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Enno Kolk; Annemai Raidjõe

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THE CONTINUITY OF SUPERPOSITION OPERATORS ON SOME SEQUENCE SPACES DEFINED BY MODULI

Enno Kolk, Tartu, and Annemai Raidjõe, Tallinn

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Abstract. Let λ and μ be solid sequence spaces. For a sequence of modulus functions $\Phi = (\varphi_k)$ let $\lambda(\Phi) = \{x = (x_k)\colon\ (\varphi_k(|x_k|)) \in \lambda\}$. Given another sequence of modulus functions $\Psi = (\psi_k)$, we characterize the continuity of the superposition operators P_f from $\lambda(\Phi)$ into $\mu(\Psi)$ for some Banach sequence spaces λ and μ under the assumptions that the moduli φ_k $(k \in \mathbb{N})$ are unbounded and the topologies on the sequence spaces $\lambda(\Phi)$ and $\mu(\Psi)$ are given by certain F-norms. As applications we consider superposition operators on some multiplier sequence spaces of Maddox type.

Keywords: sequence space, superposition operator, modulus function, continuity

MSC 2000: 47H30, 46A45

1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers, respectively. Let ω be the vector space of all real sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$. By the term *sequence space*, we shall mean any linear subspace of ω.

Let λ and μ be two sequence spaces and let $f\colon\ \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a function with $f(k, 0) = 0$ $(k \in \mathbb{N})$. A *superposition operator* $P_f\colon\ \lambda \to \mu$ is defined by

$$P_f(x) = (f(k, x_k)) \in \mu \quad (x = (x_k) \in \lambda).$$

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Let ℓ_∞, c_0, ℓ_p and $(w_0)_p$ $(1 \leqslant p < \infty)$ be the sequence spaces with their usual norms as follows:

$$\ell_\infty = \{x \in \omega\colon \sup_k |x_k| < \infty\}, \quad \|x\|_{\ell_\infty} = \sup_k |x_k|;$$
$$c_0 = \{x \in \omega\colon \lim_k x_k = 0\}, \quad \|x\|_{c_0} = \sup_k |x_k|;$$
$$\ell_p = \Big\{x \in \omega\colon \sum_{k=1}^\infty |x_k|^p < \infty\Big\}, \quad \|x\|_{\ell_p} = \Big(\sum_{k=1}^\infty |x_k|^p\Big)^{1/p};$$
$$(w_0)_p = \Big\{x \in \omega\colon \lim_n \frac{1}{n}\sum_{k=1}^n |x_k|^p = 0\Big\}, \quad \|x\|_{(w_0)_p} = \sup_{i\geqslant 0}\Big(\frac{1}{2^i}\sum_{k=2^i}^{2^{i+1}-1} |x_k|^p\Big)^{1/p}.$$

For $p = 1$ we write ℓ and w_0 instead of ℓ_1 and $(w_0)_1$, respectively. We remark that the space $(w_0)_p$ can be equipped also with the norm $\|x\| = \sup_n (n^{-1}\sum_{k=1}^n |x_k|^p)^{1/p}$ which is equivalent to $\|\cdot\|_{(w_0)_p}$ (see, for example, [9, p. 39]).

Superposition operators on sequence spaces are not studied so intensively as on spaces of functions (see, for example, [1]). Dedagich and Zabreĭko [3] (see also [12] and [14]) have investigated the continuity of superposition operators on the sequence spaces ℓ_∞, c_0 and ℓ_p for $1 \leqslant p < \infty$. Płuciennik [13] characterized continuous superposition operators from w_0 into ℓ. Continuity of superposition operators in some another sequence spaces, including Orlicz sequence spaces, is studied in [14] and [16].

We shall characterize the continuity of superposition operators on some sequence spaces defined by a sequence of modulus functions.

Following Ruckle [15] and Maddox [10], a function $\varphi\colon [0,\infty) \to [0,\infty)$ is called a *modulus function* (or simply a *modulus*), if

(i) $\varphi(t) = 0 \iff t = 0$,

(ii) $\varphi(t+u) \leqslant \varphi(t) + \varphi(u)$ $(t, u \geqslant 0)$,

(iii) φ is nondecreasing,

(iv) φ is continuous from the right at 0.

It follows from (ii) and (iv) that φ is continuous everywhere on $[0,\infty)$. For example, $\varphi(t) = t^p$ is an unbounded modulus for $0 < p \leqslant 1$ and $\varphi(t) = t/(1+t)$ is a bounded modulus.

The sequence space λ is called *solid* if $(y_k) \in \lambda$ whenever $(x_k) \in \lambda$ and $|y_k| \leqslant |x_k|$ $(k \in \mathbb{N})$. Well known examples of solid sequence spaces are ℓ_∞, c_0, ℓ_p and $(w_0)_p$ $(1 \leqslant p < \infty)$.

Let $\Phi = (\varphi_k)$ be a sequence of moduli. For a solid sequence space λ we consider new solid sequence space (cf. [6]–[8])

$$\lambda(\Phi) = \{x \in \omega\colon \Phi(x) = (\varphi_k(|x_k|)) \in \lambda\}.$$

Let $\Psi = (\psi_k)$ be an another sequence of moduli. We investigate the continuity of the superposition operators $P_f\colon \ell_p(\Phi) \to \ell_q(\Psi)$, $P_f\colon \ell_p(\Phi) \to c_0(\Psi)$, $P_f\colon c_0(\Phi) \to \ell_q(\Psi)$, $P_f\colon c_0(\Phi) \to c_0(\Psi)$, $P_f\colon \ell_\infty(\Phi) \to \ell_q(\Psi)$, $P_f\colon \ell_\infty(\Phi) \to c_0(\Psi)$ and $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ $(1 \leqslant p, q < \infty)$ under the assumption that the moduli φ_k $(k \in \mathbb{N})$ are unbounded. Our results generalize the corresponding theorems about the continuity of superposition operators from [3], [12], [13] and [16].

