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Meng Xiao Yin; Jian Hua Yin

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ON POTENTIALLY H-GRAPHIC SEQUENCES

Meng-Xiao Yin, Nanning, Jian-Hua Yin, Haikou

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Abstract. For given a graph H, a graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially H-graphic if there is a realization of π containing H as a subgraph. In this paper, we characterize the potentially $(K_5 - e)$-positive graphic sequences and give two simple necessary and sufficient conditions for a positive graphic sequence π to be potentially K_5-graphic, where K_r is a complete graph on r vertices and $K_r - e$ is a graph obtained from K_r by deleting one edge. Moreover, we also give a simple necessary and sufficient condition for a positive graphic sequence π to be potentially K_6-graphic.

Keywords: graph, degree sequence, potentially H-graphic sequence

MSC 2000: 05C07

1. Introduction

The set of all non-increasing nonnegative integer sequences $\pi = (d_1, d_2, \ldots, d_n)$ with $d_i \leqslant n-1$ for each i is denoted by NS_n. A sequence $\pi \in \mathrm{NS}_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. If each term of a graphic sequence $\pi \in \mathrm{GS}_n$ is nonzero, then π is said to be *positive graphic*. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$, define $\sigma(\pi) = d_1 + d_2 + \ldots + d_n$. For given a graph H, a sequence $\pi \in \mathrm{GS}_n$ is said to be *potentially H-graphic*, if there is a realization of π containing H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be *potentially A_{r+1}-graphic*. Erdős, Jacobson and Lehel [1] in 1991 considered an extremal problem on potentially K_{r+1}-graphic sequences: determine the smallest even integer $\sigma(K_{r+1}, n)$ such that every sequence $\pi \in \mathrm{GS}_n$ with $\sigma(\pi) \geqslant \sigma(K_{r+1}, n)$ is potentially K_{r+1}-graphic. They proved that $\sigma(K_3, n) = 2n$ for $n \geqslant 6$ and conjectured

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that $\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$ for sufficiently large n. Gould et al. [3] and Li and Song [6] independently proved it for $r = 3$. Recently, Li et al. [7], [8] proved that the conjecture is true for $r = 4$ and $n \geqslant 10$ and for $r \geqslant 5$ and $n \geqslant \binom{r}{2} + 3$. Although the Erdős-Jacobson-Lehel conjecture was confirmed, it leaves a natural open question: given a graphic sequence π, how to tell whether it is potentially K_{r+1}-graphic? In [12], Rao considered the problem of characterizing potentially K_{r+1}-graphic sequences, proved that a sequence $\pi \in \mathrm{GS}_n$ is potentially A_{r+1}-graphic if and only if it is potentially K_{r+1}-graphic, and developed a "Havel-Hakimi" type procedure as follows to determine the maximum clique number of a graph with a given degree sequence.

Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ with $d_{r+1} \geqslant r$. We define sequences $\pi_0, \ldots, \pi_{r+1}$ as follows. Let $\pi_0 = \pi$. Let

$$\pi_1 = (d_2 - 1, \ldots, d_{r+1} - 1, d_{r+2}^{(1)}, \ldots, d_n^{(1)}),$$

where $d_{r+2}^{(1)} \geqslant \ldots \geqslant d_n^{(1)}$ is the rearrangement of $d_{r+2} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. For $2 \leqslant i \leqslant r+1$, given $\pi_{i-1} = (d_i - i + 1, \ldots, d_{r+1} - i + 1, d_{r+2}^{(i-1)}, \ldots, d_n^{(i-1)})$, let

$$\pi_i = (d_{i+1} - i, \ldots, d_{r+1} - i, d_{r+2}^{(i)}, \ldots, d_n^{(i)}),$$

where $d_{r+2}^{(i)} \geqslant \ldots \geqslant d_n^{(i)}$ is the rearrangement of $d_{r+2}^{(i-1)} - 1, \ldots, d_{d_i+1}^{(i-1)} - 1, d_{d_i+2}^{(i-1)}, \ldots, d_n^{(i-1)}$.

Theorem 1.1 [12]. *Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ with $d_{r+1} \geqslant r$. Then π is potentially A_{r+1}-graphic if and only if π_{r+1} is graphic.*

Theorem 1.2 [12]. *Let $n \geqslant r+2, \pi = (d_1, d_2, \ldots, d_n)$ with $d_{r+2} \geqslant d_{r+3} \geqslant \ldots \geqslant d_n$. If there exists a graph G on the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and $\{v_1, v_2, \ldots, v_{r+1}\}$ forms a complete subgraph of G, then there is one such graph in which v_1 is joined to $v_{r+2}, v_{r+3}, \ldots, v_{d_1+1}$.*

From the proof of Theorem 1.2, it is easy to obtain the following

Remark 1.1. Let $n \geqslant r+2$ and $\pi = (d_1, d_2, \ldots, d_n)$ with $d_{r+2} \geqslant d_{r+3} \geqslant \ldots \geqslant d_n$. If there exists a graph G on the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and the subgraph of G induced by $\{v_1, v_2, \ldots, v_{r+1}\}$ contains $K_{r+1} - e$ as a subgraph, where $e = v_r v_{r+1}$, then there is one such graph in which v_1 is joined to $v_{r+2}, v_{r+3}, \ldots, v_{d_1+1}$.

In [13], Rao gave the following characterization for a sequence $\pi \in \mathrm{GS}_n$ to be potentially A_{r+1}-graphic.

Theorem 1.3 [13]. *Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$. Then π is potentially A_{r+1}-graphic if and only if the following conditions hold:*

(i) $d_{r+1} \geqslant r$,

(ii) $\sigma(\pi)$ *is even,*

(iii) *for any s and t, $0 \leqslant s \leqslant r+1$ and $0 \leqslant t \leqslant n-r-1$,*

$$L(s,t) \leqslant R(s,t),$$

where $L(s,t) = \sum\limits_{i=1}^{s} d_i + \sum\limits_{i=1}^{t} d_{r+1+i}$ *and* $R(s,t) = (s+t)(s+t-1) + \sum\limits_{i=s+1}^{r+1} \min\{s+t, d_i - r + s\} + \sum\limits_{i=r+t+2}^{n} \min\{s+t, d_i\}$.

The original proof of Theorem 1.3 remains unpublished, but recently Kézdy and Lehel [4] have given a different proof using network flows. Unfortunately, the conditions in Theorem 1.3 are not easy to check, but Luo et al. [10], [11] gave simple characterizations for a positive graphic sequence π to be potentially K_r-graphic for $r = 3$ and 4, and Yin and Li [15] also obtained two sufficient conditions for a graphic sequence π to be potentially K_r-graphic. The following are their results.

Theorem 1.4 [10]. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ be a positive graphic sequence with $n \geqslant 3$. Then π is potentially K_3-graphic if and only if $d_3 \geqslant 2$ except for two cases: $\pi = (2^4)$ and $\pi = (2^5)$, where the symbol x^y in a sequence stands for y consecutive terms, each equal to x.*

Theorem 1.5 [11]. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ be a positive graphic sequence with $n \geqslant 4$ and $d_4 \geqslant 3$. Then π is potentially K_4-graphic if and only if $\pi \neq (n-1, 3^s, 1^{n-s-1})$ for $s = 4, 5$, and π is not one the following sequences:*

$n = 5$: $(4, 3^4), (3^4, 2)$;
$n = 6$: $(4^6), (4^2, 3^4), (4, 3^4, 2), (3^6), (3^5, 1), (3^4, 2^2)$;
$n = 7$: $(4^7), (4, 3^6), (4, 3^5, 1), (3^6, 2), (3^5, 2, 1)$;
$n = 8$: $(3^7, 1), (3^6, 1^2)$.

Theorem 1.6 [15]. *Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r+1} \geqslant r$. If $d_i \geqslant 2r - i$ for $i = 1, 2, \ldots, r-1$, then π is potentially A_{r+1}-graphic.*

Theorem 1.7 [15]. *Let $n \geqslant 2r+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r+1} \geqslant r$. If $d_{2r+2} \geqslant r-1$, then π is potentially A_{r+1}-graphic.*

Recently, Eschen and Niu [2] characterized potentially $K_4 - e$-graphic sequences, and Yin and Li [15] gave two sufficient conditions for a graphic sequence π to be potentially $K_r - e$-graphic. In other words, they proved the following

Theorem 1.8 [2]. *Let $n \geqslant 4$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ be a positive graphic sequence. Then π is potentially $K_4 - e$-graphic if and only if the following conditions hold:*

(1) $d_1 \geqslant d_2 \geqslant 3, d_4 \geqslant 2$;

(2) $\pi \neq (3^6), (3^2, 2^4), (3^2, 2^3)$.

Theorem 1.9 [15]. *Let $n \geqslant r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r+1} \geqslant r-1$. If $d_i \geqslant 2r - i$ for $i = 1, 2, \ldots, r-1$, then π is potentially $K_{r+1} - e$-graphic.*

Theorem 1.10 [15]. *Let $n \geqslant 2r+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ with $d_{r-1} \geqslant r$. If $d_{2r+2} \geqslant r-1$, then π is potentially $K_{r+1} - e$-graphic.*

In this paper, we characterize potentially $K_5 - e$-positive graphic sequences, give two simple necessary and sufficient conditions for a positive graphic sequence π to be potentially K_5-graphic, and also present a simple necessary and sufficient condition for a positive graphic sequence π to be potentially K_6-graphic, which are the following four theorems.

