/(Z, f_Z)$ and $(A - X, f_{A-X})/(Y, f_Y)$ are isomorphic.*

P r o o f. The trivial blocks of the factor partial algebra $(A, f_A)/(Z, f_Z)$ are formed by the elements of the set $A - Z$, the trivial blocks of $(A - X, f_{A-X})/(Y, f_Y)$ are formed by the elements of the set $A - X - Y = A - Z$. Hence the trivial blocks in the two factor partial algebras are the same.

It follows that trivial blocks with trivial values of the operation are the same in the two factor partial algebras and the values of operations coincide.

If $x \in A - Z$ is such that $f_A(x) \in Z$, then $f_A(x) \in Y$. Indeed, if $f_A(x) \in Z$, $f_A(x) \notin Y$ is satisfied, then $x \leqslant f_A(x) \in X = (a\rangle$ and, hence, $x \in X \subseteq Z$, which is a contradiction.

Thus the trivial blocks with nontrivial values of the operation are the same in the two factor algebras and the value $\{Y\}$ in the latter factor algebra corresponds to the value $\{Z\}$ of the former factor algebra. Since the partial algebras (Y, f_Y), (Z, f_Z) are either both complete or both incomplete, the two factor partial algebras are isomorphic in view of 1.3. $\square$

Let (A, f_A) be a complete connected monounary algebra having a cycle. If $a \in A$ is arbitrary, we denote by $p(a)$ the least number $n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(a)$ is cyclic.

Let (A, f_A) be an incomplete connected partial monounary algebra, c its top. For an arbitrary $a \in A$, we denote by $r(a)$ the least number $n \in \mathbb{N} \cup \{0\}$ such that $f_A^n(a) = c$.

1.10 Lemma. *Let (A, f_A) be a connected partial monounary algebra, $a \in A$ a convenient element. The following assertions hold:*

(i) *if (A, f_A) is complete and has no cyclic elements, then $\langle f_A(a)) = \{f_A^n(a) \colon n \in \mathbb{N}\}$;*

(ii) *if (A, f_A) is complete and has a cycle with $q \geqslant 1$ elements, then $\langle f_A(a)) = \{f_A^n(a) \colon 1 \leqslant n \leqslant p(a) + q - 1\}$;*

(iii) *if (A, f_A) is incomplete, then $\langle f_A(a)) = \{f_A^n(a) \colon 1 \leqslant n \leqslant r(a)\}$.*

The algebras appearing in this lemma will play a role in the following construction. To simplify the formulations we introduce the following definitions.

(i) The algebra defined on the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with the operation $f_{\mathbb{N}_0}$ of the successor will be called an *infinite stick*.

(ii) The algebra defined on the set $S_n = \{1, 2, \ldots, n\}$ with $n \in \mathbb{N}$ such that there exists $k \in S_n$ and the operation f_{S_n} satisfies $f_{S_n}(i) = i + 1$ for $1 \leqslant i \leqslant n - 1$, $f_{S_n}(n) = k$ will be called an *(n, k)-lasso.*

(iii) The algebra defined on the set $S_n = \{1, 2, \ldots, n\}$ with $n \in \mathbb{N}$, $n \notin \operatorname{dom} f_{S_n}$ and $f_{S_n}(i) = i + 1$ for $1 \leqslant i \leqslant n - 1$ will be called an *n-element stick.*

The partial algebras defined above will be called *good algebras.*

Let (B, f_B) be a connected partial monounary algebra, (Y, f_Y) a good algebra. Then (Y, f_Y) will be said to be *suitable to* (B, f_B) if either both (B, f_B), (Y, f_Y) are complete or both (B, f_B), (Y, f_Y) are incomplete.

1.11 Construction. Let (B, f_B), (X, f_X) be connected partial monounary algebras with $B \cap X = \emptyset$, $|B| > 1$, $|X| > 1$. Suppose that (X, f_X) is incomplete and that (B, f_B) has a top.

