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Seon Mi Lyu; Juan de Dios Pérez; Young Jin Suh

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REAL HYPERSURFACES IN COMPLEX SPACE FORMS CONCERNED WITH THE LOCAL SYMMETRY

Seon Mi Lyu, Taegu Juan de Dios Pérez, Granada,
and Young Jin Suh, Taegu

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Abstract. This paper consists of two parts. In the first, we find some geometric conditions derived from the local symmetry of the inverse image by the Hopf fibration of a real hypersurface M in complex space form $M_m(4\varepsilon)$. In the second, we give a complete classification of real hypersurfaces in $M_m(4\varepsilon)$ which satisfy the above geometric facts.

Keywords: real hypersurfaces, local symmetry, derivations, Kulkarni-Nomizu product

MSC 2000: 53C40, 53C15

1. Introduction

A complex m-dimensional Kaehler manifold of constant holomorphic sectional curvature 4ε is called a complex space form, which is denoted by $M_m(4\varepsilon)$. A complete and simply connected complex space form is a complex projective space $P_m(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^m$ or a complex hyperbolic space $H_m(\mathbb{C})$, according as $\varepsilon = 1$, $\varepsilon = 0$ or $\varepsilon = -1$. The induced almost contact metric structure of a real hypersurface M of $M_m(4\varepsilon)$ is denoted by (φ, ξ, η, g). From now on, unless otherwise stated, the sign ε in $M_m(4\varepsilon)$ will be denoted 1 or -1.

There exist many studies about real hypersurfaces of $M_m(4\varepsilon)$. The classification of homogeneous real hypersurfaces of a complex projective space $P_m(\mathbb{C})$ was given by Takagi [24], who showed that these hypersurfaces of $P_m(\mathbb{C})$ could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E. Moreover, Kimura in [9] proved that they are realized as the tubes of constant radius over

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Kaehler submanifolds if the structure vector field ξ is principal. Also Berndt [1] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_m(\mathbb{C})$ are realized as tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. In $H_m(\mathbb{C})$ they are said to be of type A_0, A_1, A_2, and B. Moreover, recently Berndt and the third author ([3], [4]) have classified real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfying certain geometric conditions, which are said to be of type A and B.

Now, let us consider the following condition for the shape operator A of M in $M_m(4\varepsilon)$ could satisfy

$$(\nabla_X A)Y = -\varepsilon\{\eta(Y)\varphi X + g(\varphi X, Y)\xi\}, \tag{1.1}$$

for any tangent vector fields X and Y of M.

Maeda, [12], investigated the condition (1.1) and used it to find a lower bound of $\|\nabla A\|$ for real hypersurfaces in $P_m(\mathbb{C})$. In fact, it was shown that $\|\nabla A\|^2 \geqslant (m-1)$ for such hypersurfaces and the equality is attained if and only if the condition (1.1) holds. Moreover, in this case it was known that M is locally congruent to one of the homogeneous real hypersurfaces of type A_1 and A_2. Also Chen, Ludden and Montiel [5] generalized this inequality to real hypersurfaces in $H_m(\mathbb{C})$ and showed that the equality (1.1) holds if and only if M is congruent to one of the types A_0, A_1, and A_2. Moreover, the present authors in [11] have also found that a lower bound of $\|\nabla A\|^2$ for real hypersurfaces in quaternionic hyperbolic space $H_m(\mathbb{Q})$ is given by $24(m-1)$.

Let us denote by $S^{2m+1}(1)$ (resp. $H_1^{2m+1}(-1)$) a $(2m+1)$-dimensional unit sphere (resp. anti-de Sitter space) defined in such a way that

$$S^{2m+1}(1) = \left\{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} :\ \sum_{i=0}^{m} z_i\bar{z}_i = 1\right\}$$

(resp. $H^{2m+1}(-1) = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} :\ -z_0\bar{z}_0 + \sum_{i=1}^{m} z_i\bar{z}_i = -1\}$), which is well known bundle space of the Hopf map

$$\pi' :\ S^{2m+1}(1) \to P_m(\mathbb{C}) \quad (\text{resp. } H_1^{2m+1}(-1)) \to H_m(\mathbb{C})).$$

Then we say that $S^{2m+1}(1)$ (resp. $H_1^{2m+1}(-1)$) is a (resp. Lorentzian) Hopf hypersurface of $\mathbb{C}^{m+1}$ with Hopf vector field with a distinguished (resp. time-like) unit vector field on $S^{2m+1}(1)$ (resp. $H_1^{2m+1}(-1)$) tangent to the fibre of the Hopf map π'.

Given a real hypersurface of $M_m(4\varepsilon)$, one can construct a (resp. Lorentzian) hypersurface $\overline{M}$ in $S^{2m+1}(1)$ (resp. $H_1^{2m+1}(-1)$) which is a principal S^1-bundle (resp. S_1^1-bundle) over M with (resp. time-like) totally geodesic fibers and the projection

$\pi\colon \overline{M} \to M$ in such a way that the diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\iota'} & S^{2m+1}(1)(H_1^{2m+1}(-1)) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow[\iota]{} & P_m(\mathbb{C})(H_m(\mathbb{C})) \end{array}$$

is commutative (ι, ι' being the isometric immersions). Then it is seen (Chen, Ludden and Montiel in [5], and Okumura in [16]) that the second fundamental tensor $\bar{A}$ of $\overline{M}$ is parallel if and only if the second fundamental tensor A of M satisfies the condition (1.1) or (1.2). Thus M is congruent to real hypersurfaces of type A_1 or A_2 in $P_m(\mathbb{C})$ or real hypersurfaces of type A_0, A_1 or A_2 in $H_m(\mathbb{C})$.

On the hypersurface $\overline{M}$, we consider the condition of local symmetry $\overline{\nabla}\overline{R} = 0$, which follows from the condition $\overline{\nabla}\bar{A} = 0$ due to the Gauss equation. Here $\overline{\nabla}$ and $\overline{R}$ denote the induced Riemannian connection and the curvature tensor defined on $\overline{M}$ respectively.