Theorem 1.11. *Let $n \geqslant 5$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_3 \geqslant 4$ and $d_5 \geqslant 3$. Then π is potentially $K_5 - e$-graphic if and only if π is not one of the following sequences:*

$(n-1, 4^6, 1^{n-7})$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^3, 1^{n-6})$;

$n = 6$: (4^6), $(4^4, 3^2)$, $(4^3, 3^2, 2)$;

$n = 7$: $(4^3, 3^4)$, $(5^2, 4, 3^4)$, (4^7), $(4^5, 3^2)$, $(5, 4^3, 3^3)$, $(5^2, 4^5)$, $(5, 4^5, 3)$, $(4^3, 3^2, 2^2)$, $(4^4, 3^2, 2)$, $(5, 4^2, 3^3, 2)$, $(4^6, 2)$, $(4^3, 3^3, 1)$;

$n = 8$: (5^8), (4^8), $(5^2, 4^6)$, $(6, 4^7)$, $(4^4, 3^4)$, $(5, 4^2, 3^5)$, $(4^6, 3^2)$, $(5, 4^6, 3)$, $(4^3, 3^4, 2)$, $(4^7, 2)$, $(4^4, 3^3, 1)$, $(5, 4^2, 3^4, 1)$, $(4^3, 3^3, 2, 1)$, $(4^6, 3, 1)$, $(5, 4^6, 1)$;

$n = 9$: (4^9), $(4^3, 3^5, 1)$, $(4^8, 2)$, $(4^7, 3, 1)$, $(5, 4^7, 1)$, $(4^3, 3^4, 1^2)$, $(4^7, 1^2)$;

$n = 10$: $(4^8, 1^2)$.

Theorem 1.12. *Let $n \geqslant 14$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_5 \geqslant 4$. Then π is potentially A_5-graphic if and only if $\pi_5 \notin S$, where $S = \{(2), (2^2), (3, 1), (3^2), (3, 2, 1), (3^2, 2), (3^3, 1), (3^2, 1^2)\}$.*

Theorem 1.13. *Let $n \geqslant 18$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_6 \geqslant 5$. Then π is potentially A_6-graphic if and only if $\pi_6 \notin S$.*

Theorem 1.14. *Let n be sufficiently large and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_5 \geqslant 4$. Then π is potentially A_5-graphic if and only if $(d_1 - 4, d_2 - 4, d_3 - 4, d_4 - 4, d_5 - 4, d_6, \ldots, d_n)$ is graphic, $\pi \neq (n-a, n-b, 4^4, 2^{n-(a+b+4)}, 1^{a+b-2})$ for $1 \leqslant a \leqslant b \leqslant n-6$ and $a+b \leqslant n-4$, and $\pi \neq (n-a, n-b, 4^5, 2^{n-(a+b+5)}, 1^{a+b-2})$ for $1 \leqslant a \leqslant b \leqslant n-6$ and $a+b \leqslant n-5$.*

2. Preparations

In order to prove our main results, we need the following notations and known results.

Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ and $1 \leqslant k \leqslant n$. Let

$$\pi''_k = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n), & \text{if } d_k \geqslant k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n), & \text{if } d_k < k. \end{cases}$$

Let $\pi'_k = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_1 \geqslant d'_2 \geqslant \ldots \geqslant d'_{n-1}$ is the rearrangement of the $n-1$ terms of π''_k. π'_k is called the *residual sequence* obtained by laying off d_k from π. It is easy to see that if π'_k is graphic then so is π, since a realization G of π can be obtained from a realization G' of π'_k by adding a new vertex of degree d_k and joining it to the vertices whose degrees are reduced by one in going from π to π'_k. In fact more is true:

Theorem 2.1 [5]. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ and $1 \leqslant k \leqslant n$. Then $\pi \in \mathrm{GS}_n$ if and only if $\pi'_k \in \mathrm{GS}_{n-1}$.*

Theorem 2.2 [14]. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$, $d_1 = m$ and $\sigma(\pi)$ be even. If there exists an integer n_1, $n_1 \leqslant n$ such that $d_{n_1} \geqslant h \geqslant 1$ and $n_1 \geqslant [\frac{1}{4}(m+h+1)^2]/h$, then $\pi \in \mathrm{GS}_n$.*

Theorem 2.3 [9]. *Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ and $\sigma(\pi)$ be even. If $d_1 - d_n \leqslant 1$, then $\pi \in \mathrm{GS}_n$.*

Theorem 2.4 [3]. *If $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{GS}_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

Lemma 2.1. *If $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ is potentially $K_{r+1} - e$-graphic, then there is a realization G of π containing $K_{r+1} - e$ such that the $r+1$ vertices $v_1, v_2, \ldots, v_{r+1}$ of $K_{r+1} - e$ satisfy $d_G(v_i) = d_i$ for $i = 1, 2, \ldots, r+1$ and $e = v_r v_{r+1}$.*

Proof. According to Theorem 2.4, there is a graph G' with vertex set $V(G') = \{v_1, v_2, \ldots, d_n\}$ and $d_{G'}(v_i) = d_i$ for $i = 1, 2, \ldots, n$ such that the subgraph of G' induced by $\{v_1, v_2, \ldots, v_{r+1}\}$ contains a $K_{r+1} - e$. If $e = v_r v_{r+1}$, then the lemma holds. We now assume $e = v_i v_j$.

If $v_i, v_j \in \{v_1, \ldots, v_{r-1}\}$, then for v_i, there exists a vertex $v_i' \in G' \setminus \{v_1, v_2, \ldots, v_{r+1}\}$ such that $v_i' v_i \in E(G')$ and $v_i' v_r \notin E(G')$. Otherwise $d_r \geqslant d_i + 1$, which is a contradiction. Similarly, for v_j, there is a vertex $v_j' \in G' \setminus \{v_1, v_2, \ldots, v_{r+1}\}$ such that $v_j v_j' \in E(G')$ and $v_j' v_{r+1} \notin E(G')$. Then

$$G = G' - v_i v_i' - v_r v_{r+1} - v_j v_j' + v_i v_j + v_r v_i' + v_{r+1} v_j'$$

is also a realization of π and G satisfies the conditions of the lemma.

If $v_i \in \{v_1, \ldots, v_{r-1}\}$, without loss of generality, let $v_j = v_r$, then there exists a vertex $v_i' \in G' \setminus \{v_1, v_2, \ldots, v_{r+1}\}$ such that $v_i' v_i \in E(G')$ and $v_i' v_{r+1} \notin E(G')$ since $d_i \geqslant d_{r+1}$. Hence,

$$G = G' - v_i v_i' - v_r v_{r+1} + v_i v_r + v_{r+1} v_i'$$

is also a realization of π satisfying the conditions of the lemma.

For $v_j \in \{v_1, \ldots, v_{r-1}\}$, the proof is similar to the above and is omitted here. $\square$

Lemma 2.2. *Let $\pi = (3^x, 2^y, 1^z)$ with even $\sigma(\pi)$ and $x + y + z = n \geqslant 1$, then $\pi \in \mathrm{GS}_n$ if and only if $\pi \notin S$.*

Proof. For $n = 1$, since $\sigma(\pi)$ is even, π must be (2), which belongs to S. For $n \geqslant 2$, we consider the following cases.

Case 1: $n = 2$. Then π is one of the following sequences: $(3,1), (2^2), (3^2), (1^2)$. It is easy to check that only one sequence (1^2) is graphic.

Case 2: $n = 3$. Since $\sigma(\pi)$ is even, π may be $(3,2,1), (3^2,2), (2^3)$ or $(2,1^2)$. We can see that (2^3) and $(2,1^2)$ are graphic.

Case 3: $n = 4$. Then π is one of the following:

$$(3^3,1), (3,1^3), (3^4), (2^4), (3,2^2,1), (2^2,1^2), (3^2,2^2), (1^4), (3^2,1^2),$$

which are all graphic except $(3^2,1^2)$ and $(3^3,1)$.

Case 4: $n = 5$. It is easy to see that π must be one of the following graphic sequences:

$$(2,1^4), (3,2,1^3), (3^2,2,1^2), (3^3,2,1), (3,2^3,1), (2^5), (3^2,2^3), (2^3,1^2), (3^4,2).$$

C a s e 5: $n \geqslant 6$. If $x > 0$ and $z > 0$, then $n \geqslant [\frac{(3+1+1)^2}{4}]$. Hence, π is graphic from Theorem 2.2. Otherwise, π is graphic by Theorem 2.3. $\square$

Lemma 2.3. *Let $\pi = (d_1, \ldots, d_n) \in \mathrm{NS}_n$ with $d_n \geqslant 1$ and even $\sigma(\pi)$. (1) If $n \geqslant 9$ and $d_1 \leqslant 4$, then $\pi \in \mathrm{GS}_n$. (2) If $n \geqslant 12$ and $d_1 \leqslant 5$, then $\pi \in \mathrm{GS}_n$.*

P r o o f. (1) If $d_1 = 4$ and $d_n \leqslant 2$, then $n \geqslant 9 = \max\{[\frac{(4+1+1)^2}{4}], \frac{1}{2}[\frac{(4+2+1)^2}{4}]\}$. Therefore, π is graphic by Theorem 2.2. If $d_1 = 4$ and $d_n \geqslant 3$, then by Theorem 2.3, π is graphic. If $d_1 \leqslant 3$, then $\pi \in \mathrm{GS}_n$ by Lemma 2.2.