2. Auxiliary results

In this section we formulate some definitions and known propositions, and prove a few lemmas which are needed in the proofs of main results.

Recall that an F-*seminorm* g on a vector space V is a functional $g\colon V \to \mathbb{R}$ satisfying, for all $x, y \in V$, the axioms

(N1) $g(0) = 0$,

(N2) $g(x+y) \leqslant g(x) + g(y)$,

(N3) $g(\alpha x) \leqslant g(x)$ for all scalars α with $|\alpha| \leqslant 1$,

(N4) $\lim\limits_n g(\alpha_n x) = 0$ for every scalar sequence (α_n) with $\lim\limits_n \alpha_n = 0$.

An F-*norm* is an F-seminorm satisfying the condition

(N5) $g(x) = 0 \iff x = 0$.

A B-*space* is a complete normed space. A topological sequence space in which all coordinate functionals π_k, $\pi_k(x) = x_k$, are continuous is called a K-*space*. A BK-*space* is defined as a K-space which is also a B-space.

An F-seminorm g on a solid sequence space λ is said to be *absolutely monotone* if $g(y) \leqslant g(x)$ for all $x = (x_k), y = (y_k) \in \lambda$ with $|y_k| \leqslant |x_k|$ $(k \in \mathbb{N})$.

An F-seminormed solid sequence space (λ, g) is called an AK-*space* if for any $x = (x_k) \in \lambda$,

$$\lim_m \sum_{k=1}^m x_k e^k = x,$$

where $e^k = (\delta_{ki})_{i\in\mathbb{N}}$ $(k \in \mathbb{N})$ with $\delta_{ki} = 1$ if $k = i$ and $\delta_{ki} = 0$ otherwise.

The following mapping conditions for superposition operators P_f are contained in Theorems 3–6 of [8]. For a sequence space λ we use the notation

$$\lambda^+ = \{(x_k) \in \lambda\colon\ x_k \geqslant 0\ (k \in \mathbb{N})\}.$$

Proposition 2.1. *Let $0 < p, q < \infty$. Then $P_f\colon \ell_p(\Phi) \to \ell_q(\Psi)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\gamma \geqslant 0$, $\delta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$(\psi_k(|f(k,t)|))^q \leqslant a_k + \gamma(\varphi_k(|t|))^p \quad (\varphi_k(|t|) \leqslant \delta,\ \ k \geqslant k_0). \tag{2.1}$$

Proposition 2.2. *Let $0 < q < \infty$. If the moduli φ_k $(k \in \mathbb{N})$ are unbounded, then $P_f\colon\ c_0(\Phi) \to \ell_q(\Psi)$ if and only if there exist a sequence $(a_k) \in \ell^+$ and numbers $\delta > 0$, $k_0 \in \mathbb{N}$ such that*

$$(\psi_k(|f(k,t)|))^q \leqslant a_k \quad (\varphi_k(|t|) \leqslant \delta,\ k \geqslant k_0). \tag{2.2}$$

Proposition 2.3. *If the moduli φ_k $(k \in \mathbb{N})$ are unbounded, then $P_f\colon\ c_0(\Phi) \to c_0(\Psi)$ and $P_f\colon\ \ell_p(\Phi) \to c_0(\Psi)$ $(0 < p < \infty)$ if and only if there exist a sequence $(a_k) \in c_0^+$ and numbers $\delta > 0$, $k_0 \in \mathbb{N}$ such that*

$$\psi_k(|f(k,t)|) \leqslant a_k \quad (\varphi_k(|t|) \leqslant \delta,\ k \geqslant k_0). \tag{2.3}$$

Proposition 2.4. *Let $0 < q < \infty$. If the moduli φ_k $(k \in \mathbb{N})$ are unbounded, then $P_f\colon\ \ell_\infty(\Phi) \to \ell_q(\Psi)$ if and only if for any $\eta > 0$ there exists a sequence $(a_k) \in \ell^+$ such that for all $k \in \mathbb{N}$*

$$(\psi_k(|f(k,t)|))^q \leqslant a_k \quad (\varphi_k(|t|) \leqslant \eta). \tag{2.4}$$

Proposition 2.5. *If the moduli φ_k $(k \in \mathbb{N})$ are unbounded, then $P_f\colon\ \ell_\infty(\Phi) \to c_0(\Psi)$ if and only if for any $\eta > 0$ there exist a sequence $(a_k) \in c_0^+$ and number $k_0 \in \mathbb{N}$ such that*

$$\psi_k(|f(k,t)|) \leqslant a_k \quad (\varphi_k(|t|) \leqslant \eta,\ k \geqslant k_0). \tag{2.5}$$

Proposition 2.6. *Let $0 < p, q < \infty$. If the functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and the moduli φ_k $(k \in \mathbb{N})$ are strictly increasing and unbounded, then $P_f\colon\ (w_0)_p(\Phi) \to \ell_q(\Psi)$ if and only if there exist a number $\delta > 0$ and sequences $(c_k)_{k=0}^\infty \in \ell^+$, $(d_k) \in \ell^+$ such that*

$$(\psi_k(|f(k,t)|))^q \leqslant d_k + c_i 2^{-i}(\varphi_k(|t|))^p \tag{2.6}$$

whenever $\varphi_k(|t|)^p \leqslant 2^i\delta$, $2^i \leqslant k < 2^{i+1}$ $(i = 0, 1, \ldots)$.

If (λ, g) is an F-seminormed space, then for the topologization of $\lambda(\Phi)$ it is natural to use the functional g_Φ, where

$$g_\Phi(x) = g(\Phi(x)).$$

Soomer [17, Theorem 3] and Kolk [7, Theorems 1 and 2] proved the following statements about the topologization of $\lambda(\Phi)$.