Now let us suppose that $\overline{M}$ is a locally symmetric hypersurface in $S^{2m+1}(1)$ or in $H_1^{2m+1}(-1)$. Then we can verify that the real hypersurface M in $P_m(\mathbb{C})$ or in $H_m(\mathbb{C})$ satisfies

$$\text{(I)} \qquad \varphi * R = 0$$

and

$$\text{(II)} \qquad (\nabla^* A) \otimes A = 0,$$

where $*$ denotes an operator defined on the curvature tensor R of M as a derivation in such a way that

$$\begin{aligned} g((\varphi * R)(X,Y)Z,W) = g(R(\varphi X,Y)Z,W) + g(R(X,\varphi Y)Z,W) \\ + g(R(X,Y)\varphi Z,W) + g(R(X,Y)Z,\varphi W). \end{aligned}$$

Moreover, the tensor product $\otimes$ in the formula (II) denotes the Kulkarni-Nomizu product in $\operatorname{End}\Lambda^2 TM$ given by

$$\{(\nabla_V^* A) \otimes A\}(X,Y) = (\nabla_V^* A)X \wedge AY - (\nabla_V^* A)Y \wedge AX,$$

where $(\nabla_X^* A)Y$ denotes

$$(\nabla_X^* A)Y = (\nabla_X A)Y + \varepsilon\{\eta(Y)\varphi X + g(\varphi X, Y)\xi\}$$

and $\wedge$ denotes the wedge product defined by

$$(X \wedge Y)(Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W)$$

for any vector fields X, Y, Z, V and W on M.

From such an expression the condition (1.1) is equivalent to

$$\nabla^* A = 0. \tag{1.2}$$

Then we know that the formula (II) is weaker than the condition (1.1), which gives a lower bound of $\|\nabla A\|$ for real hypersurfaces in $M_n(4\varepsilon)$.

Now let us consider the converse problems related to such conditions and generalize a result in Maeda [12] without the assumption that the structure vector ξ is principal. We assert the following:

Theorem 1. *Let M be a real hypersurface in $M_m(4\varepsilon)$ $(m \geqslant 3)$. If it satisfies the formula* (I), *then M is locally congruent to one of the following:*

(1) *In case $M_m(4) = P_m(\mathbb{C})$*
- (A_1) *a tube of radius r over a hyperplane $P_{m-1}(\mathbb{C})$, where $0 < r < \frac{1}{2}\pi$,*
- (A_2) *a tube of radius r over a totally geodesic $P_k(\mathbb{C})$ $(1 \leqslant k \leqslant m-2)$, where $0 < r < \frac{1}{2}\pi$.*

(2) *In case $M_m(-4) = H_m(\mathbb{C})$*
- (A_0) *a horosphere in $H_m(\mathbb{C})$, i.e., a Montiel tube,*
- (A_1) *a tube of a totally geodesic hyperplane $H_k(\mathbb{C})$ ($k = 0$ or $m-1$),*
- (A_2) *a tube of a totally geodesic $H_k(\mathbb{C})$ $(1 \leqslant k \leqslant m-2)$.*

Now unless otherwise stated, we say simply that M is locally congruent to a *real hypersurface of type A* when M is locally congruent to one of the real hypersurfaces of type A_1 and A_2 for $\varepsilon = 1$ or to one of the real hypersurfaces of type A_0, A_1 and A_2 for $\varepsilon = -1$ respectively. Next, let us consider the formula (II), which is more weaker notion than the geometric condition (1.1). Then we also assert the following:

Theorem 2. *Let M be a real hypersurface in $M_m(4\varepsilon)$ $(m \geqslant 3)$. If it satisfies the formula* (II), *then M is locally congruent to a real hypersurface of type A.*

In Section 2 we recall some fundamental properties of real hypersurfaces in $M_m(4\varepsilon)$ and find some geometric conditions derived from the locally symmetry of $\overline{M}$ in $S^{2m+1}(1)$ (resp. $H_1^{2m+1}(-1)$). In Section 3 we give a proof of Theorem 1 and in Sections 4 and 5 we give the proof of Theorem 2.

2. Preliminaries

Let M be a real hypersurface of m-dimensional ($m \geqslant 2$) complex space form $M_m(4\varepsilon)$ of constant holomorphic sectional curvature 4ε and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_m(4\varepsilon)$. For a local vector field X on a neighborhood of x in M, the images of X and C under the linear transformation J can be represented as

$$JX = \varphi X + \eta(X)C, \quad JC = -\xi,$$

where φ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (φ, ξ, η, g) of tensors satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Usually, the set is said to be *almost contact metric structure*. Furthermore the covariant derivatives of the structure tensor φ and the structure vector fields ξ are given by

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX, \tag{2.1}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal vector field C on M.

Since the ambient space is of constant holomorphic sectional curvature 4ε, the equations of Gauss and Codazzi are respectively given as follows

$$\begin{aligned} R(X,Y)Z = \varepsilon\{&g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ &- 2g(\varphi X, Y)\varphi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \varepsilon\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\}. \tag{2.3}$$

Now let $\overline{M}$ be a hypersurface mentioned in the introduction. Let us note that $T_z\overline{M} = \mathrm{Span}\{\overline{F}\} \oplus \overline{F}^{\perp}$, where $z \in \overline{M}$ and $\overline{F} = Jz$ for a induced complex structure J defined on $\overline{M}$ from $S^{2m+1}(1)$ or $H_1^{2m+1}(-1)$. Moreover, $\pi_*\overline{F} = 0$ and π_* is an isomorphism on $\overline{F}^{\perp}$. For $X \in T_{\pi(z)}M$ we denote by X^L the horizontal lift of X to z. Moreover, f^L denotes the horizontal lift on $\overline{M}$ of the function f on M defined by $f^L(z) = f(\pi(z))$ for any point $z \in \overline{M}$. Then it can be easily seen that

$$g(X,Y)^L = \bar{g}(X^L, Y^L)$$

for a Riemannian metric $\bar{g}$ defined on $\overline{M}$. Moreover, the metric $\bar{g}$ on $\overline{M}$ is invariant by the fiber compatible to S^1 (or S_1^1). Then by using the formula on a Riemannian submersion given in [17] due to B. O'Neill we note that

$$\nabla_X Y = \pi_*(\overline{\nabla}_{X^L} Y^L) \tag{2.4}$$

and

$$\overline{\nabla}_{X^L} \overline{F} = \overline{\nabla}_{\overline{F}} X^L = JX^L = (\varphi X)^L \tag{2.5}$$

for any tangent vector field X orthogonal to ξ on M, where φ and ∇ (resp. $\overline{\nabla}$) denote the almost contact structure tensor and the Riemannian connection on M (resp. on $\overline{M}$).