(2) If $d_1 \leqslant 4$, then $\pi \in \mathrm{GS}_n$ from (1). For $d_1 = 5$ and $d_n \leqslant 3$, we have $n \geqslant 12 = \max\{\frac{1}{2}[\frac{(5+2+1)^2}{4}], [\frac{(5+1+1)^2}{4}], \frac{1}{3}[\frac{(5+3+1)^2}{4}]\}$. By Theorem 2.2, π is graphic. If $d_1 = 5$ and $d_n \geqslant 4$, then $\pi \in \mathrm{GS}_n$ by Theorem 2.3. $\square$

Lemma 2.4. *Let $n \geqslant 5$ and $\pi = (d_1, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_3 \geqslant 4$ and $d_5 \geqslant 3$. If π is not potentially $K_5 - e$-graphic and $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$, then $n - 2 \geqslant d_1 \geqslant \ldots \geqslant d_4 \geqslant d_5 = d_6 = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$.*

P r o o f. By way of contradiction, we assume that there exists an integer $t, 5 \leqslant t \leqslant d_1 + 1$ such that $d_t > d_{t+1}$. Since $d_3 \geqslant 4$, $d_5 \geqslant 3$ and $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$, the residual sequence $\pi_1' = (d_1', \ldots, d_{n-1}')$ satisfies the conditions in Theorem 1.8. Notice that $d_i' = d_{i+1} - 1$ for $i = 1, \ldots, t-1$. Therefore, π_1' has a realization G containing $K_4 - e$ such that the degrees of the vertices of $K_4 - e$ in G are d_1', d_2', d_3', d_4'. Thus π is potentially $K_5 - e$-graphic by $\{d_2 - 1, d_3 - 1, d_4 - 1, d_5 - 1\} = \{d_1', \ldots, d_4'\}$. $\square$

For convenience, we need the following definitions.

Let $n \geqslant 5$ and $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ with $d_3 \geqslant 4$ and $d_5 \geqslant 3$. We define sequences $\pi_0^*, \pi_1^*, \pi_2^*$ and π_3^* as follows. Let $\pi_0^* = \pi$. Let

$$\pi_1^* = (d_2 - 1, \ldots, d_5 - 1, d_6^{(1)}, \ldots, d_n^{(1)}),$$

where $d_6^{(1)} \geqslant \ldots \geqslant d_n^{(1)}$ is a rearrangement of $d_6 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. Let

$$\pi_2^* = (d_3 - 2, \ldots, d_5 - 2, d_6^{(2)}, \ldots, d_n^{(2)}),$$

where $d_6^{(2)} \geqslant \ldots \geqslant d_n^{(2)}$ is the rearrangement of $d_6^{(1)} - 1, \ldots, d_{d_2+1}^{(1)} - 1, d_{d_2+2}^{(1)}, \ldots, d_n^{(1)}$. Let

$$\pi_3^* = (d_4 - 3, d_5 - 3, d_6^{(3)}, \ldots, d_n^{(3)}),$$

where $d_6^{(3)} \geqslant \ldots \geqslant d_n^{(3)}$ is the rearrangement of $d_6^{(2)} - 1, \ldots, d_{d_3+1}^{(2)} - 1, d_{d_3+2}^{(2)}, \ldots, d_n^{(2)}$.

Lemma 2.5. *Let $n \geqslant 5$ and $\pi = (d_1, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_3 \geqslant 4$ and $d_5 \geqslant 3$. Then π is potentially $K_5 - e$-graphic if and only if π_3^* is graphic.*

Proof. The sufficient condition is obvious from the definition of π_3^*. Now we show the necessary condition. By Lemma 2.1 and Remark 1.1, π has a realization G_0 on the vertex set $V(G_0) = \{v_1, v_2, \ldots, v_n\}$ such that $d_{G_0}(v_i) = d_i$ for $i = 1, 2, \ldots, n$, the subgraph of G_0 induced by $\{v_1, v_2, v_3, v_4, v_5\}$ contains $K_5 - e$ as a subgraph, where $e = v_4v_5$, and v_1 is joined to $v_6, v_7, \ldots, v_{d_1+1}$. Let G_1' be the graph obtained from G_0 by deleting v_1. Then G_1' is a realization of π_1^*. By Remark 1.1, there exists a graph G_1 on the vertex set $V(G_1) = \{v_2, v_3, \ldots, v_n\}$ having the following properties. First, $d_{G_1}(v_i) = d_i - 1$ for $i = 2, 3, 4, 5$ and $d_{G_1}(v_i) = d_i^{(1)}$ for $i = 6, \ldots, n$. Additionally, the subgraph of G_1 induced by $\{v_2, v_3, v_4, v_5\}$ contains a $K_4 - e$ as a subgraph and $e = v_4v_5$. Finally, v_2 is joined to $v_6, v_7, \ldots, v_{d_2+1}$. Denote the graph obtained from G_1 by deleting v_2 by G_2'. Then G_2' is a realization of π_2^*. By Remark 1.1, π_2^* has a realization G_2 on the vertex set $V(G_2) = \{v_3, v_4, \ldots, v_n\}$ satisfying: (1) $d_{G_2}(v_i) = d_i - 2$ for $i = 3, 4, 5$ and $d_{G_2}(v_i) = d_i^{(2)}$ for $i = 6, \ldots, n$, (2) the subgraph of G_2 induced by $\{v_3, v_4, v_5\}$ contains a $K_3 - e$ as a subgraph, where $e = v_4v_5$, and (3) v_3 is joined to $v_6, v_7, \ldots, v_{d_3+1}$. Deleting the vertex v_3 from G_2, we get a realization of π_3^*. $\square$

Lemma 2.6. *Let $n \geqslant 9$ and $\pi = (d_1, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_1 \leqslant n-2, d_3 \geqslant 4$ and $d_5 \geqslant 3$. If the residual sequence $\pi_5' \neq (3^7, 1), (3^6, 1^2)$ and $d_3 > d_5$, then π is potentially $K_5 - e$-graphic.*

Proof. As $d_1 \leqslant n-2$ and $\pi_5' \neq (3^7, 1), (3^6, 1^2)$, there is a realization G' of π_5' containing a K_4 such that the degrees of vertices of K_4 in G' are $d_1', \ldots, d_4'$ by Theorem 1.5 and Theorem 2.4. Since $d_3 > d_5$, we have $\{d_1 - 1, d_2 - 1, d_3 - 1, \} \subseteq \{d_1', \ldots, d_4'\}$. Hence, π is potentially $K_5 - e$-graphic. $\square$

Lemma 2.7. *Let $n \geqslant 14$ and $\pi = (d_1, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $d_5 \geqslant 4$ and $n - 2 \geqslant d_1 \geqslant \ldots \geqslant d_5 = d_6 = \ldots = d_{d_1+2} \geqslant \ldots \geqslant d_n$. Then π is potentially A_5-graphic.*

Proof. Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ be a graphic sequence satisfying the conditions of the Lemma. Here, $|\pi|$ means the positive term number of π. By Theorem 1.1, we only need to verify that $\pi_5 = (d_6^{(5)}, d_7^{(5)}, \ldots, d_n^{(5)})$ is graphic. According to Theorem 1.6 and Theorem 1.7, it is sufficient to consider the following three cases:

Case 1. $d_1 \leqslant 6$ and $d_{10} \leqslant 2$. Then $d_1 = 4, 5$ or 6. We consider the following three subcases.

S u b c a s e 1.1. $d_1 = 4$. Then $d_5 = d_6 = 4$. We may assume that $\pi = (4^6, d_7, d_8, d_9, 2^x, 1^y)$ with $x + y \geqslant 5$ and even $\sigma(\pi)$. It is easy to compute that the corresponding π_5 is $(4, d_7, d_8, d_9, 2^x, 1^y)$. It follows from Lemma 2.3 that π_5 is graphic.

S u b c a s e 1.2. $d_1 = 5$. Then $d_5 = d_6 = d_7 \geqslant 4$.

If $d_5 = d_6 = d_7 = 4$, then we may assume that $\pi = (5, d_2, d_3, d_4, 4^3, d_8, d_9, 2^x, 1^y)$ with $x + y \geqslant 5$ and even $\sigma(\pi)$. Since $1 \leqslant \sum\limits_{i=1}^{5}(d_i - 4) \leqslant 4$, we have $d_6^{(5)} \leqslant 4$ and $|\pi_5| \geqslant 9$. So π_5 is graphic by Lemma 2.3.