Choose a good algebra (Y, f_Y) suitable to (B, f_B).

Construct a partial algebra (C, f_C) such that $C \cap X = \emptyset$ and $(C, f_C)/(Y, f_Y)$ is isomorphic to (B, f_B), using a critical mapping (see [3]).

Construct the algebra (A, f_A), where $A = C \cup X$, $f_A(x) = f_C(x)$ for any $x \in C \cap \operatorname{dom} f_C$, $f_A(x) = f_X(x)$ for any $x \in X \cap \operatorname{dom} f_X$, $f_A(a) = 1$ where a is the top of (X, f_X) and 1 is the least element in $Y \subseteq C$.

1.11.1 Lemma. *If partial algebras are as in* 1.11, *then* (X, f_X) *is an ideal of* (A, f_A) *and* $(A, f_A)/(X, f_X)$ *is isomorphic to* (B, f_B).

Proof. Consider (A, f_A). Then $X = (a\rangle$, $\langle f_A(a)) = \langle 1) = Y$, $Z = X \cup Y$ is the support of the subalgebra of (A, f_A) generated by the set X (according to 1.8). Clearly, (X, f_X) is an ideal of (A, f_A). Then $(A, f_A)/(X, f_X) = (A, f_A)/(Z, f_Z)$ by 1.3 and 1.8, $(A, f_A)/(Z, f_Z)$ is isomorphic to $(A - X, f_{A-X})/(Y, f_Y)$ by 1.9. Further, $(A - X, f_{A-X})/(Y, f_Y) = (C, f_C)/(Y, f_Y)$ and the last partial algebra is isomorphic to (B, f_B). $\square$

1.12 Proposition. *Let* (B, f_B), (X, f_X) *be partial monounary algebras satisfying the assumption of* 1.11. *Suppose that* (P, f_P) *is a connected partial monounary algebra. The following conditions are equivalent:*

(1) (X, f_X) *is an ideal of* (P, f_P) *and* $(P, f_P)/(X, f_X) \cong (B, f_B)$;

(2) (P, f_P) *is* (up to isomorphism) *obtained by the construction* 1.11 *from the pair* (B, f_B), (X, f_X).

Proof. The implication (2) $\Rightarrow$ (1) was shown in 1.11.1.

Assume that (1) is valid. Then (X, f_X) is a proper ideal of (P, f_P) and by 1.6 there is a convenient element $a \in P$ with $X = (a\rangle$. Then 1.10 yields that $(Y, f_Y) =$

$(\langle f_A(a)\rangle, f_{\langle f_A(a)\rangle})$ is, up to isomorphism, a good algebra which is suitable to (B, f_B). Denote $C = P - X$. Then $C \cap X = \emptyset$. By 1.9, $(C, f_C)/(Y, f_Y) \cong (P, f_P)/(X \cup Y, f_{X\cup Y})$. Further, [3; 4.3] yields that $(P, f_P)/(X, f_X) \cong (P, f_P)/(X \cup Y, f_{X\cup Y})$. Then by virtue of (1), $(C, f_C)/(Y, f_Y) \cong (B, f_B)$, thus (C, f_C) is constructed as in 1.11. From the definition of C it follows that (P, f_P) is, up to isomorphism, obtained by the construction 1.11 from (B, f_B), (X, f_X). $\square$

1.13 Theorem. *Suppose that (X, f_X), (B, f_B) are disjoint connected partial monounary algebras. An ideal extension of (B, f_B) by (X, f_X) exists if and only if one of the following conditions is satisfied:*

(i) $|X| = |B| = 1$, $(X, f_X) \cong (B, f_B)$;

(ii) $|X| = 1$, $\operatorname{dom} f_X \neq X$, $|B| > 1$ *and there is* $y \in B$ *with* $f_B^{-1}(y) = \emptyset$;