Now let us give some examples of locally symmetric hypersurfaces, that is $\overline{\nabla}\overline{R} = 0$, in $S^{2m+1}(1)$ or in $H_1^{2m+1}(-1)$ as follows:

Example 1. Let us consider a family of product hypersurfaces in the $(2m+1)$-dimensional unit sphere given by

$$S^p(c_1) \times S^{2m-p}(c_2) = \left\{ x \in S^{2m+1}(1)\colon \sum_{i=1}^{p+1} x_i^2 = \frac{1}{c_1},\ \sum_{i=p+2}^{2m+2} x_i^2 = \frac{1}{c_2} \right\},$$

where c_1 and c_2 are positive constants such that $1/c_1 + 1/c_2 = 1$. Then the second fundamental tensor of every hypersurface of this family has two eigenvalues, say λ, equal to $\pm\sqrt{c_1 - 1}$ and of multiplicity p, and μ equal to $\mp\sqrt{c_2 - 1}$ and of multiplicity $2m - p$. Then the distribution corresponding to each eigenvalue is parallel and the second fundamental tensor is parallel. So its curvature tensor $\overline{R}$ is parallel. Thus these hypersurfaces are locally symmetric hypersurfaces in $S^{2m+1}(1)$.

Example 2. Now let us consider an anti-de Sitter space given by

$$H_1^{2m+1}(-1) = \left\{ x \in R_2^{2m+2}\colon\ -x_1^2 - x_2^2 + \sum_{i=3}^{2m+2} x_i^2 = -1 \right\}.$$

Then we consider two families of product hypersurface in $H_1^{2m+1}(-1)$ given by

$$\begin{aligned} &S^r(c_1) \times H_1^{2m-r}(c_2) \\ &\quad = \left\{ x \in H_1^{2m+1}(-1)\colon \sum_{i=3}^{r+3} x_i^2 = \frac{1}{c_1}, -x_1^2 - x_2^2 + \sum_{i=r+4}^{2m+2} x_i^2 = \frac{1}{c_2} \right\}, \end{aligned}$$

$$S_1^r(c_1) \times H^{2m-r}(c_2)$$
$$= \left\{ x \in H_1^{2m+1}(-1) \colon -x_1^2 + \sum_{i=3}^{r+1} x_i^2 = \frac{1}{c_1}, -x_2^2 + \sum_{i=r+2}^{2m+2} x_i^2 = \frac{1}{c_2} \right\},$$

where c_1 and c_2 are constants such that $1/c_1 + 1/c_2 = -1$ and $c_1 > 0$, and $c_2 < 0$. Then in this kind of families the second fundamental tensor has two eigenvalues

$$\lambda = \pm\sqrt{c_1+1}, \quad \mu = \pm\sqrt{c_2+1}.$$

Then each corresponding distribution is parallel and the second fundamental tensor is parallel. So its curvature tensor $\overline{R}$ is parallel.

If $\overline{M}$ is locally symmetric in the diagram mentioned in the Introduction, we have the following.

Lemma 2.1. *Let $\overline{M}$ be a locally symmetric hypersurface in $S^{2m+1}(1)$ or in $H_1^{2m+1}(-1)$. Then a real hypersurface $M = \pi(\overline{M})$ in $P_m(\mathbb{C})$ or in $H_m(\mathbb{C})$ satisfies the following*

(I) $\varphi * R = 0$ *and*

(II) $(\nabla^* A) \otimes A = 0$,

where $\otimes$ denotes the Kulkarni-Nomizu product and π a fibration $\pi \colon \overline{M} \to M$ compatible to the Hopf fibration π' defined in the Introduction.

Proof. Now let us denote by $\overline{R}$ the curvature tensor of $\overline{M}$ in $S^{2m+1}(1)$ or in $H_1^{2m+1}(-1)$. Then by virtue of the local symmetry of $\overline{M}$ and using (2.4) and (2.5), we have the formula (I) for any vertical vector field $\overline{F}$ defined on the fiber of $\overline{M}$

$$\begin{aligned}
0 &= \overline{F}(\bar{g}(\overline{R}(X^L, Y^L)Z^L, W^L)\\
&= -\bar{g}(\overline{R}((\varphi X)^L, Y^L)Z^L, W^L) - \bar{g}(\overline{R}(X^L, (\varphi Y)^L)Z^L, W^L)\\
&\quad - \bar{g}(\overline{R}(X^L, Y^L)(\varphi Z)^L, W^L) - \bar{g}(\overline{R}(X^L, Y^L)Z^L, (\varphi W)^L)\\
&= -g(R(\varphi X, Y)Z, W)^L - g(R(X, \varphi Y)Z, W)^L - g(R(X, Y)\varphi Z, W)^L\\
&\quad - g(R(X, Y)Z, \varphi W)^L
\end{aligned}$$

for any vector fields X, Y, Z and W on M and X^L (resp. Y^L, Z^L and W^L) denoting the horizontal lift of X (resp. Y, Z and W) to $\overline{M}$.

On the other hand, the equation of Gauss for a hypersurface $\overline{M}$ in S^{2m+1} or in H_1^{2m+1} is given by

$$\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \varepsilon\{\bar{g}(\overline{Y}, \overline{Z})\overline{X} - \bar{g}(\overline{X}, \overline{Z})\overline{Y}\} + \bar{g}(\bar{A}\overline{Y}, \overline{Z})\bar{A}\overline{X} - \bar{g}(\bar{A}\overline{X}, \overline{Z})\bar{A}\overline{Y}$$

for any tangent vector fields $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ on $\overline{M}$. Then from the local symmetry of $\overline{M}$ we have the following

$$\begin{aligned}0 &= \bar{g}((\overline{\nabla}_{X^L}\overline{R})(Y^L, Z^L)V^L, W^L)\\ &= \bar{g}((\overline{\nabla}_{X^L}\bar{A})Z^L, V^L)\bar{g}(\bar{A}Y^L, W^L) + \bar{g}(\bar{A}Z^L, V^L)\bar{g}((\bar{\nabla}_{X^L}\bar{A})Y^L, W^L)\\ &\quad - \bar{g}((\overline{\nabla}_{X^L}\bar{A})Y^L, V^L)\bar{g}(\bar{A}Z^L, W^L) - \bar{g}(\bar{A}Y^L, V^L)\bar{g}((\overline{\nabla}_{X^L}\bar{A})Z^L, W^L)\end{aligned}$$

for any vector fields X, Y, Z, V and W on M, where X^L (resp. Y^L, Z^L, V^L and W^L) also denotes the horizontal lift of X (resp. Y, Z and W) to $\overline{M}$. From this, if we substitute the following