If $d_5 = d_6 = d_7 = 5$, then we assume that $\pi = (5^7, d_8, d_9, 2^x, 1^y)$. Notice that $\sum\limits_{i=1}^{5}(d_i - 4) = 5$, we have $d_6^{(5)} \leqslant 4$ and $|\pi_5| \geqslant 9$. It follows from Lemma 2.3 that π_5 is graphic.

S u b c a s e 1.3. $d_1 = 6$. Then $d_5 = d_6 = d_7 = d_8 \geqslant 4$. The general form for π is $(6, d_2, \ldots, d_9, 2^x, 1^y)$ with $x + y \geqslant 5$ and even $\sigma(\pi)$.

If $d_5 = 4$, then $d_6^{(5)} \leqslant 4$ and $\sum\limits_{i=1}^{5}(d_i - 4) \leqslant 8$. Therefore, $|\pi_5| \geqslant 9$, and so π_5 is graphic by Lemma 2.3.

If $d_5 = 5$, then $6 \leqslant \sum\limits_{i=1}^{5}(d_i - 4) \leqslant 9$. Thus, $d_6^{(5)} \leqslant 4$ and $|\pi_5| \geqslant 9$. By Lemma 2.3, π_5 is graphic.

If $d_5 = 6$, then $\sum\limits_{i=1}^{5}(d_i - 4) = 10$ and $d_6 = d_7 = d_8 = 6$. Therefore, $d_6^{(5)} \leqslant 4$ and $|\pi_5| \geqslant 9$. It follows from Lemma 2.3 that π_5 is graphic.

C a s e 2. $d_2 \leqslant 5, d_1 \geqslant 7$ and $d_{10} \leqslant 2$.

Then $d_2 = 4$ or 5. Since $d_{10} \leqslant 2$, we have $d_1 = 7$. Thus $d_5 = d_6 = d_7 = d_8 = d_9 \geqslant 4$.

If $d_2 = 4$, then we may assume that $\pi = (7, 4^8, 2^x, 1^y)$ with $x+y \geqslant 5$ and even $\sigma(\pi)$. It is easy to compute that the corresponding π_5 is $(4, 3^3, 2^x, 1^y)$, which is graphic by Lemma 2.3.

If $d_2 = 5$ and $d_5 = 4$, then we may assume $\pi = (7, 5, d_3, d_4, 4^5, 2^x, 1^y)$ with $x+y \geqslant 5$ and even $\sigma(\pi)$. Since $\sum\limits_{i=1}^{5}(d_i - 4) \leqslant 6$, we have $d_6^{(5)} \leqslant 4$ and $|\pi_5| \geqslant 9$. It follows from Lemma 2.3 that π_5 is graphic.

If $d_2 = 5$ and $d_5 = 5$, then we assume $\pi = (7, 5^8, 2^x, 1^y)$ with $x + y \geqslant 5$ and even $\sigma(\pi)$. Since $\sum\limits_{i=1}^{5}(d_i - 4) = 7$ and $d_6 = d_7 = d_8 = d_9 = 5$, we have $d_6^{(5)} \leqslant 4$ and $|\pi_5| \geqslant 9$. Thus π_5 is graphic by Lemma 2.3.

C a s e 3. $d_3 = 4, d_1 \geqslant 7, d_2 \geqslant 6$ and $d_{10} \leqslant 2$. Then $d_1 = 7$ and $d_2 = 6$ or 7. The general form for π is either $(7, 6, 4^7, 2^x, 1^y)$ or $(7^2, 4^7, 2^x, 1^y)$ with $x + y \geqslant 5$ and even $\sigma(\pi)$. It is easy to compute that the corresponding π_5 is $(3^3, 2, 2^x, 1^y)$ or $(3^2, 2^2, 2^x, 1^y)$. From Lemma 2.2, both of them are graphic. $\square$

Lemma 2.8. *Let $n \geqslant 18$ and $\pi = (d_1, \ldots, d_n) \in \mathrm{NS}_n$ be a positive graphic sequence with $n-2 \geqslant d_1 \geqslant \ldots \geqslant d_6 = d_7 = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$ and $d_6 \geqslant 5$. Then π is potentially A_6-graphic.*

Proof. Let $\pi = (d_1, d_2, \ldots, d_n) \in \mathrm{NS}_n$ be a graphic sequence satisfying the conditions of the Lemma. By Theorem 1.1, it is sufficient to show that $\pi_6 = (d_7^{(6)}, d_8^{(6)}, \ldots, d_n^{(6)})$ is graphic. According to Theorem 1.6 and Theorem 1.7, we only need to consider the following four cases:

Case 1. $d_1 \leqslant 8$ and $d_{12} \leqslant 3$. Then the general form for π is $(d_1, d_2, \ldots, d_{11}, 3^x, 2^y, 1^z)$ with $x+y+z \geqslant 7$ and even $\sigma(\pi)$. Consider the following four subcases.

Subcase 1.1. $d_1 = 5$. Then $d_6 = d_7 = 5$. We may assume that $\pi = (5^7, d_8, d_9, d_{10}, d_{11}, 3^x, 2^y, 1^z)$. It is easy to compute that π_6 is $(5, d_8, \ldots, d_{11}, 3^x, 2^y, 1^z)$. By Lemma 2.3, π_6 is graphic.

Subcase 1.2. $d_1 = 6$. Then $d_6 = d_7 = d_8 \geqslant 5$.

If $d_6 = d_7 = d_8 = 5$, then $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$ by $1 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 5$. Thus by Lemma 2.3, π_6 is graphic.

If $d_6 = d_7 = d_8 = 6$, then $\pi = (6^8, d_9, d_{10}, d_{11}, 3^x, 2^y, 1^z)$. Since $\sum\limits_{i=1}^{6}(d_i - 5) = 6$, we have $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$. Therefore, π_6 is graphic from Lemma 2.3.

Subcase 1.3. $d_1 = 7$. Then $d_6 = d_7 = d_8 = d_9 \geqslant 5$.

If $d_6 = 5$, then $\sum\limits_{i=1}^{6}(d_i - 5) \leqslant 10$. Thus $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 5$. It follows from Lemma 2.3 that π_6 is graphic.

If $d_6 = 6$, then $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$ by $7 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 11$. Therefore, π_6 is graphic by Lemma 2.3.

If $d_6 = 7$, then we assume that $\pi = (7^9, d_{10}, d_{11}, 3^x, 2^y, 1^z)$. Since $\sum\limits_{i=1}^{6}(d_i - 5) = 12$, we know that $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$. By Lemma 2.3, π_6 is graphic.

Subcase 1.4. $d_1 = 8$. Then $d_6 = d_7 = d_8 = d_9 = d_{10} \geqslant 5$.

If $d_6 = 5$, then $|\pi_6| \geqslant 12$ by $\sum\limits_{i=1}^{6}(d_i - 5) \leqslant 15$. Thus π_6 is graphic by Lemma 2.3.

If $d_6 = 6$, then $8 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 16$. Hence $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$, and so π_6 is graphic by Lemma 2.3.

If $d_6 = 7$, then $13 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 17$. Therefore, $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$. By Lemma 2.3, π_6 is graphic.

If $d_6 = 8$, then $d_7^{(6)} \leqslant 5$ and $|\pi_6| \geqslant 12$ by $\sum\limits_{i=1}^{6}(d_i - 5) = 18$. It follows from Lemma 2.3 that π_6 is graphic.

Case 2. $d_2 \leqslant 7$, $d_1 \geqslant 9$ and $d_{12} \leqslant 3$. Then $d_1 = 9$ and $d_6 = d_7 = d_8 = d_9 = d_{10} = d_{11} \geqslant 5$. The general form for π is $(9, d_2, \ldots, d_{11}, 3^x, 2^y, 1^z)$ with $x+y+z \geqslant 7$ and even $\sigma(\pi)$. Consider the following three subcases.

Subcase 2.1. $d_2 = 5$. Then $d_6 = d_7 = \ldots = d_{11} = 5$ and $\pi = (9, 5^{10}, 3^x, 2^y, 1^z)$. The corresponding sequence π_6 is $(5, 4^4, 3^x, 2^y, 1^z)$, which is graphic by Lemma 2.3.

Subcase 2.2. $d_2 = 6$. Then $d_6 = d_7 = \ldots = d_{11} = 5$ or 6.

If $d_6 = 5$, then $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 5$ by $5 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 8$. From Lemma 2.3, π_6 is graphic.

If $d_6 = 6$, then $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 5$ by $\sum\limits_{i=1}^{6}(d_i - 5) = 9$. Therefore, π_6 is graphic by Lemma 2.3.

Subcase 2.3. $d_2 = 7$. Then $d_6 = d_7 = \ldots = d_{11} = 5, 6$ or 7.

If $d_6 = 5$, then $6 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 12$. Therefore, $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 5$. By Lemma 2.3, π_6 is graphic.

If $d_6 = 6$, then $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 4$ by $10 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 12$. It follows from Lemma 2.3 that π_6 is graphic.

If $d_6 = 7$, then $\pi = (9, 7^{10}, 3^x, 2^y, 1^z)$. The corresponding sequence is $\pi_6 = (5, 4^4, 3^x, 2^y, 1^z)$, which is graphic by Lemma 2.3.