(iii) $|X| > 1$, $|B| = 1$ *and* (X, f_X) *is complete iff* (B, f_B) *is complete;*

(iv) $|X| > 1$, $|B| > 1$, $\operatorname{dom} f_X \neq X$ *and* (B, f_B) *contains a top.*

If (i) *or* (iii) *is valid then* (A, f_A) *is an ideal extension of* (B, f_B) *by* (X, f_X) *iff* $(A, f_A) = (X, f_X)$. *If* (ii) *holds then* (A, f_A) *is an ideal extension of* (B, f_B) *by* (X, f_X) *iff* $(A, f_A) \cong (B, f_B)$. *If* (iv) *is valid then* (A, f_A) *is an ideal extension of* (B, f_B) *by* (X, f_X) *iff* (A, f_A) *is obtained (up to isomorphism) by the construction* 1.11 *from the pair* (B, f_B), (X, f_X).

Proof. First let $|X| = 1$. If $\operatorname{dom} f_X = X$, then for any ideal extension (A, f_A) of (Y, f_Y) by (X, f_X) we get $(A, f_A) = (X, f_X)$ and $(Y, f_Y) \cong (X, f_X)$. If $\operatorname{dom} f_X \neq X$, then for any ideal extension (A, f_A) of (Y, f_Y) by (X, f_X), the congruence θ_X is trivial, i.e. $(A, f_A) \cong (Y, f_Y)$, and (X, f_X) is isomorphic to an ideal of (Y, f_Y), i.e. $f_Y^{-1}(y) = \emptyset$ for some $y \in Y$. If $\operatorname{dom} f_X \neq X$, $|Y| = 1$, then also $|A| = 1$, $A = X$ and $(Y, f_Y) \cong (X, f_X)$.

Let $|Y| = 1$, $|X| > 1$. Then for any ideal extension (A, f_A) of (Y, f_Y) by (X, f_X) we have $(A, f_A) = (X, f_X)$. Next, $(A, f_A)/(X, f_X)$ is complete iff (X, f_X) is complete.

Suppose that $|X| > 1$, $|Y| > 1$. If $(A, f_A)/(X, f_X) \cong (Y, f_Y)$, then the algebra (Y, f_Y) contains a top. Since $\operatorname{dom} f_X \neq X$, the algebra (X, f_X) contains a top. Thus by 1.12 we obtain that (A, f_A) is an ideal extension of (Y, f_Y) by (X, f_X) iff (A, f_A) is obtained (up to isomorphism) by the construction 1.11 from the pair (Y, f_Y), (X, f_X). $\square$

2. The general case

In the present section we will generalize the above results to the case when the partial monounary algebras under consideration are not assumed to be connected.

2.1 Notation. Let $(S, f_A) \in \mathscr{U}$ and let $\{A_j\}_{j\in J}$ be the system of connected components of (A, f_A); we will express this by writing

$$A = \sum_{j\in J} A_j, \quad (A, f_A) = \sum_{j\in J} (A_j, f_{A_j}).$$

2.2 Lemma. *Let $(A, f_A) = \sum\limits_{j\in J} (A_j, f_{A_j}) \in \mathscr{U}$, let (X, f_X) be an ideal of (A, f_A) and let $(Y, f_Y) = (A, f_A)/(X, f_X)$. Then $Y = \sum\limits_{j\in J} Y_j$, $X = \sum\limits_{l\in L} X_l$, $L \subseteq J$. Further,*

(1) *if $j \in J - L$, then $(Y_j, f_{Y_j}) \cong (A_j, f_{A_j})$,*
(2) *if $j \in L$, then (A_j, f_{A_j}) is an ideal extension of (Y_j, f_{Y_j}) by (X_j, f_{X_j}).*