Proposition 2.7. *Let* (λ, g) *be an* F*-seminormed space. If* g *is absolutely monotone and the sequence of moduli* $\Phi = (\varphi_k)$ *satisfies one of conditions* (M) *and* (M′), *where*

(M) *there exists a function* ν *such that* $\varphi_k(ut) \leqslant \nu(u)\varphi_k(t)$ $(0 \leqslant u < 1,\ t \geqslant 0)$ *and* $\lim\limits_{u\to 0+} \nu(u) = 0$,

(M′) $\lim\limits_{u\to 0+} \sup\limits_{t>0} \sup\limits_{k} \varphi_k(ut)/\varphi_k(t) = 0$,

then g_Φ *is an absolutely monotone* F*-seminorm on* $\lambda(\Phi)$.

Proposition 2.8. *Let* (λ, g) *be an* F*-seminormed* AK*-space. If* g *is absolutely monotone, then* g_Φ *is an absolutely monotone* F*-seminorm on* $\lambda(\Phi)$ *for an arbitrary sequence of moduli* $\Phi = (\varphi_k)$*. Moreover,* $(\lambda(\Phi), g_\Phi)$ *is an* AK*-space.*

It is known that the spaces ℓ_p, c_0 and $(w_0)_p$ $(1 \leqslant p < \infty)$ are BK-AK-spaces with absolutely monotone norms $\|\cdot\|_{\ell_p}$, $\|\cdot\|_{c_0}$ and $\|\cdot\|_{(w_0)_p}$, respectively. By Proposition 2.8, the topology on the sequence space $\lambda(\Phi)$ with $\lambda \in \{\ell_p, c_0, (w_0)_p\}$ is given by the F-norm

$$g_\Phi(x) = \|\Phi(x)\|_\lambda.$$

Since $(\ell_\infty, \|\cdot\|_{\ell_\infty})$ is not an AK-space, on $\ell_\infty(\Phi)$ the same F-norm topology can be given by Proposition 2.7.

For the proof of main theorems we need the following lemmas.

Lemma 2.9. *Let* φ *be an unbounded modulus. The function* φ^{-1}*, defined by*

$$\varphi^{-1}(t) = \sup\{u\colon\ \varphi(u) = t\},$$

is nondecreasing and continuous from the right at 0*. Moreover,* $\varphi(\varphi^{-1}(t)) = t$ *and* $t \leqslant \varphi^{-1}(\varphi(t))$.

P r o o f. Continuity of φ^{-1} from the right at 0 follows from the fact that $\varphi(u) \to 0$ if and only if $u \to 0+$. The assertions $\varphi(\varphi^{-1}(t)) = t$ and $t \leqslant \varphi^{-1}(\varphi(t))$ are clear by the definition of φ^{-1}. □

Lemma 2.10. *Let* λ *be a solid Banach sequence space with* $e^k \in \lambda$ $(k \in \mathbb{N})$ *and let* μ *be a solid* BK*-space. Let* $\Phi = (\varphi_k)$ *and* $\Psi = (\psi_k)$ *be two sequences of moduli such that* $\lambda(\Phi)$ *and* $\mu(\Psi)$ *are topologized by* F*-norms* g_Φ *and* g_Ψ*. If the superposition operator* $P_f\colon\ \lambda(\Phi) \to \mu(\Psi)$ *is continuous, then all functions* $f(k,\cdot)$ $(k \in \mathbb{N})$ *are continuous.*

P r o o f. Suppose that the superposition operator $P_f\colon\ \lambda(\Phi) \to \mu(\Psi)$ is continuous. Let $i_k\colon\ \mathbb{R} \to \lambda(\Phi)$ be the embedding defined for every $u \in \mathbb{R}$ by the formula

$i_k(u) = ue^k \in \lambda(\Phi)$. Then for every $k \in \mathbb{N}$ the function $f(k,\cdot)$ factors as follows

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f(k,\cdot)"] \arrow[d, "i_k"'] & \mathbb{R} \\ \lambda(\Phi) \arrow[r, "P_f"'] & \mu(\Psi) \arrow[u, "\pi_k"'] \end{tikzcd} .$$

Here the coordinate functionals π_k are continuous for every $k \in \mathbb{N}$, since by Proposition 3 from [7] the space $\mu(\Psi)$ is a K-space.

To show that i_k is continuous at $u_0 \in \mathbb{R}$, let $\varepsilon > 0$ be given. Since the moduli φ_k are continuous from the right at 0, there exists $\delta > 0$ such that $0 \leqslant t \leqslant \delta$ implies $|\varphi_k(t)| < \varepsilon(\|e^k\|_\lambda)^{-1}$. If now $|u-u_0| < \delta$, then by (iii) we have $\varphi_k(|u-u_0|) \leqslant \varphi_k(\delta) < \varepsilon(\|e^k\|_\lambda)^{-1}$. Then

$$g_\Phi(i_k(u) - i_k(u_0)) = \|\Phi(i_k(u) - i_k(u_0))\|_\lambda = \varphi_k(|u-u_0|)\|e^k\|_\lambda < \varepsilon.$$

Hence i_k is continuous.

Consequently, all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous as compositions of continuous functions π_k, P_f and i_k. □

Lemma 2.11. *Let $\Phi = (\varphi_k)$ be a sequence of moduli and let $(\lambda, \|\cdot\|_\lambda)$ be a solid Banach sequence space such that $\lambda \subseteq c_0$ and $|y_k| \leqslant \|y\|_\lambda$ $(k \in \mathbb{N})$ for all $y = (y_k) \in \lambda$. For every fixed sequence $x = (x_k^0) \in \lambda(\Phi)$ and for a number $\delta > 0$ there exists $m \in \mathbb{N}$ such that*

$$\max\{\varphi_k(|x_k^0|),\ \varphi_k(|x_k|)\} < \delta \quad (k \geqslant m) \tag{2.7}$$

for all $x \in \lambda(\Phi)$ with $g_\Phi(x - x_0) < \delta/2$.