Case 3. $d_3 \leqslant 6, d_2 \geqslant 8, d_1 \geqslant 9$ and $d_{12} \leqslant 3$. Then $d_1 = 9$ and $d_6 = \ldots = d_{11} \geqslant 5$. We may assume that $\pi = (9, d_2, \ldots, d_{11}, 3^x, 2^y, 1^z)$ with $x+y+z \geqslant 7$ and even $\sigma(\pi)$.

If $d_6 = 5$, then $|\pi_6| \geqslant 12$ by $\sum\limits_{i=1}^{6}(d_i - 5) \leqslant 11$. By Lemma 2.3, π_6 is graphic.

If $d_6 = 6$, then $11 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 12$. Therefore, $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 4$. From Lemma 2.3, π_6 is graphic.

Case 4. $d_4 = 5, d_3 \geqslant 7, d_2 \geqslant 8, d_1 \geqslant 9$ and $d_{12} \leqslant 3$. Then $d_1 = 9$ and $d_5 = d_6 = \ldots = d_{11} = 5$. Since $9 \leqslant \sum\limits_{i=1}^{6}(d_i - 5) \leqslant 12$, we have $|\pi_6| \geqslant 12$ and $d_7^{(6)} \leqslant 4$. It follows Lemma 2.3 that π_6 is graphic. □

3. Proofs of Theorems

Proof of Theorem 1.11. Assume that π is one of the following sequences:
$(n-1, 4^6, 1^{n-7})$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^3, 1^{n-6})$;
$n = 6$: (4^6), $(4^4, 3^2)$, $(4^3, 3^2, 2)$;
$n = 7$: $(4^3, 3^4)$, $(5^2, 4, 3^4)$, (4^7), $(4^5, 3^2)$, $(5, 4^3, 3^3)$, $(5^2, 4^5)$, $(5, 4^5, 3)$, $(4^3, 3^2, 2^2)$, $(4^4, 3^2, 2)$, $(5, 4^2, 3^3, 2)$, $(4^6, 2)$, $(4^3, 3^3, 1)$;

$n=8$: (5^8), (4^8), $(5^2,4^6)$, $(6,4^7)$, $(4^4,3^4)$, $(5,4^2,3^5)$, $(4^6,3^2)$, $(5,4^6,3)$, $(4^3,3^4,2)$, $(4^7,2)$, $(4^4,3^3,1)$, $(5,4^2,3^4,1)$, $(4^3,3^3,2,1)$, $(4^6,3,1)$, $(5,4^6,1)$;

$n=9$: (4^9), $(4^3,3^5,1)$, $(4^8,2)$, $(4^7,3,1)$, $(5,4^7,1)$, $(4^3,3^4,1^2)$, $(4^7,1^2)$;

$n=10$: $(4^8,1^2)$.

Then, it is easy to compute that the corresponding π_3^* of π is one of the following sequences: $(1^2,3^2,0^{n-7})$, $(0^2,2^2,0^{n-7})$, $(0^2,2,0^{n-6})$, $(1^2,4)$, $(1,0,3)$, $(0^2,2)$, $(0^2,3^2)$, $(0^2,2^2)$, $(1^2,4^2)$, $(1^2,3^2)$, $(1,0,3,2)$, $(1^2,4,2)$, $(0^2,3,1)$, $(2^2,4^3)$, $(1^2,4^3)$, $(1^2,4,3^2)$, $(1,0,3^3)$, $(0^2,3^2,2)$, $(1^2,4^2,2)$, $(1,0,3^2,1)$, $(0^2,3,2,1)$, $(1^2,4,3,1)$, $(1^2,4^4)$, $(0^2,3^3,1)$, $(1^2,4^3,2)$, $(1^2,4^2,3,1)$, $(0^2,3^2,1^2)$, $(1^2,4^2,1^2)$, $(1^2,4^3,1^2)$. It is easy to check that all of the above sequences are not graphic. By Lemma 2.5, π is not potentially K_5-e-graphic. Now, we show the sufficient condition.

If $d_1=n-1$, then π is potentially K_5-e-graphic by Lemma 2.4. If $n=5$, then π is either $(4^3,3^2)$ or (4^5), and it is easy to see that they both have realizations containing K_5-e. Assume that $d_1\leqslant n-2$ and $n\geqslant 6$. According to Lemma 2.5, it is enough to prove that π_3^* is graphic. We consider the following cases:

Case 1. $n=6$. Then $d_1=d_2=d_3=4$. As $\pi\neq(4^6),(4^4,3^2),(4^3,3^2,2)$, π must be either $(4^5,2)$ or $(4^4,3,1)$, each of which is potentially K_5-e-graphic.

Case 2. $n=7$. Then $d_1\leqslant 5$. We consider the following two subcases.

Subcase 2.1. $d_1=4$. Then $d_1=d_2=d_3=4$. If $\pi_1'=(3^6)$ or $(3^2,2^4)$, then $\pi=(4^5,3^2)$ or $(4^3,3^2,2^2)$, which is impossible. Since π_1' has six positive terms, $\pi_1'\neq(3^2,2^3)$. By Lemma 2.4, we may assume that $d_5=d_6\geqslant 3$. Notice that $d_4+d_5+d_6+d_7$ is even. If $d_5=d_6=3$, then (d_4,d_7) is one of the following: $(4,2),(3^2),(3,1)$; if $d_5=d_6=4$, then (d_4,d_7) is either $(4,2)$ or (4^2). Thus π is one of the following sequences:

$$(4^4,3^2,2),(4^3,3^4),(4^3,3^3,1),(4^6,2),(4^7)$$

which is impossible.

Subcase 2.2. $d_1=5$. If $\pi_1'=(3^2,2^3)$, then the residual sequence π_1' must contain 1 as a term. Therefore, $\pi_1'\neq(3^2,2^3)$. If $\pi_1'=(3^6)$ or $(3^2,2^4)$, then π is either $(5,4^5,3)$ or $(5,4^2,3^3,2)$, which is impossible. By Lemma 2.4, we may assume that $d_5=d_6=d_7\geqslant 3$. Since $\sigma(\pi)$ is even, we have $d_5\neq 5$.

If $d_5=d_6=d_7=3$, then $d_2+d_3+d_4$ is even. Thus $(d_2,d_3,d_4)=(4^3)$ or $(5^2,4)$ or $(5,4,3)$. If $d_5=d_6=d_7=4$, then (d_2,d_3,d_4) is either $(5,4^2)$ or (5^3) by $d_2+d_3+d_4$ being odd. As $\pi\neq(5^2,4^5),(5,4^3,3^3),(5^2,4,3^4)$, π is either $(5^4,4^3)$ or $(5^3,4,3^3)$. The corresponding π_3^* is $(2,1,3,2)$ or $(1,0,2,1)$, which are both graphic. Hence π is potentially K_5-e-graphic from Lemma 2.5.

Case 3. $n=8$. Then $d_1\leqslant 6$. We consider the following three subcases.

S u b c a s e 3.1. $d_1 = 4$. Then $d_1 = d_2 = d_3 = 4$. As $d_8 \geqslant 1$ and $d_5 \geqslant 3$, the residual sequence $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$. According to Lemma 2.4, we may assume that $d_5 = d_6 \geqslant 3$. Consider the residual sequence $\pi_5' = (d_1', d_2', \ldots, d_{n-1}')$.

If $d_5 = 3$ and $\pi_5' \neq (4, 3^6), (4, 3^5, 1), (3^6, 2), (3^5, 2, 1)$, then there is a realization G' of π_5' containing a K_4 such that the degrees of vertices of K_4 in G' are d_1', d_2', d_3', d_4' by Theorem 1.5 and Theorem 2.4. Therefore, π is potentially $K_5 - e$-graphic from $\{d_1 - 1, d_2 - 1, d_3 - 1\} \subseteq \{d_1', d_2', d_3', d_4'\}$. If π_5' is one of the following sequences: $(4, 3^6), (4, 3^5, 1), (3^6, 2), (3^5, 2, 1)$, then π must be one of the following sequences:$(4^4, 3^4), (4^4, 3^3, 1), (4^3, 3^4, 2), (4^3, 3^3, 2, 1)$, which is impossible.

Assume that $d_5 = 4$. Then $d_1 = \ldots = d_6 = 4$. If $\pi_5' \neq (4, 3^6), (4, 3^5, 1)$, then π_5' is potentially A_4-graphic by Theorem 1.5 and Theorem 2.4. If $d_7 \leqslant 3$, then π is potentially $K_5 - e$-graphic by $\{d_1 - 1, d_2 - 1, d_3 - 1\} \subseteq \{d_1', d_2', d_3', d_4'\}$. If $d_7 = 4$, then π is either $(4^7, 2)$ or (4^8), which is impossible. If $\pi_5' = (4, 3^6)$ or $(4, 3^5, 1)$, then $\pi = (4^6, 3^2)$ or $(4^6, 3, 1)$, which is also impossible.

S u b c a s e 3.2. $d_1 = 5$. Then π_1' has at most seven positive terms. If π_1' has at most six positive terms, then it must contain 1 as a term. Thus, $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$. By Lemma 2.4, we assume that $d_5 = d_6 = d_7 \geqslant 3$. Consider the residual sequence π_5'.