$$\begin{aligned}\bar{g}((\bar{\nabla}_{X^L}\bar{A})Y^L, Z^L) &= X^L(\bar{g}(\bar{A}Y^L, Z^L) - \bar{g}(\bar{A}\overline{\nabla}_{X^L}Y^L, Z^L) - \bar{g}(\bar{A}Y^L, \overline{\nabla}_{X^L}Z^L)\\ &= (X(g(AY, Z)))^L - \bar{g}(\bar{A}(\nabla_X Y)^L, Z^L) - \bar{g}(\bar{A}Y^L, (\nabla_X Z)^L)\\ &\quad - g(\varphi X, Y)^L\bar{g}(\bar{A}\overline{F}, Z^L) - \bar{g}(\bar{A}Y^L, \overline{F})g(\varphi X, Z)^L\\ &= g((\nabla_X A)Y, Z)^L + \varepsilon g(\varphi X, Y)^L\eta(Z)^L + \varepsilon g(\varphi X, Z)^L\eta(Y)^L\end{aligned}$$

into the above equation, we have the following

$$\{(\nabla^*_X A)\otimes A\}(Z, Y) = (\nabla^*_X A)Z \wedge AY - (\nabla^*_X A)Y \wedge AZ = 0$$

for any vector fields X, Y and Z on M. From this, together with the definition of the wedge product $\wedge$ again, we have the formula (II). This completes the proof of our Lemma. □

3. Real hypersurfaces satisfying the formula (I)

In this section we will give a complete classification of real hypersurfaces M in $M_m(4\varepsilon)$ satisfying the formula (I). Then this formula (I) can be written as

(3.1)
$$\begin{aligned}&g(AY, Z)g((A\varphi - \varphi A)X, W) + g(AX, W)g((A\varphi - \varphi A)Y, Z)\\ &-g(AY, W)g((A\varphi - \varphi A)X, Z) - g(AX, Z)g((A\varphi - \varphi A)Y, W) = 0\end{aligned}$$

for any vector fields X, Y, Z and W on M.

Now we assert the following

Lemma 3.1. *Let M be a real hypersurface in $M_m(4\varepsilon)$ satisfying the formula* (I). *Then the structure vector field ξ is principal.*

P r o o f. Let us suppose that there is a point where the vector ξ is not principal. Then there exists a neighborhood M_0 of this point, on which we can define a unit vector field U orthogonal to ξ in such a way that

$$\beta U = A\xi - g(A\xi, \xi)\xi = A\xi - \alpha\xi,$$

where β denotes the length of the vector field $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point $x \in M_0$.

Let T_0 be the distribution defined by the subspace $T_0(x) = \{X \in T_xM\colon\ X \perp \xi_x\}$ in the tangent subspace T_xM of the real hypersurface M in $M_m(4\varepsilon)$. Then we can write

$$A\xi = \alpha\xi + \beta U,$$

where U is a unit vector field in T_0 and α and β are smooth functions on M. Then we consider an open set $M_0 = \{x \in M\colon\ \beta(x) \neq 0\}$, on which we continue our discussion.

Putting $Y = Z = \xi$ in (3.1), we get

$$\alpha g((A\varphi - \varphi A)X, W) - \eta(AW)g(A\varphi X, \xi) - \eta(AX)g(A\varphi W, \xi) = 0. \tag{3.2}$$

On the other hand, we calculate

$$\eta(AW) = g(A\xi, W) = \alpha\eta(W) + \beta g(U, W)$$

and

$$g(A\varphi X, \xi) = g(\varphi X, A\xi) = \beta g(\varphi X, U).$$

Substituting these into (3.2), we have

$$\begin{aligned} &\alpha g((A\varphi - \varphi A)X, W) - \beta\{\alpha\eta(W) + \beta g(U, W)\}g(\varphi X, U) \\ &\quad -\beta\{\alpha\eta(X) + \beta g(U, X)\}g(\varphi W, U) = 0. \end{aligned} \tag{3.3}$$

Let $L(\xi, U)$ be the distribution defined by the subspace $L_x(\xi, U)$ in the tangent space T_xM spanned by vectors ξ_x and U_x at any point x in M_0. Then we consider the following two cases:

Case I: $\alpha \neq 0$. Then by (3.3) we know that

$$g((A\varphi - \varphi A)X, W) = 0$$

for any $X, W \in L(\xi, U)^{\perp}$, where $L(\xi, U)^{\perp}$ denotes the orthogonal complement of the subspace $L(\xi, U)$. Of course we know the following formulas:

$$g((A\varphi - \varphi A)U, \xi) = g(A\varphi U, \xi) = \beta g(\varphi U, U) = 0,$$
$$g((A\varphi - \varphi A)\xi, \xi) = 0,$$

and

$$g((A\varphi - \varphi A)\xi, U) = 0.$$

Replacing X and W by U in (3.3) and using $\alpha \neq 0$, we get

$$g((A\varphi - \varphi A)U, U) = 0.$$

Summing up the fomulas mentioned above, we have

$$g((A\varphi - \varphi A)Y, Z) = 0$$

for any tangent vector fields Y and Z on M. That is, the shape operator A commutes with the structure tensor φ. Now we have $\varphi A\xi = \beta\varphi U$ and hence $\beta\varphi U = A\varphi\xi = 0$. Because $\varphi U \neq 0$, we get $\beta = 0$, which makes a contradiction on M_0. Hence the vector field ξ is principal, which concludes the proof of Lemma 3.1 in Case I.

Case II: $\alpha = 0$. Then in this case from (3.2) we know that

$$\eta(AW)g(A\varphi X, \xi) + \eta(AX)g(A\varphi W, \xi) = 0.$$

From this, substituting $A\xi = \beta U$, we have

$$\beta^2\{g(U, W)g(\varphi X, U) + g(U, X)g(U, \varphi W)\} = 0.$$

Now putting $X = U$, $W = \varphi U$ gives $\beta^2 = 0$, which makes also a contradiction. In this case ξ is a principal vector with zero principal curvature.