Proof. Let $x = (x_k^0) \in \lambda(\Phi)$ and let $\delta > 0$. Since $\Phi(x_0) = (\varphi_k(|x_k^0|)) \in \lambda \subseteq c_0$, there exists $m \in \mathbb{N}$ with

$$\varphi_k(|x_k^0|) < \frac{\delta}{2} \quad (k \geqslant m). \tag{2.8}$$

If $x = (x_k) \in \lambda(\Phi)$ satisfies $g_\Phi(x - x_0) < \delta/2$, then

$$\varphi_k(|x_k|) \leqslant \varphi_k(|x_k + x_k^0|) + \varphi_k(|x_k^0|) < \|\Phi(x - x_0)\|_\lambda + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta \tag{2.9}$$

for $k \geqslant m$. By (2.8) and (2.9) we get (2.7). □

Lemma 2.12. *Let $\Phi = (\varphi_k)$ be a sequence of unbounded moduli and $\Psi = (\psi_k)$ be a sequence of moduli. Let $x = (x_k^0)$ be a given sequence and $1 \leqslant q < \infty$. If the functions $f(k, \cdot)$ $(k = 1, \dots, m)$ are continuous, then for an arbitrary $\varepsilon > 0$ there exists a number $\delta' > 0$ such that*

$$\max_{k \leqslant m} \psi_k(|f(k,t) - f(k, x_k^0)|) < \varepsilon m^{-1/q} \tag{2.10}$$

whenever

$$\varphi_k(|t - x_k^0|) < \delta' \quad (k = 1, \dots, m).$$

Proof. Let $\varepsilon > 0$. Since the moduli ψ_k $(k = 1, \dots, m)$ are continuous from the right at 0, there exists $\alpha > 0$ such that

$$\psi_k(\alpha) < \varepsilon m^{-1/q} \tag{2.11}$$

for all $k = 1, \dots, m$. By the continuity of functions $f(k, \cdot)$ $(k = 1, \dots, m)$ there exists $\beta > 0$ such that

$$|t - x_k^0| < \beta$$

implies

$$|f(k,t) - f(k, x_k^0)| < \alpha. \tag{2.12}$$

Further, using Lemma 2.9, we can find $\delta' > 0$ such that

$$\varphi_k^{-1}(\delta') < \beta \quad (k = 1, \dots, m).$$

If now $\varphi_k(|t - x_k^0|) < \delta'$ $(k = 1, \dots, m)$, then

$$|t - x_k^0| \leqslant \varphi_k^{-1}(\varphi_k(|t - x_k^0|)) \leqslant \varphi_k^{-1}(\delta') < \beta.$$

Since the moduli ψ_k $(k = 1, \dots, m)$ are nondecreasing, from (2.12) we deduce that

$$\psi_k(|f(k,t) - f(k, x_k^0)|) \leqslant \psi_k(\alpha),$$

which together with (2.11) gives (2.10). □

3. Superposition operators on $\ell_p(\Phi)$, $c_0(\Phi)$, $\ell_\infty(\Phi)$ and $(w_0)_p(\Phi)$ $(1 \leqslant p < \infty)$

In the following let $\Phi = (\varphi_k)$ be a sequence of unbounded moduli and $\Psi = (\psi_k)$ be an arbitrary sequence of moduli.

First we characterize the continuity of superposition operators from $\ell_p(\Phi)$ and $c_0(\Phi)$ into $\ell_q(\Psi)$.

Theorem 3.1. *Let $1 \leqslant p, q < \infty$. A superposition operator $P_f\colon \ell_p(\Phi) \to \ell_q(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

P r o o f. If P_f is continuous, then all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous by Lemma 2.10.

Conversely, suppose that P_f maps $\ell_p(\Phi)$ into $\ell_q(\Psi)$ and all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous. Let $x = (x_k^0) \in \ell_p(\Phi)$ and $\varepsilon > 0$. If the numbers $\delta > 0$, $\gamma \geqslant 0$, $k_0 \in \mathbb{N}$ and the sequence $(a_k) \in \ell^+$ are determined by Proposition 2.1, then, using also Lemma 2.11, we may choose a number $m \in \mathbb{N}$ such that $m \geqslant k_0$,

$$\sum_{k=m+1}^{\infty} a_k < \varepsilon^q, \tag{3.1}$$

$$\Big(\sum_{k=m+1}^{\infty} (\varphi_k(|x_k^0|))^p\Big)^{1/p} < \frac{1}{2}\Big(\frac{\varepsilon^q}{\gamma+1}\Big)^{1/p} \tag{3.2}$$

and the condition (2.7) is satisfied whenever

$$g_\Phi(x - x_0) < \varrho = \min\Big\{\frac{\delta}{2}, \frac{1}{2}\Big(\frac{\varepsilon^q}{\gamma+1}\Big)^{1/p}\Big\}.$$

Thus we get

$$\begin{aligned} &\Big(\sum_{k=m+1}^{\infty} (\varphi_k(|x_k|))^p\Big)^{1/p} \\ &\quad\leqslant \Big(\sum_{k=m+1}^{\infty} (\varphi_k(|x_k - x_k^0|))^p\Big)^{1/p} + \Big(\sum_{k=m+1}^{\infty} (\varphi_k(|x_k^0|))^p\Big)^{1/p} \\ &\quad\leqslant \varrho + \frac{1}{2}\Big(\frac{\varepsilon^q}{\gamma+1}\Big)^{1/p} < \Big(\frac{\varepsilon^q}{\gamma+1}\Big)^{1/p}. \end{aligned} \tag{3.3}$$