If $d_5 = d_6 = d_7 = 3$, then $d_1 - 1, d_2 - 1, d_3 - 1, d_4$ are the four largest degrees in π_5'. If $\pi_5' \neq (4, 3^6), (4, 3^5, 1)$, then π_5' is potentially A_4-graphic by Theorem 1.5 and Theorem 2.4. Thus π is potentially $K_5 - e$-graphic. If $\pi_5' = (4, 3^6)$ or $(4, 3^5, 1)$, then π is either $(5, 4^2, 3^5)$ or $(5, 4^2, 3^4, 1)$, which is impossible.

If $d_5 = d_6 = d_7 = 4$ and $\pi_5' \neq (4^7)$, then π_5' is potentially A_4-graphic by Theorem 1.5 and Theorem 2.4. If $d_3 \geqslant 5$, then π is potentially $K_5 - e$-graphic by $\{d_1 - 1, d_2 - 1, d_3 - 1\} \subseteq \{d_1', d_2', d_3', d_4'\}$. If $d_3 = 4$, then $\pi = (5^2, 4^5, 2)$ since $\pi \neq (5, 4^6, 1), (5, 4^6, 3), (5^2, 4^6)$. The corresponding π_3^* is graphic sequence $(1^2, 3^2, 2)$. If $d_5 = d_6 = d_7 = 4$ and $\pi_5' = (4^7)$, then $\pi = (5^4, 4^4)$. The corresponding sequence $\pi_3^* = (2, 1, 3^3)$, which is graphic.

If $d_5 = d_6 = d_7 = 5$, then $\pi = (5^7, 1)$ or $(5^7, 3)$ by $\pi \neq (5^8)$. The corresponding π_3^* is $(2^2, 4, 3, 1)$ or $(2^2, 4, 3^2)$, which are both graphic.

S u b c a s e 3.3. $d_1 = 6$. Then the residual sequence π_1' has at most seven positive terms. If π_1' has at most six positive terms, then it should contain 1 as a term. Therefore, $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$. We may assume that $d_5 = d_6 = d_7 = d_8 \geqslant 3$ by Lemma 2.4. Consider the residual sequence $\pi_5' = (d_1', d_2', \ldots, d_{n-1}')$.

If $d_5 = d_6 = d_7 = d_8 = 3$, then $d_1 - 1, d_2 - 1, d_3 - 1, d_4$ are the four largest degrees in π_5'. Since $d_1 - 1 = 5$, π_5' is potentially A_4-graphic by Theorem 1.5 and Theorem 2.4. Therefore, π is potentially $K_5 - e$-graphic.

If $d_5 = d_6 = d_7 = d_8 = 4$ and $d_4 \geqslant 5$, then π is potentially $K_5 - e$-graphic by $\{d_1 - 1, d_2 - 1, d_3 - 1, d_4 - 1\} = \{d'_1, d'_2, d'_3, d'_4\}$ and Theorem 1.5. If $d_4 = d_5 = d_6 = d_7 = d_8 = 4$, then $\pi = (6, 5^2, 4^5)$ or $(6^3, 4^5)$ since $\pi \neq (6, 4^7)$. It is easy to see that $(6, 5^2, 4^5)$ and $(6^3, 4^5)$ are both potentially $K_5 - e$-graphic.

If $d_5 = d_6 = d_7 = d_8 = 5$, then (d_2, d_3, d_4) is either (6^3) or $(6, 5^2)$ since $d_2+d_3+d_4$ is even. That is, $\pi = (6^4, 5^4)$ or $(6^2, 5^6)$. The corresponding π_3^* is $(3, 2, 3^3)$ or $(2^2, 4, 3^2)$, which are both graphic.

If $d_5 = d_6 = d_7 = d_8 = 6$, then $\pi = (6^8)$ and π_3^* is graphic sequence $(3^2, 4^3)$.

Case 4. $n = 9$. Then the residual sequence π'_1 has at most eight positive terms. If π'_1 has at most seven positive terms, then it must contain 1 as a term. Therefore $\pi'_1 \neq (3^6), (3^2, 2^4), (3^2, 2^3)$. Assume that $d_5 = d_6 = \ldots = d_{d_1+2} \geqslant 3$ by Lemma 2.4. We consider the following four subcases.

Subcase 4.1. $d_1 = 4$. Then $d_5 = d_6 \geqslant 3$. Consider the residual sequence π'_5.

If $d_5 = d_6 = 3$ and $\pi'_5 \neq (3^7, 1), (3^6, 1^2)$, then π is potentially $K_5 - e$-graphic according to Lemma 2.6. If $d_5 = d_6 = 3$ and $\pi'_5 = (3^7, 1)$ or $(3^6, 1^2)$, then $\pi = (4^3, 3^5, 1)$ or $(4^3, 3^4, 1^2)$, which is impossible.

If $d_5 = d_6 = 4$, then $\pi'_5 \neq (3^7, 1), (3^6, 1^2)$. Thus there is a realization G of π'_5 containing a K_4 such that the degrees of vertices of K_4 in G are d'_1, d'_2, d'_3, d'_4 by Theorem 1.5 and Theorem 2.4. If $d_7 \leqslant 3$, then π is potentially $K_5 - e$-graphic by $\{d_6, d_1 - 1, d_2 - 1, d_3 - 1\} = \{d'_1, d'_2, d'_3, d'_4\}$. If $d_7 = 4$, then $d_8 + d_9$ is even, and (d_8, d_9) is one of the following: $(1^2), (2^2), (3^2), (4^2), (3, 1), (4, 2)$. Therefore, $\pi = (4^7, 2^2)$ or $(4^7, 3^2)$ by $\pi \neq (4^7, 3, 1), (4^8, 2), (4^9), (4^7, 1^2)$. The corresponding $\pi_3^* = (1^2, 4^2, 2^2)$ or $(1^2, 4^2, 3^2)$, which are both graphic.

Subcase 4.2. $d_1 = 5$. Then $d_5 = d_6 = d_7 \geqslant 3$ and the residual sequence $\pi'_5 \neq (3^7, 1), (3^6, 1^2)$.

If $d_5 = d_6 = d_7 = 3$, then π is potentially $K_5 - e$-graphic from Lemma 2.6.

If $d_5 = d_6 = d_7 = 4$ and $d_3 = 5$, then π is potentially $K_5 - e$-graphic by Lemma 2.6. If $d_2 = d_3 = 4$, then $d_4 = 4$ and $d_8 + d_9$ is odd. Therefore (d_8, d_9) is $(2, 1)$ or $(3, 2)$ or $(4, 1)$ or $(4, 3)$. Since $\pi \neq (5, 4^7, 1)$, $\pi = (5, 4^6, 2, 1)$ or $(5, 4^6, 3, 2)$ or $(5, 4^7, 3)$. The corresponding π_3^* is one of the following graphic sequences:

$$(1^2, 4, 3, 2, 1), (1^2, 4, 3^2, 2), (1^2, 4^2, 3^2).$$

In this case, if $d_3 = 4$ and $d_2 = 5$, then $d_8 + d_9$ is even, and (d_8, d_9) is one of the following:

$$(1^2), (2^2), (3^2), (4^2), (3, 1), (4, 2).$$

Therefore, π must be one of the following sequences:

$$(5^2, 4^5, 1^2), (5^2, 4^5, 2^2), (5^2, 4^5, 3^2), (5^2, 4^7), (5^2, 4^5, 3, 1), (5^2, 4^6, 2)$$

and the corresponding π_3^* is one of the following graphic sequences:

$$(1^2, 3^2, 1^2), (1^2, 3^2, 2^2), (1^2, 3^4), (1^2, 4^2, 3^2), (1^2, 3^3, 1), (1^2, 4, 3^2, 2).$$

If $d_5 = d_6 = d_7 = 5$, then π is one of the following sequences:

$$(5^7, 2, 1), (5^7, 4, 1), (5^7, 3, 2), (5^8, 2), (5^8, 4), (5^7, 4, 3)$$

and it is easy to compute that the corresponding π_3^* is one of the following graphic sequences:

$$(2^2, 4, 3, 2, 1), (2^2, 4^2, 3, 1), (2^2, 4, 3^2, 2), (2^2, 4^3, 2), (2^2, 4^4), (2^2, 4^2, 3^2).$$

Subcase 4.3. $d_1 = 6$. Then $d_5 = d_6 = d_7 = d_8 \geqslant 3$ and $\pi_5' \neq (3^7, 1), (3^6, 1^2)$.

If $d_5 = d_6 = d_7 = d_8 = 3$, then π is potentially $K_5 - e$-graphic by Lemma 2.6.

