Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 2, 689–696

Persistent URL: `http://dml.cz/dmlcz/128198`

Czechoslovak Mathematical Journal, 57 (132) (2007), 689–696

THE AP-DENJOY AND AP-HENSTOCK INTEGRALS

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(Received April 8, 2005)

Abstract. In this paper we define the ap-Denjoy integral and show that the ap-Denjoy integral is equivalent to the ap-Henstock integral and the integrals are equal.

Keywords: approximate Lusin function, ap-Denjoy integral, ap-Henstock integral, choice

MSC 2000: 26A39, 28B05

1. Introduction

For a measurable set E of real numbers we denote by $|E|$ its Lebesgue measure. Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_cE = \lim_{h\to 0^+} \frac{|E\cap(c-h,c+h)|}{2h}$$

provided the limit exists. The point c is called a *point of density* of E if $d_cE = 1$ and a *point of dispersion* of E if $d_cE = 0$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A function $F\colon [a,b] \to \mathbb{R}$ is said to be *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and $\lim\limits_{\substack{x\to c\\ x\in E}} \frac{F(x)-F(c)}{x-c}$ exists. The approximate derivative of F at c is denoted by $F'_{\rm ap}(c)$.

An *approximate neighborhood* (or ap-nbd) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x as a point of density. For every $x \in E \subseteq [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x\colon x \in E\}$ is a *choice* on E. A tagged interval $(x,[c,d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathscr{P} = \{(x_i,[c_i,d_i])\colon 1 \leqslant i \leqslant n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i,[c_i,d_i])$ is subordinate to a choice S for each i, then we say that $\mathscr{P}$ is subordinate to S. If $\mathscr{P}$ is subordinate to S and $[a,b]=\bigcup_{i=1}^{n}[c_i,d_i]$, then we say that $\mathscr{P}$ is a tagged partition of $[a,b]$ that is subordinate to S.

2. The ap-Denjoy integral

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

For a function $F\colon [a,b]\to\mathbb{R}$, F can be treated as a function of intervals by defining $F([c,d])=F(d)-F(c)$.

Definition 2.1. Let $F\colon [a,b]\to\mathbb{R}$ be a function. The function F is an *approximate Lusin function* (or F is an AL function) on $[a,b]$ if for every measurable set $E\subseteq[a,b]$ of measure zero and for every $\varepsilon>0$ there exists a choice S on E such that $|(\mathscr{P})\sum F(I)|<\varepsilon$ for every finite collection $\mathscr{P}$ of non-overlapping tagged intervals that is subordinate to S.

Recall that $F\colon [a,b]\to\mathbb{R}$ is AC_s on a measurable set $E\subseteq[a,b]$ if for each $\varepsilon>0$ there exist a positive number δ and a choice S on E such that $|(\mathscr{P})\sum F(I)|<\varepsilon$ for every finite collection $\mathscr{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(\mathscr{P})\sum|I|<\delta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

Lemma 2.2. *If $F\colon [a,b]\to\mathbb{R}$ is ACG_s on $[a,b]$, then F is an AL function on $[a,b]$.*

Proof. Suppose that $E\subseteq[a,b]$ is a measurable set of measure zero. Let $E=\bigcup_{n=1}^{\infty}E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. Let $\varepsilon>0$. For each positive integer n there exist a choice $S^n=\{S_x^n\colon x\in E_n\}$ on E_n and a positive number δ_n such that $|(\mathscr{P})\sum F(I)|<\varepsilon/2^n$ whenever $\mathscr{P}$ is subordinate to S^n and $(\mathscr{P})\sum|I|<\delta_n$. For each positive integer n, choose an open set O_n such that $E_n\subseteq O_n$ and $|O_n|<\delta_n$. Let $S_x=S_x^n\cap(x-\varrho(x,O_n^c),x+\varrho(x,O_n^c))$ for each $x\in E_n$, where $\varrho(x,O_n^c)$ is the distance from x to $O_n^c=[a,b]-O_n$. Then $S=\{S_x\colon x\in E\}$ is a choice on E. Suppose that $\mathscr{P}$ is subordinate to S. Let $\mathscr{P}_n$ be a subset of $\mathscr{P}$ that has tags in E_n and note that $(\mathscr{P}_n)\sum|I|<|O_n|<\delta_n$. Hence, we have

$$\Big|(\mathscr{P})\sum F(I)\Big|\leqslant\sum_{n=1}^{\infty}\Big|(\mathscr{P}_n)\sum F(I)\Big|<\sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}=\varepsilon.$$

□

Definition 2.3. A function $f\colon [a,b]\to\mathbb{R}$ is *ap-Denjoy integrable* on $[a,b]$ if there exists an AL function F on $[a,b]$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{\rm ap} = f$ almost everywhere on $[a,b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a,b]$.