Moreover, by the inequality (2.1), because of (2.7), for all $k > m$ we have

$$\begin{aligned} (\psi_k(|f(k,x_k)|))^q &\leqslant a_k + \gamma(\varphi_k(|x_k|))^p, \\ (\psi_k(|f(k,x_k^0)|))^q &\leqslant a_k + \gamma(\varphi_k(|x_k^0|))^p. \end{aligned} \tag{3.4}$$

Further, since the functions $f(k,\cdot)$ are continuous, by Lemma 2.12 there exists $\delta' > 0$ with $\delta' \leqslant \varrho$ such that $g_\Phi(x - x_0) < \delta'$ implies

$$\psi_k(|f(k,x_k) - f(k,x_k^0)|) < \varepsilon m^{-1/q} \quad (k = 1, 2, \dots, m). \tag{3.5}$$

Now, by (3.1)–(3.5) we get

$$
\begin{aligned}
&\|\Psi(P_f(x)-P_f(x_0))\|_{\ell_q}\\
&\quad\leqslant\Bigl(\sum_{k=1}^{m}(\psi_k(|f(k,x_k)-f(k,x_k^0)|))^q\Bigr)^{1/q}+\Bigl(\sum_{k=m+1}^{\infty}(\psi_k(|f(k,x_k)|))^q\Bigr)^{1/q}\\
&\qquad+\Bigl(\sum_{k=m+1}^{\infty}(\psi_k(|f(k,x_k^0)|))^q\Bigr)^{1/q}\\
&\quad\leqslant\Bigl(\sum_{k=1}^{m}(\varepsilon m^{-1/q})^q\Bigr)^{1/q}+2\Bigl(\sum_{k=m+1}^{\infty}a_k\Bigr)^{1/q}\\
&\qquad+\Bigl(\sum_{k=m+1}^{\infty}\gamma(\varphi_k(|x_k|))^p\Bigr)^{1/q}+\Bigl(\sum_{k=m+1}^{\infty}\gamma(\varphi_k(|x_k^0|))^p\Bigr)^{1/q}\\
&\quad<\varepsilon+2\varepsilon+\varepsilon+\varepsilon=5\varepsilon.
\end{aligned}
$$

Consequently, $h_\Psi(P_f(x)-P_f(x_0))<5\varepsilon$ whenever $g_\Phi(x-x_0)<\delta'$. □

Theorem 3.2. *Let $1\leqslant q<\infty$. A superposition operator $P_f\colon c_0(\Phi)\to\ell_q(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k\in\mathbb{N})$ are continuous.*

Proof. If P_f is continuous, then the continuity of the functions $f(k,\cdot)$ $(k\in\mathbb{N})$ is clear by Lemma 2.10.

Conversely, if all functions $f(k,\cdot)$ $(k\in\mathbb{N})$ are continuous, $x_0=(x_k^0)\in c_0(\Phi)$ and $\varepsilon>0$ is arbitrarily given, then, using Proposition 2.2 and Lemmas 2.11 and 2.12, similarly to the proof of Theorem 3.1, we may find a sequence $(a_k)\in\ell^+$ and numbers $m\in\mathbb{N}$, $\delta'>0$ such that (3.1) holds and $g_\Phi(x-x_0)<\delta'$ yields (3.5) and

$$(\psi_k(|f(k,x_k)|))^q\leqslant a_k,\quad(\psi_k(|f(k,x_k^0)|))^q\leqslant a_k\quad(k>m).\tag{3.6}$$

Consequently, by (3.1), (3.5) and (3.6) we get

$$
\begin{aligned}
&\|\Psi(P_f(x)-P_f(x_0))\|_{\ell_q}\\
&\quad\leqslant\Bigl(\sum_{k\leqslant m}(\psi_k(\|f(k,x_k)-f(k,x_k^0)|))^q\Bigr)^{1/q}\\
&\qquad+\Bigl(\sum_{k>m}(\psi_k(|f(k,x_k)|))^q\Bigr)^{1/q}+\Bigl(\sum_{k>m}(\psi_k(|f(k,x_k^0)|))^q\Bigr)^{1/q}\\
&\quad\leqslant\Bigl(\sum_{k\leqslant m}(\varepsilon m^{-1/q})^q\Bigr)^{1/q}+2\Bigl(\sum_{k>m}a_k\Bigr)^{1/q}\\
&\quad<\varepsilon+2\varepsilon=3\varepsilon
\end{aligned}
$$

whenever $g_\Phi(x-x_0)<\delta'$. □

The continuity of superposition operators from $\ell_p(\Phi)$ $(1 \leqslant p < \infty)$ and $c_0(\Phi)$ into $c_0(\Psi)$ is described by

Theorem 3.3. *Let $1 \leqslant p < \infty$. The superposition operators $P_f\colon \ell_p(\Phi) \to c_0(\Psi)$, $P_f\colon c_0(\Phi) \to c_0(\Psi)$ are continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

Proof. Lemma 2.10 shows that the continuity of the functions $f(k,\cdot)$ $(k \in \mathbb{N})$ is necessary for the continuity of P_f.