If $d_5 = d_6 = d_7 = d_8 = 4$ and $d_3 \geqslant 5$, then π is potentially $K_5 - e$-graphic by Lemma 2.6. If $d_3 = 4$, then π is one of the following sequences:

$$(6^2, 4^6, 2), (6^2, 4^7), (6, 5, 4^6, 3), (6, 5, 4^6, 1), (6, 4^7, 2), (6, 4^8)$$

and it is easy to compute that the corresponding π_3^* is one of the following graphic sequences:

$$(1^2, 3^2, 2^2), (1^2, 3^4), (1^2, 3^3, 1), (1^2, 4, 3^2, 2), (1^2, 4^2, 3^2).$$

If $d_5 = d_6 = d_7 = d_8 = 5$ and $d_3 = 6$, then π is potentially $K_5 - e$-graphic by Lemma 2.6. If $d_3 = d_5 = d_6 = d_7 = d_8 = 5$, then π is one of the following sequences:

$$(6^2, 5^6, 2), (6^2, 5^6, 4), (6, 5^7, 1), (6, 5^7, 3), (6, 5^8)$$

and the corresponding π_3^* is one of the following graphic sequences:

$$(2^2, 4, 3^2, 2), (2^2, 4^2, 3^2), (2^2, 4^2, 3, 1), (2^2, 4^4).$$

If $d_5 = d_6 = d_7 = d_8 = 6$, then π is $(6^8, 2)$ or $(6^8, 4)$ or (6^9). The corresponding π_3^* are $(3^2, 4^3, 2), (3^2, 4^4)$ and $(3^2, 5^2, 4^2)$, respectively, all of which are graphic.

Subcase 4.4. $d_1 = 7$. Then $d_5 = d_6 = d_7 = d_8 = d_9 \geqslant 3$.

If $d_5 = d_6 = d_7 = d_8 = d_9 = 3$, then π is potentially $K_5 - e$-graphic by Lemma 2.6.

If $d_5 = d_6 = d_7 = d_8 = d_9 = 4$ and $d_3 \geqslant 5$, then π is potentially $K_5 - e$-graphic by Lemma 2.6. If $d_3 = d_5 = d_6 = d_7 = d_8 = d_9 = 4$, then $\pi = (7, 5, 4^7)$ or $(7^2, 4^7)$. The corresponding $\pi_3^* = (1^2, 3^4)$ or $(1^2, 3^2, 2^2)$, both of which are graphic.

If $d_5 = d_6 = d_7 = d_8 = d_9 = 5$ and $d_3 \geqslant 6$, then π is potentially $K_5 - e$-graphic by Lemma 2.6. If $d_3 = d_5 = d_6 = d_7 = d_8 = d_9 = 5$, then $\pi = (7, 6, 5^7)$. The corresponding sequence π_3^* is $(2^2, 4^2, 3^2)$, which is graphic.

If $d_5 = d_6 = d_7 = d_8 = d_9 = 6$ and $d_3 \geqslant 7$, then π is potentially $K_5 - e$-graphic by Lemma 2.6. If $d_3 = d_5 = d_6 = d_7 = d_8 = d_9 = 6$, then $\pi = (7^2, 6^7)$, $\pi_3^* = (3^2, 4^4)$ is graphic.

Case 5. $n = 10$. Then $d_1 \leqslant 8$. The residual sequence π_1' has at most nine positive terms. If π_1' has at most eight positive terms, then it must contains 1 as a term. Therefore, $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$. We may assume that $d_5 = d_6 = \ldots = d_{d_1+2} \geqslant 3$ by Lemma 2.4. We consider the following two subcases.

Subcase 5.1. $d_3' \geqslant 4$ in the residual sequence π_{10}'.

If $\pi_{10}' \neq (4^9)$, $(4^3, 3^5, 1)$, $(4^8, 2)$, $(4^7, 3, 1)$, $(5, 4^7, 1)$, $(4^3, 3^4, 1^2)$, $(4^7, 1^2)$, then π_{10}' is potentially $K_5 - e$-graphic by Case 4, and so is π.

If $\pi_{10}' = (4^9)$, then $d_{10} \leqslant 4$. Thus π is one of the following sequences:

$$(5, 4^8, 1), (5^2, 4^7, 2), (5^3, 4^6, 3), (5^4, 4^6)$$

and it is easy to compute that the corresponding π_3^* is one of the following graphic sequences:

$$(1^2, 4^3, 3, 1), (1^2, 4^2, 3^2, 2), (1^2, 4, 3^4), (2, 1, 4^2, 3^3).$$

If $\pi_{10}' = (4^8, 2)$, then $d_{10} \leqslant 2$. Therefore π is either $(5, 4^7, 2, 1)$ or $(5^2, 4^6, 2^2)$. The corresponding sequence π_3^* is $(1^2, 4^2, 3, 2, 1)$ or $(1^2, 4, 3^2, 2^2)$, both of which are graphic.

If $\pi_{10}' = (4^3, 3^5, 1)$, then $d_{10} = 1$. Hence, $\pi = (5, 4^2, 3^5, 1^2)$ or $(4^4, 3^4, 1^2)$. The corresponding $\pi_3^* = (0^2, 3^2, 2, 1^2)$ or $(1, 0, 3^3, 1^2)$, which are both graphic.

If $\pi_{10}' = (4^7, 3, 1)$, then $d_{10} = 1$. Since $\pi \neq (4^8, 1^2)$, $\pi = (5, 4^6, 3, 1^2)$. The sequence $\pi_3^* = (1^2, 4, 3^2, 1^2)$, which is graphic.

If $\pi_{10}' = (5, 4^7, 1)$, then $d_{10} = 1$. Thus $\pi = (6, 4^7, 1^2)$ or $\pi = (5^2, 4^6, 1^2)$. The sequences π_3^* are both $(1^2, 4, 3^2, 1^2)$, which is graphic.

If $\pi_{10}' = (4^3, 3^4, 1^2)$, then $d_{10} = 1$. Therefore, $\pi = (5, 4^2, 3^4, 1^3)$ or $\pi = (4^4, 3^3, 1^3)$. The corresponding sequence π_3^* is $(0^2, 3, 2, 1^3)$ or $(1, 0, 3^2, 1^3)$, which are both graphic.

If $\pi_{10}' = (4^7, 1^2)$, then $\pi = (5, 4^6, 1^3)$ by $d_{10} = 1$. The sequence $\pi_3^* = (1^2, 4, 3, 1^3)$, which is graphic.

Subcase 5.2. $d_3' \leqslant 3$ in the residual sequence π_{10}'. Then $d_3' = d_4' = d_5' = 3$ by $d_5' \geqslant 3$. Since $d_3' = 3$, we have $d_{10} \leqslant 3$ and $d_5 = d_6 = 3$. It follows from Lemma 2.6 that π is potentially $K_5 - e$-graphic.

Case 6. $n \geqslant 11$. Then $\pi_1' \neq (3^6), (3^2, 2^4), (3^2, 2^3)$. Otherwise, each of the three sequences should contain 1 as a term, which is a contradiction. Assume that $d_5 =$

$d_6 = \ldots = d_{d_1+2} \geqslant 3$. Consider the residual sequence π'_n. Obviously, $d'_5 \geqslant 3$ in π'_n. We use induction on n to prove this case. We first prove the case $n = 11$.

If $d'_3 \geqslant 4$ in the residual sequence π'_{11} and $\pi'_{11} \neq (4^8, 1^2)$, then π'_{11} is potentially $K_5 - e$-graphic by Case 5 and so is π. If $\pi'_{11} = (4^8, 1^2)$, then $\pi = (5, 4^7, 1^3), \pi^*_3 = (1^2, 4^2, 3, 1^3)$, which is graphic.

If $d'_3 = 3$ in π'_{11}, then $d_5 = 3$. From Lemma 2.6, π is potentially $K_5 - e$-graphic.

Now we assume that for $n - 1 \geqslant 11$ the result is true. If $d'_3 \geqslant 4$ in the residual sequence π'_n, then π'_n is potentially $K_5 - e$-graphic by the induction hypothesis, and so is π. If $d'_3 = 3$ in π'_n, then $d_5 = 3$. We consider the residual sequence π'_5. According to Lemma 2.6, π is potentially $K_5 - e$-graphic. □

Proof of Theorem 1.12. If $d_1 = n - 1$ or there exists an integer $t, 5 \leqslant t \leqslant d_1 + 1$ such that $d_t > d_{t+1}$, then π is potentially A_5-graphic if and only if $\pi_5 \notin S$ by Theorem 1.5 and Theorem 2.4. If $n - 2 \geqslant d_1 \geqslant \ldots \geqslant d_4 \geqslant d_5 = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$, then π is potentially A_5-graphic by Lemma 2.7. Therefore, π is potentially A_5-graphic if and only if $\pi_5 \notin S$. □

Proof of Theorem 1.13. If $d_1 = n - 1$ or there exists an integer t, $6 \leqslant t \leqslant d_1 + 1$ such that $d_t > d_{t+1}$, then π is potentially A_6-graphic if and only if $\pi_6 \notin S$ from Theorem 1.12 and Theorem 2.4. If $n - 2 \geqslant d_1 \geqslant \ldots \geqslant d_5 \geqslant d_6 = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$, then π is potentially A_6-graphic by Lemma 2.8. Hence π is potentially A_6-graphic if and only if $\pi_6 \notin S$. □