Proof. For $j \in J$ we denote $X_j = X \cap A_j$. Let $L = \{j \in J\colon X_j \neq \emptyset\}$. Then $(X_l, f_A \restriction X_l)$ for $l \in L$ is an ideal of (A_l, f_{A_l}) and $(X, f_X) = \sum\limits_{l\in L} (X_l, f_{X_l})$. From the definition of θ_X it follows that if $(x, y) \in \theta_X$, $x \neq y$, then x, y belong to the same connected component of (A, f_A). Therefore $(Y, f_Y) = \sum\limits_{j\in J} (Y_j, f_{Y_j})$. The assertions (1) and (2) then hold in view of the definition. $\square$

2.3 Theorem. *Let $(X, f_X) = \sum\limits_{l\in L} (X_l, f_{X_l})$, $(Y, f_Y) = \sum\limits_{j\in J} (Y_j, f_{Y_j})$, $(P, f_P) \in \mathscr{U}$. The following conditions are equivalent:*

(i) *(P, f_P) is an ideal extension of (Y, f_Y) by (X, f_X);*
(ii) *$P = \sum\limits_{j\in J} P_j$ and there is an injection $\tau\colon L \to J$ such that, for $j \in J$,*
   (1) *if $j \neq \tau(l)$ for each $l \in L$, then $(Y_j, f_{Y_j}) \cong (P_j, f_{P_j})$,*
   (2) *if $j = \tau(l)$, $l \in L$, then (P_j, f_{P_j}) is an ideal extension of (Y_j, f_{Y_j}) by (X_l, f_{X_l}).*

Proof. Assume that (P, f_P) is an ideal extension of (Y, f_Y) by (X, f_X). In view of 2.2, the number of connected components of (P, f_P) is the same as the number of connected components of (Y, f_Y), $P = \sum\limits_{j\in J} P_j$. Since (X, f_X) is an ideal of (P, f_P), for each $l \in L$ there is a uniquely determined $j \in J$ such that (X_l, f_{X_l}) is an ideal of (P_j, f_{P_j}); put $\tau(l) = j$. Then $\tau\colon L \to J$ is an injection and (2) holds. Let $j \in J$ and suppose that $\tau(l) \neq j$ for each $l \in L$. Then $(P_j, f_{P_j}) \cong (Y_j, f_{Y_j})$ by 2.2; hence (1) is valid.

Conversely, assume that the condition (ii) is satisfied. Then (X, f_X) is an ideal of (P, f_P). Denote $(D, f_D) = (P, f_P)/(X, f_X)$. According to 2.2, $D = \sum\limits_{j\in J} D_j$. Further, by 2.2, (X, f_X) can be written in the form $(X, f_X) = \sum\limits_{k\in K} (E_k, f_{E_k})$, $K \subseteq J$ and

(3) if $j \in J - K$, then $(D_j, f_{D_j}) \cong (P_j, f_{P_j})$,

(4) if $j \in K$, then (P_j, f_{P_j}) is an ideal extension of (D_j, f_{D_j}) by (E_j, f_{E_j}).

By the assumption, $(X, f_X) = \sum_{l \in L} (X_l, f_{X_l})$, thus τ is a bijection $L \to K$; we can suppose that $X_l = E_{\tau(l)}$ for each $l \in L$.

Let $j \in J - K$, i.e., $j \neq \tau(l)$ for each $l \in L$. By (1) and (3) we obtain

(5) $(Y_j, f_{Y_j}) \cong (P_j, f_{P_j}) \cong (D_j, f_{D_j})$.

Let $j \in K$, i.e., $j = \tau(l)$ for some $l \in L$. From (2) and (4) we obtain

(6) (P_j, f_{P_j}) is an ideal extension of (Y_j, f_{Y_j}) by (X_j, f_{X_j}),

(7) (P_j, f_{P_j}) is an ideal extension of (D_j, f_{D_j}) by $(E_{\tau(l)}, f_{E_{\tau(l)}}) = (X_l, f_{X_l})$.