Conversely, suppose that all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and let $x = (x_k^0)$ be an element from $\ell_p(\Phi)$ or $c_0(\Phi)$. By Proposition 2.3 there exist numbers $\delta > 0$, $k_0 \in \mathbb{N}$ and a sequence $(a_k) \in c_0^+$ such that (2.3) holds. Now, in view of Lemma 2.11, for an arbitrary number $\varepsilon > 0$ we may choose an index $m \in \mathbb{N}$, $m \geqslant k_0$, such that

$$a_k < \frac{\varepsilon}{2} \quad (k > m)$$

and (2.7) is true whenever $g_\Phi(x - x_0) < \delta/2$. So by (2.3) we have, for all $k > m$,

$$\begin{aligned}\psi_k(|f(k,x_k) - f(k,x_k^0)|) &\leqslant \psi_k(|f(k,x_k)|) + \psi_k(|f(k,x_k^0)|)\\ &\leqslant a_k + a_k < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $g_\Phi(x - x_0) < \delta/2$ yields

$$\sup_{k>m} \psi_k(|f(k,x_k) - f(k,x_k^0)|) < \varepsilon. \tag{3.7}$$

Further, using Lemma 2.12, we fix a number $\delta' \leqslant \delta/2$ such that (3.5) holds for $g_\Phi(x - x_0) < \delta'$. But (3.5) immediately gives

$$\sup_{k\leqslant m} \psi_k(|f(k,x_k) - f(k,x_k^0)|) < \varepsilon. \tag{3.8}$$

Finally, by (3.7) and (3.8) we obtain

$$\begin{aligned}h_\Psi(P_f(x) - P_f(x_0)) &= \|\Psi(P_f(x) - P_f(x_0))\|_{c_0} = \sup_k \psi_k(|f(k,x_k) - f(k,x_k^0)|)\\ &= \max\Big\{\sup_{k\leqslant m} \psi_k(|f(k,x_k) - f(k,x_k^0)|), \sup_{k>m} \psi_k(|f(k,x_k) - f(k,x_k^0)|)\Big\} < \varepsilon\end{aligned}$$

whenever $g_\Phi(x - x_0) < \delta'$. □

To investigate the continuity of superposition operators on $\ell_\infty(\Phi)$, we equip the space $\ell_\infty(\Phi)$ with the F-norm

$$g_\Phi(x) = \|\Phi(x)\|_{\ell_\infty}$$

on the ground of Proposition 2.7.

Theorem 3.4. *Let $1 \leqslant q < \infty$. If the sequence of moduli $\Phi = (\varphi_k)$ satisfies one of the conditions* (M) *and* (M′)*, then $P_f\colon \ell_\infty(\Phi) \to \ell_q(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

P r o o f. If P_f is continuous, then the functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous by Lemma 2.10.

Conversely, suppose that all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous. If $x = (x_k^0) \in \ell_\infty(\Phi)$, then for some $\eta > 0$ we have

$$\varphi_k(|x_k^0|) \leqslant \frac{\eta}{2}. \tag{3.9}$$

By Proposition 2.4, for this number $\eta > 0$ we can find a sequence $(a_k) \in \ell^+$ such that the condition (2.4) is valid for every $k \in \mathbb{N}$. Since $(a_k) \in \ell^+$, for a given $\varepsilon > 0$ we may choose $m \in \mathbb{N}$ such that (3.1) holds. On the other hand, (3.9) together with (2.9) (for $\delta = \eta$) gives

$$\varphi_k(|x_k|) \leqslant \eta$$

if $g_\Phi(x - x_0) < \eta/2$. So (2.4) yields (3.6) whenever $g_\Phi(x - x_0) < \eta/2$.

Further, using the continuity of functions $f(k,\cdot)$ $(k = 1,\dots,m)$, by Lemma 2.12 there exists $\delta' > 0$ with $\delta' \leqslant \eta/2$ such that (3.5) is true if

$$\varphi_k(|x_k - x_k^0|) < \delta'.$$

Now, as in the proof of Theorem 3.2, from (3.1), (3.5) and (3.6) we deduce the continuity of P_f at x_0. □

Theorem 3.5. *If the sequence of moduli $\Phi = (\varphi_k)$ satisfies one of the conditions* (M) *and* (M′)*, then $P_f\colon \ell_\infty(\Phi) \to c_0(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

P r o o f. The continuity of the functions $f(k,\cdot)$ $(k \in \mathbb{N})$ is necessary for the continuity of P_f by Lemma 2.10.

If all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous and $P_f\colon \ell_\infty(\Phi) \to c_0(\Psi)$, then by Proposition 2.5 we can find, for $\eta = 1$, a sequence $(a_k) \in c_0^+$ and a number $k_0 \in \mathbb{N}$ such that (2.5) is satisfied. Now, putting $\delta = 1$, continuity of P_f follows in the same way as in Theorem 3.3. □

Our last theorem describes the continuity of superposition operators on the space $(w_0)_p(\Phi)$.

Theorem 3.6. *Let $1 \leqslant p, q < \infty$. If the moduli φ_k $(k \in \mathbb{N})$ are strictly increasing, then a superposition operator $P_f\colon (w_0)_p(\Phi) \to \ell_q(\Psi)$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

Proof. If P_f is continuous, then the continuity of the functions $f(k,\cdot)$ $(k \in \mathbb{N})$ follows by Lemma 2.10.

Conversely, suppose that all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous, P_f maps $(w_0)_p(\Phi)$ into $\ell_q(\Psi)$ and $x = (x_k^0) \in (w_0)_p(\Phi)$. By Proposition 2.6 there exist a number $\delta > 0$ and sequences $(c_k)_{k=0}^{\infty} \in \ell^+$ and $(d_k) \in \ell^+$ such that the condition (2.6) holds whenever $\varphi_k(|t|)^p \leqslant 2^i\delta$, $2^i \leqslant k < 2^{i+1}$ $(i = 0, 1, \ldots)$. It is known (see, for example, [11, p. 523]) that $x = (x_k^0) \in (w_0)_p(\Phi)$ if and only if

$$\lim_{i\to\infty} 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k^0|))^p = 0.$$

For a fixed $\varepsilon > 0$ we denote by i_ε the least of all numbers s such that

$$\sup_{i\geqslant s} 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k^0|))^p < \frac{\delta}{2^p}, \quad \sum_{k=2^s}^{\infty} d_k < \Bigl(\frac{\varepsilon}{2}\Bigr)^q \quad \text{and} \quad \sum_{i=s}^{\infty} c_i < \frac{\varepsilon^q}{\delta}.$$