Proof of Theorem 1.14. If π is potentially A_5-graphic, then it is obvious that $(d_1 - 4, d_2 - 4, \ldots, d_5 - 4, d_6, \ldots, d_n)$ is graphic. If π is $(n - a, n - b, 4^4, 2^{n-(a+b+4)}, 1^{a+b-2})$ or $(n-a, n-b, 4^5, 2^{n-(a+b+5)}, 1^{a+b-2})$, then the corresponding π_5 is $(2, 0^{n-6})$ or $(2^2, 0^{n-7})$, neither of which is graphic. Thus π is not potentially A_5-graphic by Theorem 1.1. Now we verify the sufficient condition. According to Theorem 1.6 and Theorem 1.7, we only need to consider the following three cases:

Case 1. $d_1 \leqslant 6$ and $d_{10} \leqslant 2$. Let G be a realization of the sequence $(d_1 - 4, d_2 - 4, \ldots, d_5 - 4, d_6, \ldots, d_n)$ with $V(G) = \{v_1, \ldots, v_n\}$, $d(v_i) = d_i - 4$ for $i = 1, \ldots, 5$ and $d(v_i) = d_i$ for $i = 6, \ldots, n$. Let $A = \{v_1, \ldots, v_5\}$ and $B = V(G) \setminus A$. Moreover, G minimizes the edge number $|E(G[A])|$ of the induced subgraph $G[A]$. If $|E(G[A])| = 0$, then π is potentially A_5-graphic. Otherwise, there exists at least one edge $e = uv$ in $G[A]$. Without loss of generality, we may assume that $d_G(u) \geqslant d_G(v)$. Then u and v are respectively adjacent to at most one vertex u'' and v'' of B. Since n is sufficiently large and π is positive graphic, we may find an edge $e' = u'v'$ with $u', v' \in B$ and $u', v' \neq u'', v''$. Since $d_1 \leqslant 6$, u and v are not adjacent to u' and v'. We may obtain another realization G' of $(d_1 - 4, d_2 - 4, \ldots, d_5 - 4, d_6, \ldots, d_n)$ by swapping the edges e and e' with the non-edges uu' and vv'. Clearly, $|E(G'[A])|$ is less than $|E(G[A])|$.

Case 2. $d_2 \leqslant 5, d_1 \geqslant 7$ and $d_{10} \leqslant 2$. If $d_2 = 4$, then π is potentially A_5-graphic since $(d_1-4, d_2-4, \ldots, d_5-4, d_6, \ldots, d_n)$ is graphic. If $d_2 = 5$ and $|E(G[A])| = 0$, then π is potentially A_5-graphic, where the definition of G is the same as that in Case 1. If $d_2 = 5$ and $|E(G[A])| \neq 0$, we assume that $e = uv$ in $G[A]$ and $d_G(u) \geqslant d_G(v)$. Then u is not adjacent to at least one vertex u' of B. Since π is positive graphic, there exists a vertex $v' \in N(u')$, where $N(u')$ is the neighbor set of the vertex u'. As the vertex v has degree at most one in G, v is not adjacent to u' and v'. Thus $G' = G - uv - u'v' + uu' + vv'$ is also a realization of $(d_1-4, d_2-4, \ldots, d_5-4, d_6, \ldots, d_n)$ with $|E(G'[A])| < |E(G[A])|$.

Case 3. $d_3 = 4, d_2 \geqslant 6, d_1 \geqslant 7$ and $d_{10} \leqslant 2$. Then we assume that $\pi = (d_1, d_2, 4^3, d_6, d_7, d_8, d_9, 2^x, 1^y)$ with $x + y = n - 9$. By Theorem 1.1, it is enough to prove that $\pi_5 = (d_6^{(5)}, d_7^{(5)}, \ldots, d_n^{(5)})$ is graphic. If $d_6 \leqslant 2$, then π_5 is graphic by Theorem 2.3. If $d_6 = 3$, then $(d_1 - 4) + (d_2 - 4) \geqslant 5$. Thus $d_6^{(5)} \leqslant 2$ and $h(\pi_5) = 1$, where $h(\pi_5)$ means the smallest positive term of π_5. It follows from Theorem 2.3 that π_5 is graphic. For $d_6 = 4$, we consider the following three subcases.

Subcase 3.1. $d_7 \leqslant 2$. Assume $\pi = (d_1, d_2, 4^4, 2^x, 1^y)$ with $x + y = n - 6$. Since π is graphic, we have $(d_1 - 4) + (d_2 - 4) \leqslant 2 + 2x + y$, that is, $d_1 + d_2 \leqslant n + 4 + x$.

If $d_1 + d_2 = n + 4 + x$, then $\pi_5 = (2, 0^{n-6})$, which is not graphic. Hence π is not potentially A_5-graphic. Let $d_1 = n - a$ and $d_2 = n - b$. Then $x = n - (a + b + 4)$ and $y = a + b - 2$. Since $x \geqslant 0$ and $d_2 \geqslant 6$, we have $a + b \leqslant n - 4$ and $b \leqslant n - 6$. That is, $\pi = (n - a, n - b, 4^4, 2^{n-(a+b+4)}, 1^{a+b-2})$, which is impossible.

If $d_1 + d_2 < n + 4 + x$, then $h(\pi_5) = 1$ and $d_6^{(5)} = 2$ by $\sum\limits_{i=1}^{5}(d_i - 4) \geqslant 5$. Thus π_5 is graphic by Theorem 2.3.

Subcase 3.2. $d_7 = 3$. Assume $\pi = (d_1, d_2, 4^4, 3, d_8, d_9, 2^x, 1^y)$ with $x + y = n - 9$. Since $(d_1 - 4) + (d_2 - 4) \geqslant 5$, we have $d_6^{(5)} = 2$. If $d_1 \geqslant 8$, then $h(\pi_5) = 1$ by $d_7 = 3$. Thus by Theorem 2.3, π_5 is graphic. If $d_1 = 7$ and $d_8 = 3$, then π_5 has at least three positive terms. If $d_1 = 7$ and $d_8 \leqslant 2$, then $h(\pi_5) = 1$. Therefore, π_5 is graphic by Lemma 2.2.

Subcase 3.3. $d_7 = 4$.

(1) If $d_8 \leqslant 2$, then we assume that $\pi = (d_1, d_2, 4^5, 2^x, 1^y)$ with $x + y = n - 7$. Since π is graphic, we know that $(d_1 - 4) + (d_2 - 4) \leqslant 2 + 2 + 2x + y = n - 3 + x$, that is, $d_1 + d_2 \leqslant n + 5 + x$.

If $d_1 + d_2 = n + 5 + x$, then $\pi_5 = (2^2, 0^{n-7})$, which is not graphic. Since $x \geqslant 0, d_2 \geqslant 6, x = n - (a + b + 5)$ and $y = a + b - 2$, we have $a + b \leqslant n - 5$ and $b \leqslant n - 6$. Therefore, $\pi = (n - a, n - b, 4^5, 2^{n-(a+b+5)}, 1^{a+b-2})$, which is a contradiction.

If $d_1+d_2 < n+5+x$, then $h(\pi_5)=1$. As $(d_1-4)+(d_2-4) \geqslant 5$, we have $d_6^{(5)}=2$. It follows from Theorem 2.3 that π_5 is graphic.

(2) If $d_8 \geqslant 3$ and $d_9 \leqslant 2$, then we assume that $\pi = (d_1, d_2, 4^5, d_8, 2^x, 1^y)$ with $x+y=n-8$.

If $(d_1-4)+(d_2-4) \geqslant 6$ and $d_2 \geqslant 7$, then $d_6^{(5)}=2$ and π_5 has at least three positive terms; if $(d_1-4)+(d_2-4) \geqslant 6, d_2=6$ and $d_8=4$, then $\pi_5=(3,2^2,2^{x'},1^{y'},0^{z'})$ with $x'+y'+z'=n-8$; if $(d_1-4)+(d_2-4) \geqslant 6$, $d_2=6$ and $d_8=3$, then $d_6^{(5)}=2$ and π_5 has at least three positive terms. By Lemma 2.2, π_5 is graphic.

If $(d_1-4)+(d_2-4)=5$, then $d_1=7$ and $d_2=6$. If $d_8=3$, then $\pi_5=(2^3,2^x,1^y)$. If $d_8=4$, then $\pi_5=(3,2^2,2^x,1^y)$. By Lemma 2.2, π_5 is graphic.

(3) If $d_8 \geqslant 3$ and $d_9 \geqslant 3$, then $\pi=(d_1,d_2,4^5,d_8,d_9,2^x,1^y)$ with $x+y=n-9$. Since $(d_1-4)+(d_2-4) \geqslant 5, \pi_5$ has at least four positive terms and $d_6^{(5)} \leqslant 3$. If π_5 has at least five positive terms, then π_5 is graphic by Lemma 2.2. If π_5 has exact four positive terms, then $d_6^{(5)}=2$, and π_5 is also graphic by Lemma 2.2. □

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Author's address: Meng-Xiao Yin, College of Computer and Electronics Information, Guangxi University, Nanning, Guangxi 530004, China; Jian-Hua Yin, Department of Applied Mathematics, College of Information Science and Technology, Hainan University, Haikou, Hainan 570228, China, e-mail: yinjh@ustc.edu.