Therefore

(8) (X_l, f_{X_l}) is an ideal of (P_j, f_{P_j}) and $(P_j, f_{P_j})/(X_l, f_{X_l}) \cong (Y_j, f_{Y_j})$,

(9) (X_l, f_{X_l}) is an ideal of (P_j, f_{P_j}) and $(P_j, f_{P_j})/(X_l, f_{X_l}) \cong (D_j, f_{D_j})$,

hence

(10) $(Y_j, f_{Y_j}) \cong (D_j, f_{D_j})$.

Then (5) and (10) imply that $(Y, f_Y) \cong (D, f_D)$ and that (P, f_P) is an ideal extension of (Y, f_Y) by (X, f_X). □

2.4 Proposition. *Let $(X, f_X) = \sum_{l \in L} (X_l, f_{X_l})$, $(Y, f_Y) = \sum_{j \in J} (Y_j, f_{Y_j})$. An ideal extension (P, f_P) of (Y, f_Y) by (X, f_X) exists if and only if there is an injection τ: $L \to J$ such that for $l \in L$*

(α) *if (X_l, f_{X_l}) is complete then $(Y_{\tau(l)}, f_{Y_{\tau(l)}})$ is a one-element cycle,*

(β) *if (X_l, f_{X_l}) is incomplete, $(Y_{\tau(l)}, f_{Y_{\tau(l)}})$ is complete and contains no one-element cycle, then $|X_l| = 1$.*

Proof. Let (P, f_P) be an ideal extension of (Y, f_Y) by (X, f_X) and let τ be as in 2.3.

Let $l \in L$. By 2.3, (X_l, f_{X_l}) is an ideal of $(P_{\tau(l)}, f_{P_{\tau(l)}})$, $(P_{\tau(l)}, f_{P_{\tau(l)}})/(X_l, f_{X_l}) \cong (Y_{\tau(l)}, f_{Y_{\tau(l)}})$. If (X_l, f_{X_l}) is complete, then $(P_{\tau(l)}, f_{P_{\tau(l)}}) = (X_l, f_{X_l})$, hence $(Y_{\tau(l)}, f_{Y_{\tau(l)}})$ is a one-element cycle.

Assume that (X_l, f_{X_l}) is incomplete, $|X_l| > 1$, $(Y_{\tau(l)}, f_{Y_{\tau(l)}})$ is complete and contains no one-element cycle. Since (X_l, f_{X_l}) is incomplete, $|X_l| > 1$, hence $(Y_{\tau(l)}, f_{Y_{\tau(l)}})$ has a top, which is a contradiction to the other assumptions.

Conversely, suppose conditions (α) and (β) are satisfied. Let us define (P_j, f_{P_j}) for each $j \in J$.

Let $j \in J - L$. Then we put $(P_j, f_{P_j}) = (Y_j, f_{Y_j})$. Now let $j \in L$. By the assumption, $j = \tau(l)$ for some $l \in L$. If (X_l, f_{X_l}) is complete, then we put $(P_j, f_{P_j}) = (X_l, f_{X_l})$. Further suppose that (X_l, f_{X_l}) is incomplete. If $|X_l| = 1$, then we put $(P_j, f_{P_j}) = (Y_j, f_{Y_j})$. If $|X_l| \neq 1$, then (β) yields that (Y_j, f_{Y_j}) contains a top and either it has at least two elements or it is incomplete. According to 1.13 there exists an ideal extension (P_j, f_{P_j}) of (Y_j, f_{Y_j}) by (X_l, f_{X_l}).

Now put $(P, f_P) = \sum\limits_{j\in J}(P_j, f_{P_j})$. From the above construction and in view of 2.3 it follows that (P, f_P) is an ideal extension of (Y, f_Y) by (X, f_X), which completes the proof. $\square$

3. Isomorphic systems of ideals

For a partial monounary algebra (A, f_A) let $\mathrm{Id}(A, f_A)$ be the set of all ideals of (A, f_A).