So, if $x = (x_k) \in (w_0)_p(\Phi)$ satisfies

$$g_\Phi(x - x_0) < \frac{1}{2}\,\delta^{1/p},$$

by (ii) and Minkowski's inequality, for $i \geqslant i_\varepsilon$, we get

$$\begin{aligned}
&\Biggl(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p\Biggr)^{1/p} \\
&\quad\leqslant \Biggl(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k - x_k^0|))^p\Biggr)^{1/p} + \Biggl(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k^0|))^p\Biggr)^{1/p} \\
&\quad\leqslant \|\Phi(x - x_0)\|_{(w_0)_p} + \Biggl(2^{-i} \sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k^0|))^p\Biggr)^{1/p} \\
&\quad\leqslant 2^{-1}\delta^{1/p} + 2^{-1}\delta^{1/p} = \delta^{1/p}.
\end{aligned}$$

Thus, if $i \geqslant i_\varepsilon$, then

$$\sum_{k=2^i}^{2^{i+1}-1} (\varphi_k(|x_k|))^p \leqslant 2^i\delta$$

which yields $(\varphi_k(|x_k|))^p \leqslant 2^i\delta$ $(2^i \leqslant k < 2^{i+1})$. Therefore, (2.6) implies that

$$
\begin{aligned}
(3.10)\qquad & (\psi_k(|f(k,x_k^0)|))^q \leqslant d_k + c_i 2^{-i}(\varphi_k(|x_k^0|))^p, \\
& (\psi_k(|f(k,x_k)|))^q \leqslant d_k + c_i 2^{-i}(\varphi_k(|x_k|))^p \quad (i \geqslant i_\varepsilon).
\end{aligned}
$$

Further, using the continuity of functions $f(k,\cdot)$, by Lemma 2.12 (for $m = 2^{i_\varepsilon}$) we may choose $\delta' > 0$ with $\delta' \leqslant 2^{-1}\delta^{1/p}$ such that

$$
(3.11)\qquad \max_{k<2^{i_\varepsilon}} \psi_k(|f(k,x_k) - f(k,x_k^0)|) < \varepsilon 2^{-i_\varepsilon/q}
$$

if $g_\Phi(x - x_0) < \delta'$. Now, by (3.10) and (3.11) we conclude that

$$
\begin{aligned}
&\|\Psi(P_f(x) - P_f(x_0))\|_{\ell_q} \\
&\quad \leqslant \Big(\sum_{k=1}^{2^{i_\varepsilon}-1} (\psi_k(|f(k,x_k) - f(k,x_k^0)|))^q\Big)^{1/q} + \Big(\sum_{k=2^{i_\varepsilon}}^{\infty} (\psi_k(|f(k,x_k)|))^q\Big)^{1/q} \\
&\qquad + \Big(\sum_{k=2^{i_\varepsilon}}^{\infty} (\psi_k(|f(k,x_k^0)|))^q\Big)^{1/q} \\
&\quad \leqslant \Big(\sum_{k=1}^{2^{i_\varepsilon}-1} (\varepsilon 2^{-i_\varepsilon/q})^q\Big)^{1/q} + \Big(\sum_{i=i_\varepsilon}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,x_k)|))^q\Big)^{1/q} \\
&\qquad + \Big(\sum_{i=i_\varepsilon}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} (\psi_k(|f(k,x_k^0)|))^q\Big)^{1/q} \\
&\quad < \varepsilon((2^{i_\varepsilon}-1)2^{-i_\varepsilon})^{1/q} + 2\Big(\sum_{k=2^{i_\varepsilon}}^{\infty} d_k\Big)^{1/q} + \Big(\sum_{i=i_\varepsilon}^{\infty} c_i 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} \varphi_k(|x_k|)^p\Big)^{1/q} \\
&\qquad + \Big(\sum_{i=i_\varepsilon}^{\infty} c_i 2^{-i} \sum_{k=2^i}^{2^{i+1}-1} \varphi_k(|x_k^0|)^p\Big)^{1/q} \\
&\quad < \varepsilon + 2\frac{\varepsilon}{2} + 2\Big(\frac{\varepsilon^q}{\delta}\delta\Big)^{1/q} = 4\varepsilon.
\end{aligned}
$$

Consequently, $h_\Psi(P_f(x) - P_f(x_0)) < 4\varepsilon$ whenever $g_\Phi(x - x_0) < \delta'$. □

Remark. Lemma 2.10 shows that in Theorems 3.1–3.6 the continuity of functions $f(k,\cdot)$ $(k \in \mathbb{N})$ is necessary for the continuity of $P_f\colon \lambda(\Phi) \to \mu(\Psi)$ without restrictions on the moduli φ_k $(k \in \mathbb{N})$. But the converse we are able to prove only under the assumption that the moduli φ_k $(k \in \mathbb{N})$ are unbounded, because Propositions 2.2–2.6 and Lemma 2.12 are true in this case. The validity of Theorems 3.1–3.6 for an arbitrary sequence of moduli $\Phi = (\varphi_k)$ remains open.

4. Corollaries and applications

It is clear that the classical sequence spaces ℓ_∞, ℓ_p, c_0, $(w_0)_p$ can be considered as the spaces $\ell_\infty(\Phi)$, $\ell_p(\Phi)$, $c_0(\Phi)$, $(w_0)_p(\Phi)$, where $\Phi = (\varphi_k)$ with $\varphi_k(t) = t$ $(k \in \mathbb{N})$. So, letting $\Psi = \Phi$, our Theorems 3.1–3.5 reduce to the known characterizations of the continuity of superposition operators from ℓ_∞, ℓ_p and c_0 into ℓ_q and c_0 for $1 \leqslant p, q < \infty$ ([3, Theorems 2, 7 and 8]; [12, Theorems 2.4 and 2.5]).