Summing up the above cases, we see that a real hypersurface M in $M_m(4\varepsilon)$ satisfying the condition (I) is a Hopf hypersurface, that is, its structure vector ξ is principal. □

Next we suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is seen in [1], [6] and [12] that α is constant on M and satisfies

$$A\varphi A = \varepsilon\varphi + \frac{1}{2}\alpha(A\varphi + \varphi A), \tag{3.4}$$

and hence, by (2.1) and (2.3), we get

$$\nabla_\xi A = -\frac{1}{2}\alpha(A\varphi - \varphi A). \tag{3.5}$$

If the function α vanishes, (3.4) implies

$$A\varphi AX = \varepsilon\varphi X. \tag{3.6}$$

Moreover, by virtue of the equation of Codazzi (2.3), (3.5) implies the following for $\alpha = 0$

$$(\nabla_X A)\xi = -\varepsilon\varphi X. \tag{3.7}$$

Putting $Y = Z = \xi$ in (3.1) and using that ξ is principal, we have

$$\alpha g((A\varphi - \varphi A)X, W) = 0. \tag{3.8}$$

So for a non-zero constant function α the shape operator A commutes with the structure tensor φ. Then by virtue of theorems given by Okumura [16] for $\varepsilon = 1$ and by Montiel and Romero [15] for $\varepsilon = -1$ we know that M is locally congruent to type A_1 or A_2, or respectively, type A_0, A_1 or A_2.

Now it remains only to consider the case $\alpha = 0$. This means $A\xi = 0$. So we can consider an orthonormal basis of eigenvectors of T_0. Then by (3.6), we have the following

Lemma 3.2. *Let M be a real hypersurface in a complex space form $M_m(4)$, $m \geqslant 3$ satisfying the formula* (I). *If $\alpha = 0$, then for some fixed eigenvalue λ of A on the orthogonal complement of ξ and for the corresponding eigenspace V_λ a vector X exist in V_λ such that $\varphi X \in V_\lambda$. Moreover, such an eigenvalue λ is always nonzero.*

Proof. Now let us consider the case $\varepsilon = 1$. Then by (3.6) we know

$$\varphi X \in V_{1/\lambda}$$

for any $X \in V_\lambda$. Here the eigenvalue λ can not be vanishing on M.

In fact, if the eigenvalue $\lambda = 0$ on a subset $\mathscr{U}$ in M, then (3.6) gives $0 = \varphi X$ for $\varepsilon = 1$, which makes a contradiction. So such a subset $\mathscr{U}$ should be empty.

On the other hand, contracting Y and Z in (3.1), we have

$$h(A\varphi - \varphi A)X - (A^2\varphi - \varphi A^2)X = 0, \tag{3.9}$$

where h denotes the trace of the shape operator A of M.

Now let us take an orthonormal basis $\{e_1, e_2, \dots, e_{m-1}, \varphi e_1, \dots, \varphi e_{m-1}, \xi\}$ in such a way that $Ae_i = \lambda_i e_i$ and $A\varphi e_i = (1/\lambda_i)\varphi e_i$, $i = 1, \dots, m-1$. Then putting $X = e_i$ in (3.9), we have

$$\Big(\frac{1}{\lambda_i} - \lambda_i\Big)\Big\{h - \Big(\frac{1}{\lambda_i} + \lambda_i\Big)\Big\} = 0.$$

Now suppose $\lambda_i \neq 1/\lambda_i$ for each $i = 1, \dots, m-1$. Then it follows that

$$h = \frac{1}{\lambda_i} + \lambda_i.$$

On the other hand, we know that

$$h = \sum_{i=1}^{m-1} g(Ae_i, e_i) + \sum_{i=1}^{m-1} g(A\varphi e_i, \varphi e_i) = \sum_{i=1}^{m-1}\Big(\lambda_i + \frac{1}{\lambda_i}\Big) = (m-1)h.$$

So it follows that $h = 0$ for $m \geqslant 3$, hence we have $1/\lambda_i = -\lambda_i$ for at least one $i \in \{1, \dots, m-1\}$, which concludes the proof of Lemma 3.2. □

Then by Lemma 3.2 we can take a principal curvature vector $X \in V_\beta$ such that $\beta^2 = 1$. Moreover, putting $Y = X_i \in V_{\lambda_i}$ in (3.1), we have

$$\beta\Big(\frac{1}{\lambda_i} - \lambda_i\Big)\{g(X, W)g(\varphi X_i, Z) - g(X, Z)g(\varphi X_i, W)\} = 0.$$

From this, it follows that $1/\lambda_i = \lambda_i$ for each $i = 1, \dots, m-1$. So the structure tensor φ commutes with the shape operator A. Thus in this case M is locally congruent to a tube of radius $r = \frac{1}{4}\pi$ over $P_{m-1}(\mathbb{C})$ (see [16] and [24]).

Now it remains to check the case where $\varepsilon = -1$ and $\alpha = 0$. Then for every $i = 1, \dots, m-1$ we have

$$\lambda_i \neq -\frac{1}{\lambda_i}.$$

Thus the formula (3.1) and (3.9) for such a case $\varepsilon = -1$ imply

$$h = -\frac{1}{\lambda_i} + \lambda_i \tag{3.10}$$

for all $i = 1, \dots, m-1$. This means that every principal curvature satisfies the quadratic equation $\lambda^2 - h\lambda - 1 = 0$. Moreover, by (3.10) and using the method given in Lemma 3.2 for the definition of h, we know that $h = 0$.

In fact, for $\varepsilon = -1$ we see by (3.6) that whenever $X \in V_\lambda$, $\lambda \neq 0$, then $\varphi X \in V_{-1/\lambda}$. Summing up in (3.10) over the indices $i = 1, \dots, m-1$, we obtain, analogously as in the proof of Lemma 3.2, $h = (m-1)h$ and hence $h = 0$.

So the quadratic equation reduces to $\lambda^2-1=0$. This means that M has 3 distinct constant principal curvatures 0, 1 and -1 with multiplicities 1, $m-1$, and $m-1$ respectively. Thus by a theorem of Berndt [1] M is locally congruent to a real hypersurface of type B. Then its Weingarten endomorphism A is given by

$$A=\begin{pmatrix} 2\tanh 2r & & & & & & \mathbf{0} \\ & \coth r & & & & & \\ & & \ddots & & & & \\ & & & \coth r & & & \\ & & & & \tanh r & & \\ & & & & & \ddots & \\ \mathbf{0} & & & & & & \tanh r \end{pmatrix}$$

where both principal curvatures $\coth r$ and $\tanh r$ have multiplicities $m-1$. Here we can see that $\lambda\mu-1=0$, where λ and μ denote $\coth r$ and $\tanh r$ respectively. But we know that $\lambda=1$ and $\mu=-1$, which gives a contradiction. So we conclude that there does not exist any real hypersurface M in complex hyperbolic space $H_m(\mathbb{C})$ satisfying the formula (I) when the function α vanishes identically.