Let us consider the following question:

(Q_1) Is a partial monounary algebra uniquely, up to isomorphism, determined by the system of its ideals?

In a more detailed formulation: Let (A, f_A), (B, f_B) be partial monounary algebras and let $\varphi\colon \mathrm{Id}(A, f_A) \to \mathrm{Id}(B, f_B)$ be a bijection such that if $(X, f_X) \in \mathrm{Id}(A, f_A)$, then $(X, f_X) \cong \varphi((X, f_X))$. Does this assumption imply that (A, f_A) and (B, f_B) are isomorphic?

Similarly, let (Q_2) be the following question:

(Q_2) Is a (complete) monounary algebra uniquely, up to isomorphism, determined by the system of its ideals?

(The detailed form of (Q_2) is analogous to the case of (Q_1).)

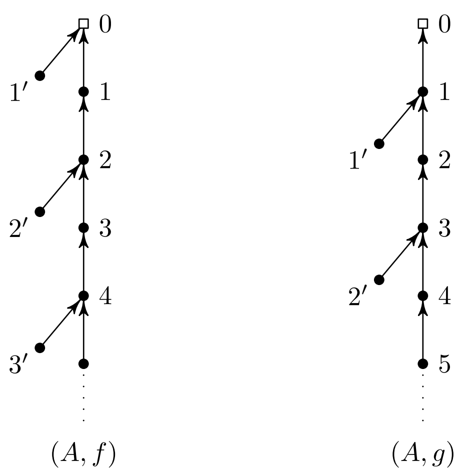

Figure 1.

3.1 Example. Let $\mathbb{N}' = \{n'\colon\ n \in \mathbb{N}\}$, $A = \mathbb{N} \cup \mathbb{N}' \cup \{0\}$. We define partial unary operations f, g on A by putting $\operatorname{dom} f = \operatorname{dom} g = \mathbb{N} \cup \mathbb{N}'$ and

$$f(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{N}, \\ 2n - 2 & \text{if } x \in \mathbb{N}',\ x = n'; \end{cases}$$

$$g(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{N}, \\ 2n - 1 & \text{if } x \in \mathbb{N}',\ x = n'. \end{cases}$$

In (A, f) there are ideals $(\langle b\rangle, f_{\langle b\rangle})$ of three types:

(a) trivial (incomplete partial monounary algebras possessing one element): for each $b \in \mathbb{N}'$,

(b) isomorphic to (A, f): for each $b = 2n$, where $n \in \mathbb{N} \cup \{0\}$,

(c) isomorphic to (A, g): for each $b = 2n - 1$, where $n \in \mathbb{N}$.

Similarly, in (A, g) there are ideals $(\langle b\rangle, g(b\rangle)$ of the same types:

(a) for each $b \in \mathbb{N}'$,

(b) for each $b = 2n - 1$, $n \in \mathbb{N}$,

(c) for each $b = 2n$, $n \in \mathbb{N}$.

We obtain

(1) $(A, f) \ncong (A, g)$,

(2) there is a bijection $\varphi\colon\ \mathrm{Id}(A, f_A) \to \mathrm{Id}(B, f_B)$

such that if $(X, f_X) \in \mathrm{Id}(A, f_A)$, then $\varphi((X, f_X)) \cong (X, f_X)$.

This implies the following assertion:

3.2 Proposition. *The answer to* (Q_1) *is negative.*

3.3 Proposition. *The answer to* (Q_2) *is affirmative.*

P r o o f. Let (A, f_A) be a monounary algebra. Then the system of all ideals (X, f_X) of (A, f_A) such that (X, f_X) is connected and complete, coincides with the system of all connected components of (A, f_A). Since a monounary algebra is determined by the system of its connected components, the answer to (Q_2) is affirmative. $\square$

4. Extensions by means of initial segments

A consideration similar to that performed for subalgebra extensions or for ideal extensions can be done for extensions by means of initial segments; this question was proposed by P. Burmeister at the Conference on Universal Algebra AAA68, Dresden, 2004 as follows: for given partial monounary algebras (X, f_X), (B, f_B), describe by means of an initial segment (X, f_X) all extensions of (B, f_B), i.e. all partial monounary algebras (P, f_P) such that (X, f_X) is an initial segment of (P, f_P) and $(P, f_P)/(X, f_X) \cong (B, f_B)$.