Theorem 3.6 allows to formulate an extension of a result of Płuciennik [13, Theorem 5] about the continuity of the superposition operator $P_f\colon w_0 \to \ell$ as follows.

Proposition 4.1. *Let $1 \leqslant p, q < \infty$. A superposition operator $P_f\colon (w_0)_p \to \ell_q$ is continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

As generalizations of the spaces ℓ_∞, c_0, ℓ_p, and $(w_0)_p$ we consider the multiplier sequence spaces of Maddox type

$$\ell_\infty(p,u) = \Big\{x \in \omega\colon \sup_k |u_k x_k|^{p_k} < \infty\Big\},$$
$$c_0(p,u) = \Big\{x \in \omega\colon \lim_k |u_k x_k|^{p_k} = 0\Big\},$$
$$\ell(p,u) = \Big\{x \in \omega\colon \sum_{k=1}^{\infty} |u_k x_k|^{p_k} < \infty\Big\},$$
$$w_0(p,u) = \Big\{x \in \omega\colon \lim_n \frac{1}{n}\sum_{k=1}^{n} |u_k x_k|^{p_k} = 0\Big\},$$

where $u = (u_k)$ is a sequence with $u_k \neq 0$ $(k \in \mathbb{N})$ and $p = (p_k)$ is a bounded sequence of positive numbers (cf. [5]).

In the case of $u_k = 1$ $(k \in \mathbb{N})$ the spaces $\ell_\infty(p,u)$, $c_0(p,u)$, $\ell(p,u)$ and $w_0(p,u)$ are known as the sequence spaces of Maddox type $\ell_\infty(p)$, $c_0(p)$, $\ell(p)$ and $w_0(p)$, respectively (see, for example, [4] and [9]). Some authors ([2], [16]) consider the spaces $\ell_\infty(p,u)$, $c_0(p,u)$ and $\ell(p,v)$ for

$$u_k = k^{-\alpha/p_k}, \quad v_k = k^{\alpha/p_k} \quad (\alpha > 0). \tag{4.1}$$

To apply our theorems to the multiplier spaces of Maddox type, we put $r = \max\{1, \sup_k p_k\}$ and define the sequence of moduli $\Phi = (\varphi_k)$ by

$$\varphi_k(t) = (|u_k| t)^{p_k/r} \quad (k \in \mathbb{N}).$$

Then we may consider the spaces $\ell_\infty(p,u)$, $c_0(p,u)$, $\ell(p,u)$ and $w_0(p,u)$ as the spaces $\ell_\infty(\Phi)$, $c_0(\Phi)$, $\ell_r(\Phi)$ and $(w_0)_r(\Phi)$, respectively. So, by Propositions 2.7

and 2.8, the F-norm

$$g_\Phi(x) = \sup_k |u_k x_k|^{p_k/r}$$

is defined on $c_0(p,u)$ for any p and on $\ell_\infty(p,u)$ under the restriction $\inf_k p_k > 0$. The corresponding F-norms on $\ell(p,u)$ and $w_0(p,u)$ are determined, respectively, by

$$g_\Phi(x) = \left(\sum_{k=1}^{\infty} |u_k x_k|^{p_k}\right)^{1/r}$$

and

$$g_\Phi(x) = \sup_{i\geqslant 0}\left(\frac{1}{2^i}\sum_{k=2^i}^{2^{i+1}-1} |u_k x_k|^{p_k}\right)^{1/r}.$$

Further, using Propositions 2.1–2.6, it is not difficult to formulate the mapping conditions for superposition operators on multiplier sequence spaces of Maddox type. Thereby, for the multipliers (4.1) we get the known characterizations of the operators $P_f\colon \ell_\infty(p,u) \to \ell$ and $P_f\colon \ell(p,v) \to \ell$ ([16, Theorems 1 and 8]; [18, Theorems 2.1 and 2.2], the case $p_k = 1$ $(k \in \mathbb{N})$).

Let $q = (q_k)$ be another bounded sequence of positive numbers and $v = (v_k)$ a sequence such that $v_k \neq 0$ $(k \in \mathbb{N})$. Now, putting $s = \max\{1, \sup_k q_k\}$ and defining the sequence of moduli $\Psi = (\psi_k)$ by

$$\psi_k(t) = (|v_k|t)^{q_k/s} \quad (k \in \mathbb{N}),$$

from Theorems 3.1–3.6 we get the following statements about the continuity of superposition operators on multiplier sequence spaces of Maddox type.

Proposition 4.2. *Superposition operators $P_f\colon \ell(p,u) \to \ell(q,v)$, $P_f\colon \ell(p,u) \to c_0(q,v)$, $P_f\colon c_0(p,u) \to c_0(q,v)$, $P_f\colon c_0(p,u) \to \ell(q,v)$ and $P_f\colon w_0(p,u) \to \ell(q,v)$ are continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

Proposition 4.3. *If $\inf_k p_k > 0$, then $P_f\colon \ell_\infty(p,u) \to \ell(q,v)$ and $P_f\colon \ell_\infty(p,u) \to c_0(q,v)$ are continuous if and only if all functions $f(k,\cdot)$ $(k \in \mathbb{N})$ are continuous.*

Sama-ae [16, Theorems 6 and 14] studied the continuity of superposition operators $P_f\colon \ell_\infty(p,u) \to \ell$ and $P_f\colon \ell(p,v) \to \ell$ in the case of multipliers (4.1).

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Authors' address: Enno Kolk, Institute of Pure Mathematics, University of Tartu, 50090 Tartu, Estonia, e-mails: `enno.kolk@ut.ee`; Annemai Raidjõe, Institute of Mathematics, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia, e-mails: `annemai.raidjoe@ttu.ee`.