Summing up all the above situations, we complete the proof of Theorem 1. $\square$

4. A Key Lemma and a Proposition

Let us consider another geometric condition derived from the local symmetry of $\overline{M}$ given in the formula (II) of Lemma 2.1. Then the formula (II) can be written as follows:

$$\{(\nabla^*_V A)\otimes A\}(X,Y)(Z,W)=\{(\nabla^*_V A)X\wedge AY\}(Z,W)-\{(\nabla^*_V A)Y\wedge AX\}(Z,W)$$

for any X, Y, Z, V and W on M in $M_m(4\varepsilon)$. Then by the expression of the derivative ∇^* and using the definition of the wedge product $\wedge$ the above formula can be rewritten as the following

$$\begin{aligned}\{g((\nabla_V A)X,Z)+\varepsilon g(\varphi V,Z)\eta(X)+\varepsilon g(\varphi V,X)\eta(Z)\}g(AY,W)\\ +\{g((\nabla_V A)Y,W)+\varepsilon g(\varphi V,W)\eta(Y)+\varepsilon g(\varphi V,Y)\eta(W)\}g(AX,Z)\\ -\{g((\nabla_V A)Y,Z)+\varepsilon g(\varphi V,Z)\eta(Y)+\varepsilon g(\varphi V,Y)\eta(Z)\}g(AX,W)\\ -\{g((\nabla_V A)X,W)+\varepsilon g(\varphi V,W)\eta(X)+\varepsilon g(\varphi V,X)\eta(W)\}g(AY,Z)=0\end{aligned} \tag{4.1}$$

for any vector fields X, Y, Z, V and W on M. Putting $V=\xi$ into (4.1), we get

$$\begin{aligned}g((\nabla_\xi A)Y,Z)AX-g((\nabla_\xi A)X,Z)AY+g(AY,Z)(\nabla_\xi A)X\\ -g(AX,Z)(\nabla_\xi A)Y=0.\end{aligned} \tag{4.2}$$

Then from (4.2), also putting $Z=\xi$ and taking $X, Y \in T_0$, we have

$$(4.3)\qquad \begin{aligned} &g((\nabla_\xi A)\xi, Y)AX - g((\nabla_\xi A)\xi, X)AY \\ &+\beta\{g(Y,U)(\nabla_\xi A)X - g(X,U)(\nabla_\xi A)Y\} = 0, \end{aligned}$$

where we have put $A\xi = \alpha\xi + \beta U$ and $U \in T_0$ is a certain vector field defined on $M_0 = \{x \in M\colon\ \beta(x) \neq 0\}$.

Next putting $Y = Z = \xi$ and taking $X \in T_0$ in (4.2), we have

$$(4.4)\qquad \alpha(\nabla_\xi A)X = g((\nabla_\xi A)\xi, X)A\xi + \beta g(X,U)(\nabla_\xi A)\xi - d\alpha(\xi)AX,$$

where $d\alpha(\xi) = g((\nabla_\xi A)\xi, \xi)$. Multiplying (4.3) by α and taking account of (4.4), we have

$$(4.5)\qquad \begin{aligned} &\{\alpha g((\nabla_\xi A)\xi, Y) - \beta d\alpha(\xi)g(Y,U)\}AX \\ &-\{\alpha g((\nabla_\xi A)\xi, X) - \beta d\alpha(\xi)g(X,U)\}AY \\ &+\beta\{g(Y,U)g((\nabla_\xi A)\xi, X) - g(X,U)g((\nabla_\xi A)\xi, Y)\}A\xi = 0. \end{aligned}$$

From this, on the open subset $M_0 = \{x \in M\colon\ \beta(x) \neq 0\}$ we get

Lemma 4.1. *Let M be a real hypersurface in $M_m(4\varepsilon)$ ($m \geqslant 3$) satisfying the formula* (II). *Then the function $\alpha = g(A\xi, \xi)$ identically vanishes.*

Proof. Let us consider the open set M_1 in M_0 defined by $\{x \in M_0\colon\ \alpha(x) \neq 0\}$. Of course on such an open subset M_0 the structure vector ξ is not principal and the function β is non-vanishing. Putting $Y = \xi$ in (4.5), we get

$$(4.6)\qquad d\alpha(\xi)AX = g((\nabla_\xi A)\xi, X)A\xi$$

for any $X \in T_0$. So if we suppose $d\alpha(\xi) \neq 0$, which is equivalent to $g((\nabla_\xi A)\xi, \xi) \neq 0$, then for any $X \in T_0$

$$AX = \gamma g((\nabla_\xi A)\xi, X)A\xi = f(X)\xi + g(X)U,$$

where we have put $\gamma = d\alpha(\xi)^{-1}$ and $f(X)$ (resp. $g(X)$) denotes $\alpha\gamma g((\nabla_\xi A)\xi, X)$ (resp. $\beta\gamma g((\nabla_\xi A)\xi, X)$). From this, together with $A\xi = \alpha\xi + \beta U$, we know that $\operatorname{rank} A \leqslant 2$. Then for the case where $\varepsilon = 1$ by a theorem of the third author in [22] we see that M is congruent to a ruled real hypersurface in $P_m(\mathbb{C})$. For $\varepsilon = -1$ by theorems of Ortega, Sohn and two last named authors in [18] and [21] we know that

M is also congruent to a ruled real hypersurface in $H_m(\mathbb{C})$. Then from the expression of the shape operator of ruled real hypersurfaces it follows that

$$AU = \beta\xi, \quad \beta \neq 0.$$

By putting $X = U$ in (4.6), we know that $d\alpha(\xi)AU = \alpha\delta\xi + \beta\delta U$, $\delta = g((\nabla_\xi A)\xi, U)$. From this, together with the above fomula we have $\beta\delta = 0$. So it follows that $\delta = 0$. This and (4.6) imply $AU = 0$, which makes a contradiction. Thus we should have $d\alpha(\xi) = 0$. So by (4.6) on M_1, we know that

$$g((\nabla_\xi A)\xi, X) = 0 \tag{4.7}$$

for any $X \in T_0$. From this, together with $d\alpha(\xi) = g((\nabla_\xi A)\xi, \xi) = 0$, we can deduce

$$(\nabla_\xi A)\xi = 0.$$

From this, together with (4.2) we have for any Y, Z on M_1

$$g((\nabla_\xi A)Y, Z)A\xi - g(A\xi, Z)(\nabla_\xi A)Y = 0.$$

Taking the inner product with ξ and using the assumption $\alpha \neq 0$ on M_1, we have

$$\nabla_\xi A = 0. \tag{4.8}$$

Then by virtue of a Lemma given by Kimura and Maeda [10] we know that the formula (4.8) implies that the structure vector ξ is principal. Thus such an open subset $M_1 = \{x \in M_0\colon \alpha(x) \neq 0\}$ can not exist. Then on the open set M_0 we conclude that the function α vanishes identically. □

By virtue of Lemma 4.1 we are going to prove the following Proposition.