If $(P, \leqslant)$ is a quasiordered set, then $\emptyset \neq X \subseteq P$ is called an *initial segment* of $(P, \leqslant)$ if $p \in P$, $x \in X$, $p \leqslant x$ implies $p \in X$.

For a partial monounary algebra (A, f_A), $\emptyset \neq X \subseteq A$, we will say that the relative subalgebra (X, f_X) is an *initial segment* of (A, f_A) if X is an *initial segment* of the corresponding quasiordered set $(A, \leqslant)$.

4.0 Lemma. *Let (X, f_X) be an initial segment of a connected partial monounary algebra (A, f_A), $X \neq A$. Then X can be written as a union $X = \bigcup\limits_{m \in M} (x_m\rangle$ with pairwise disjoint summands.*

P r o o f. Put $M = \{x \in X \colon x \in \operatorname{dom} f_A,\ f_A(x) \notin X\}$. Then $X = \bigcup\limits_{m \in M} (m\rangle$ and the summands are obviously *mutually* disjoint. $\square$

Good algebras defined above can be characterized as partial algebras with one generator. Similarly we introduce

4.1 Definition. Let $U = \{u_i \colon i \in I\}$ be any set with $u_i \neq u_j$ for $i, j \in I$, $i \neq j$. A partial monounary algebra (Y, f_Y) will be called *U-good* if $U \subseteq Y$ and $Y = \bigcup\limits_{i \in I} \langle u_i)$.

If (B, f_B) is a connected partial monounary algebra, then a U-good algebra (Y, f_Y) is said to be *suitable* to (B, f_B) if either both (B, f_B), (Y, f_Y) are complete or both (B, f_B), (Y, f_Y) are incomplete.

4.2 Construction. Let (B, f_B) be a connected partial monounary algebra, $|B| > 1$. Further, let $(X, f_X) = \sum\limits_{i \in I} (X_i, f_{X_i}) \in \mathscr{U}$ be such that for each $i \in I$ we have $|X_i| > 1$ and (X_i, f_{X_i}) is incomplete with a top x_i.

Take a U-good algebra (Y, f_Y) (for arbitrary $U = \{u_i \colon i \in I\}$ suitable to (B, f_B)). Using critical mappings as in [3], construct a partial monounary algebra (C, f_C) such that $C \cap X = \emptyset$ and $(C, f_C)/(Y, f_Y) \cong (B, f_B)$.

Now construct a partial monounary algebra (A, f_A) such that $A = C \cup X$, $f_A(x) = f_X(x)$ for each $x = X \cap \operatorname{dom} f_X$, $f_A(c) = f_C(c)$ for each $c \in C \cap \operatorname{dom} f_C$, $f_A(x_i) = u_i$.

Now the following assertions can be proved analogously to Section 2:

4.2.1 Lemma. *Let all partial algebras be as in* 4.1*. Then* (X, f_X) *is an initial segment of* (A, f_A) *and* $(A, f_A)/(X, f_X) \cong (B, f_B)$.

4.3 Lemma. *Let* (B, f_B) *and* (X, f_X) *satisfy the assumption of* 4.2*. Suppose that* $(P, f_P) \in \mathscr{U}$*. The following conditions are equivalent:*

(1) (P, f_P) *is an extension of* (B, f_B) *by means of the initial segment* (X, f_X)*;*

(2) (P, f_P) *can be obtained, up to isomorphism, by the construction* 4.2.