Proposition 4.2. *Let M be a real hypersurface in $M_m(4\varepsilon)$ $(m \geqslant 3)$ satisfying the formula* (II)*. Then the structure vector ξ is principal.*

P r o o f. Now let us continue our discussion on the open set M_0. By Lemma 4.1 we know that the function α vanishes and the function β is non-vanishing on M_0. Thus we may write

$$A\xi = \beta U.$$

Now from (4.2) we know that

$$(\nabla_\xi A)\xi = -g((\nabla_\xi A)\xi, U)U. \tag{4.9}$$

Differentiating $A\xi = \beta U$ gives

$$(\nabla_X A)\xi + A\varphi AX = (X\beta)U + \beta\nabla_X U,$$

from this putting $X = \xi$, we have

$$(\nabla_\xi A)\xi + \beta A\varphi U = (\xi\beta)U + \beta\nabla_\xi U.$$

Now if we put $Y = W = V = \xi$ in (4.1) and use (2.1), we get

$$g((\nabla_\xi A)\xi, Z)g(AX, \xi) + g((\nabla_\xi A)X, \xi)g(A\xi, Z) = 0,$$

from which, using (4.9) in this obtained equation, we get

$$\beta g((\nabla_\xi A)\xi, U) = 0.$$

From this, together with (4.9), we have on M_0

$$(\nabla_\xi A)\xi = 0.$$

Then substituting this into (4.3), we have

$$\beta\{g(Y, U)(\nabla_\xi A)X - g(X, U)(\nabla_\xi A)Y\} = 0$$

for any $X, Y \in T_0$. So it follows for any $X \in T_0$

$$(\nabla_\xi A)X = g(X, U)(\nabla_\xi A)U.$$

This gives $(\nabla_\xi A)X = 0$ for any $X \in L(\xi, U)^\perp$, where $L(\xi, U)^\perp$ denotes the orthogonal complement of the subspace $L(\xi, U)$ spanned by ξ and U in T_xM at any point x in M_0.

Now we want to show that $\nabla_\xi A = 0$. In order to do this it suffices to show that

$$(\nabla_\xi A)U = 0.$$

Since we know that$g((\nabla_\xi A)U, \xi) = g((\nabla_\xi A)\xi, U) = 0$, we can put

$$(\nabla_\xi A)U = \lambda U = g((\nabla_\xi A)U, U)U.$$

Thus by the equation of Codazzi (2.3), we have

$$(\nabla_U A)\xi = \lambda U - \varepsilon\varphi U. \tag{4.10}$$

In order to prove the result it remains to show that $\lambda = 0$. Now let us put $V = U$ and $X = \xi$ in (4.1). Then we have

$$\begin{aligned}
&\{g((\nabla_U A)\xi, Z) + \varepsilon g(\varphi U, Z)\}g(AY, W)\\
&+\{g((\nabla_U A)Y, W) + \varepsilon g(\varphi U, W)\eta(Y) + \varepsilon g(\varphi U, Y)\eta(W)\}g(A\xi, Z)\\
&-\{g((\nabla_U A)Y, Z) + \varepsilon g(\varphi U, Z)\eta(Y) + \varepsilon g(\varphi U, Y)\eta(Z)\}g(A\xi, W)\\
&-\{g((\nabla_U A)\xi, W) + \varepsilon g(\varphi U, W)\}g(AY, Z) = 0.
\end{aligned}$$

Substituting (4.9) into this equation, we have

$$\begin{aligned}
&\lambda g(U, Z)g(AY, W) + \{g((\nabla_U A)Y, W) + \varepsilon g(\varphi U, W)\eta(Y)\\
&+\varepsilon g(\varphi U, Y)\eta(W)\}g(A\xi, Z) - \{g((\nabla_U A)Y, Z) + \varepsilon g(\varphi U, Z)\eta(Y)\\
&\varepsilon g(\varphi U, Y)\eta(Z)\}g(A\xi, W) - \lambda g(U, W)g(AY, Z) = 0.
\end{aligned}$$

From this, taking skew symmetric Y and Z, and putting $W = U$ and replacing Z by W in the obtained equation, we get

$$\begin{aligned}
&g((\nabla_U A)Y, U)g(A\xi, W) + \lambda g(U, W)g(AY, U)\\
&-g((\nabla_U A)W, U)g(A\xi, Y) - \lambda g(U, Y)g(AW, U) = 0.
\end{aligned}$$

From this, putting $Y = \xi$ and using Lemma 4.1 and the fact that the function β is non-vanishing on M_0, we have for any tangent vector W on M

$$g(U, W)\{g((\nabla_U A)\xi, U) + \lambda\} = 0,$$

from this, using (4.10), we have $\lambda = 0$. Then from (2.3) and (4.10) it follows that $(\nabla_\xi A)U = 0$. From this, together with the above fact, we conclude that $\nabla_\xi A = 0$. So by a theorem of Kimura and Maeda [10], the structure vector ξ is principal. This proves our assertion. □

5. Real hypersurfaces satisfying the formula (II)

By Proposition 4.2 we know that the structure vector ξ of any real hypersurface in $M_m(4\varepsilon)$ satisfying the condition (II) is principal. So in this section by virtue of this Proposition we will completely determine all real hypersurfaces in $P_m(\mathbb{C})$ or in $H_m(\mathbb{C})$ satisfying the formula (II).

Putting $Y = W = \xi$ in (4.1) and using (3.5), we have

$$\alpha g((\nabla_V A)X, Z) = \alpha\{\alpha g(\varphi AV, Z) - g(A\varphi AV, Z)\}\eta(X) + \alpha\{\alpha g(\varphi AV, X) - g(A\varphi AV, X)\}\eta(Z). \tag{5.1}$$

From this we can consider the following two cases.