4.4 Theorem. *Let* $(X, f_X) = \sum\limits_{l \in L} (X_l, f_{X_l}) \in \mathscr{U}$, $L_1 = \{l \in L\colon (X_l, f_{X_l})$ *is complete*$\}$, $L_2 = \{l \in L - L_1\colon |X_l| = 1\}$*. Further, let* $(Y, f_Y) = \sum\limits_{j \in J} (Y_j, f_{Y_j}) \in \mathscr{U}$, $J_1 = \{j \in J\colon |Y_j| = 1$, (Y_j, f_{Y_j}) *is complete*$\}$, $Y^0 = \{y \in Y\colon y$ *is convenient*, $f^{-1}(y) = \emptyset\}$.

(a) *An extension of* (Y, f_Y) *by means of an initial segment* (X, f_X) *exists if and only if* $|L_1| = |J_1|$, $|L_2| \leqslant |Y^0|$.

(b) *Assume that* $L_1 = J_1$ *and that* φ *is an injection of* $\{x_l\colon l \in L_2\}$ *into* Y^0*, where* $X_l = \{x_l\}$ *for* $l \in L_2$*. Let* $(P, f_P) \in \mathscr{U}$*. Then* (P, f_P) *is, up to isomorphism, an extension of* (Y, f_Y) *by means of an initial segment* (X, f_X) *if and only if there is a mapping* $\tau\colon L \to J$ *such that*

(i) $(P, f_P) = \sum\limits_{j \in J} (P_j, f_{P_j})$,

(ii) *if* $l \in L$*, then* $\tau(l) = l$, $(P_l, f_{P_l}) = (X_l, f_{X_l})$,

(iii) *if* $j \in J$, $\tau^{-1}(j) = \emptyset$*, then* $(P_j, f_{P_j}) \cong (Y_j, f_{Y_j})$,

(iv) *if* $j \in J$, $\tau^{-1}(j) \neq \emptyset$*, then* $\tau^{-1}(j) \subseteq L - (L_1 \cup L_2)$ *and* (P_j, f_{P_j}) *is an extension of* (Y_j, f_{Y_j}) *by means of an initial segment* $\sum\limits_{l \in \tau^{-1}(j)} (X_l, f_{X_l})$.

Remark. Notice that if (i)–(iv) are valid, $X' = \sum\limits_{l \in L - L_2} X_l + \sum\limits_{l \in L_2} \{\varphi(X_l)\}$, then $(X', f_{X'})$ is isomorphic to (X, f_X) and $(X', f_{X'})$ is an initial segment of (P, f_P).

References

[1] *A. H. Clifford*: Extensions of semigroups. Trans. Am. Math. Soc. *68* (1950), 165–173. zbl

[2] *A J. Hulin*: Extensions of ordered semigroups. Czech. Math. J. *26(101)* (1976), 1–12. zbl

[3] *D. Jakubíková-Studenovská*: Subalgebra extensions of partial monounary algebras. Czech. Math. J. *56(131)* (2006), 845–855.

[4] *B. Jónsson*: Topics in Universal Algebra. Springer, Berlin-Heidelberg-New York, 1972. zbl

[5] *N. Kehayopulu, P. Kiriakuli*: The ideal extension of lattices. Simon Stevin *64* (1990), 51–60. zbl

[6] *N. Kehayopulu, M. Tsingelis*: The ideal extensions of ordered semigroups. Commun. Algebra *31* (2003), 4939–4969. zbl

[7] *J. Martinez*: Torsion theory of lattice ordered groups. Czech. Math. J. *25(100)* (1975), 284–299. zbl
[8] *M. Novotný*: Mono-unary algebras in the work of Czechoslovak mathematicians. Arch. Math., Brno *26* (1990), 155–164. zbl

Author's address: Danica Jakubíková-Studenovská, Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 041 54 Košice, Slovakia, e-mail: danica.studenovska@upjs.sk.