Case 1: $\alpha \neq 0$. Then in this case we know from (5.1) that

$$(\nabla_V A)X = \alpha\{\eta(X)\varphi AV + g(\varphi AV, X)\xi\} - \{\eta(X)A\varphi AV + g(A\varphi AV, X)\xi\}.$$

Now by skew-symmetry and using the equation of Codazzi, we have

$$\begin{aligned}\varepsilon\{\eta(V)\varphi X - \eta(X)\varphi V - 2g(\varphi V, X)\xi\} &= \alpha\{\eta(X)\varphi AV - \eta(V)\varphi AX\} + \alpha\{g(\varphi AV, X) - g(\varphi AX, V)\}\xi \\ &\quad - \{\eta(X)A\varphi AV - \eta(V)A\varphi AX\} - \{g(A\varphi AV, X) - g(A\varphi AX, V)\}\xi.\end{aligned}$$

From this, putting $X = \xi$ and taking symmetric part, we have for any vector field V on M

$$\alpha(A\varphi - \varphi A)V = 0.$$

So in this case we see that the structure tensor φ commutes with the second fundamental tensor A. Thus by a theorem of Okumura [16] for $\varepsilon = 1$, M is locally congruent to a real hypersurface of type A_1 or A_2 and by a theorem of Montiel and Romero [15] for $\varepsilon = -1$, M is of type A_0, A_1 or A_2.

Case II: $\alpha = 0$. Differentiating (4.1) along the vector E and then putting $Y = \xi$ in the obtained equation, we have

$$\begin{aligned}&\{g((\nabla_E\nabla_V A)\xi, W) - 2\varepsilon g(AE, V)\eta(W) + \varepsilon\eta(V)g(AE, W)\}g(AX, Z) \\ &-\{g((\nabla_E\nabla_V A)\xi, Z) - 2\varepsilon g(AE, V)\eta(Z) + \varepsilon\eta(V)g(AE, Z)\}g(AX, W) \\ &-\{g((\nabla^*_V A)X, Z)g(\varphi E, W) + g((\nabla^*_V A)X, W)g(\varphi E, Z)\} = 0\end{aligned} \tag{5.2}$$

for any vector fields E, V, W, X and Z on M, where we have used (2.1) and the formula $(\nabla_X A)\xi = -\varepsilon\varphi X$ in (3.7).

On the other hand, the well-known Ricci-identity gives

$$\begin{aligned}(\nabla_X\nabla_Y A)\xi - (\nabla_Y\nabla_X A)\xi &= R(X, Y)(A\xi) - A(R(X, Y)\xi) \\ &= -A(R(X, Y)\xi) \\ &= -\varepsilon\{\eta(Y)AX - \eta(X)AY\}\end{aligned} \tag{5.3}$$

for any vector fields X and Y on M, where we have used the equation of Gauss (2.2) and the fact that $A\xi = 0$ in this case. So by taking skew-symmetry of (5.2) with respect to E and V and using the Ricci-identity (5.3), we have

$$\begin{aligned}&-g((\nabla^*_V A)X, Z)g(\varphi E, W) + g((\nabla^*_E A)X, Z)g(\varphi V, W)\\&+g((\nabla^*_V A)X, W)g(\varphi E, Z) - g((\nabla^*_E A)X, W)g(\varphi V, Z) = 0.\end{aligned}$$

From this, putting $W = \varphi E$, we have

$$\begin{aligned}&-g((\nabla^*_V A)X, Z)g(\varphi E, \varphi E) + g((\nabla^*_E A)X, Z)g(\varphi V, \varphi E)\\&+g((\nabla^*_V A)X, \varphi E)g(\varphi E, Z) - g((\nabla^*_E A)X, \varphi E)g(\varphi V, Z) = 0.\end{aligned} \tag{5.4}$$

Now we consider an orthonormal basis given by $\{e_1, \ldots, e_{m-1}, e_m, \ldots, e_{2m-2}, e_{2m-1}\}$, where $\xi = e_{2m-1}$. Then taking $E = e_i$ and summing up from $i = 1$ to $i = 2m-1$ in (5.4), we have

$$\begin{aligned}&(2m-4)g((\nabla^*_V A)X, Z) + \eta(V)g((\nabla^*_\xi A)X, Z) + \eta(Z)g((\nabla^*_V A)X, \xi)\\&+\sum_{i=1}^{2m-1} g((\nabla^*_{e_i} A)X, \varphi e_i)g(\varphi V, Z) = 0.\end{aligned} \tag{5.5}$$

The second term of (5.5) becomes

$$\begin{aligned}\eta(V)g((\nabla^*_\xi A)X, Z) &= \eta(V)g((\nabla_\xi A)X, Z)\\&= \eta(V)g((\nabla_X A)\xi + \varepsilon\varphi X, Z) = 0,\end{aligned}$$

where we have used (3.7). Similarly, by (3.7) the third term also vanishes.

On the other hand, by the equation of Codazzi (2.3) we have

$$\begin{aligned}\sum_{i=1}^{2m-1} g((\nabla_{e_i} A)X, \varphi e_i) &= \sum_{i=1}^{m-1} g((\nabla_{e_i} A)X, \varphi e_i) - \sum_{i=1}^{m-1} g((\nabla_{\varphi e_i} A)X, e_i)\\&= \sum_{i=1}^{m-1} g((\nabla_{e_i} A)\varphi e_i - (\nabla_{\varphi e_i} A)e_i, X)\\&= -2(m-1)\varepsilon\eta(X),\end{aligned}$$

where we have taken an orthonormal basis that $e_1, \ldots, e_{m-1}$, $e_m = \varphi e_1, \ldots, e_{2m-2} = \varphi e_{m-1}$, and $\xi = e_{2m-1}$. So it follows that

$$\begin{aligned}&\sum_{i=1}^{2m-1} g((\nabla^*_{e_i} A)X, \varphi e_i)g(\varphi V, Z)\\&= \sum_{i=1}^{2m-1} \{g((\nabla_{e_i} A)X, \varphi e_i) + \varepsilon\eta(X)g(\varphi e_i, \varphi e_i)\}g(\varphi V, Z) = 0.\end{aligned}$$

Accordingly, substituting these formulas into (5.5), we have for $m \geqslant 3$

$$\nabla_V^* A = 0.$$

Thus for $\varepsilon = 1$ by a theorem of Maeda [12] M is congruent to real hypersurfaces of type A_1 or A_2. For $\varepsilon = -1$ by a theorem of Chen, Ludden and Montiel [5] M is congruent to real hypersurfaces of type A_0, A_1, A_2. This completes the proof of Theorem 2 in the Introduction.

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Authors' addresses: Juan De Dios Pérez, Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, 18071-Granada, Spain, e-mail: jdperez@ugr.es; Young Jin Suh, Seon Mi Lyu, Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea, e-mail: yjsuh@mail.knu.ac.kr.