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by $(\mathrm{AD})\int_a^x f$.

It is easy to show that if $f\colon [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$. This gives rise to an interval function F such that $F(I) = (\mathrm{AD})\int_I f$ for every subinterval $I \subseteq [a,b]$. The function F is called the primitive of f.

Recall that a function $F\colon [a,b] \to \mathbb{R}$ is AC_* on a measurable set $E \subseteq [a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $(\mathscr{P})\sum \omega(F,I) < \varepsilon$ for every finite collection $\mathscr{P}$ of non-overlapping intervals that have endpoints in E and satisfy $(\mathscr{P})\sum |I| < \delta$, where $\omega(F,I) = \sup\{|F(y)-F(x)|\colon x,y \in I\}$. A function F is ACG_* on E if $F|_E$ is continuous on E, $E = \bigcup\limits_{n=1}^{\infty} E_n$ and F is AC_* on each E_n. It is easy to show that if F is ACG_* on $[a,b]$, then F is ACG_s on $[a,b]$. A function $f\colon [a,b] \to \mathbb{R}$ is *Denjoy integrable* on $[a,b]$ if there exists an ACG_* function $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$.

The following theorem shows that the ap-Denjoy integral is an extension of the Denjoy integral.

Theorem 2.4. *If $f\colon [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on $[a,b]$.*

Proof. Suppose that $f\colon [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$. Then there exists an ACG_* function $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$. Since F is ACG_s on $[a,b]$, by Lemma 2.2 F is an AL function on $[a,b]$ and $F'_{\rm ap} = F' = f$ almost everywhere on $[a,b]$. Hence, f is ap-Denjoy integrable on $[a,b]$. □

There exists a function that is ap-Denjoy integrable on $[a,b]$, but not Denjoy integrable on $[a,b]$.

Example 2.5. Let $\{(a_n,b_n)\}$ be a sequence of disjoint open intervals in (a,b) with the following properties:

(1) $b_1 < b$ and $b_{n+1} < b_n$ for all n;
(2) $\{a_n\}$ converges to a;
(3) a is a point of dispersion of $O = \bigcup\limits_{n=1}^{\infty} (a_n,b_n)$.

Define $F\colon [a,b] \to \mathbb{R}$ by $F(x) = 0$ for all $x \in [a,b] - O$ and

$$F(x) = \sin^2\Big(\frac{x-a_n}{b_n-a_n}\pi\Big)$$

for $x \in (a_n, b_n)$. Then it is easy to show that the function F is differentiable on $(a, b]$ and approximately differentiable at a, but F is not continuous at a. Hence $F' = F_{\rm ap}$ almost everywhere on $[a, b]$, but $F'_{\rm ap}$ is not Denjoy integrable on $[a, b]$, since F is not continuous on $[a, b]$.

To show that $F'_{\rm ap}$ is ap-Denjoy integrable on $[a, b]$, it is sufficient to show that F is an AL function on $[a, b]$. Let E be a measurable set in $[a, b]$ of measure zero and let $\varepsilon > 0$. For each positive integer n, choose an open set O_n such that $E \cap [a_n, b_n] \subseteq O_n$ and $|O_n| < (b_n - a_n)\varepsilon/\pi 2^n$.

For each $x \in E$, define

$$S_x = \begin{cases} [a,b] - \bigcup\limits_{n=1}^{\infty}(a_n, b_n) & \text{if } x = a; \\ (b_{n+1}, a_n) & \text{if } b_{n+1} < x < a_n,\ \ n = 1,2,3,\ldots; \\ (x - \varrho(x, O_n^c), x + \varrho(x, O_n^c)) & \text{if } a_n \leqslant x \leqslant b_n,\ \ n = 1,2,3,\ldots. \end{cases}$$

Then $S = \{S_x \colon x \in E\}$ is a choice on E. Let $\mathscr{P} = \{(x, [a, b])\}$ be a finite collection of non-overlapping tagged intervals that is subordinate to S. Then we have

$$\begin{aligned} (\mathscr{P})\sum |F([c,d])| &= \sum_{n=1}^{\infty} \sum_{x \in (b_{n+1}, a_n)} |F([c,d])| + \sum_{n=1}^{\infty} \sum_{x \in [a_n, b_n]} |F([c,d])| \\ &\leqslant \sum_{n=1}^{\infty} \sum_{x \in [a_n, b_n]} \frac{\pi(d-c)}{b_n - a_n} \leqslant \sum_{n=1}^{\infty} \frac{\pi}{b_n - a_n}|O_n| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon. \end{aligned}$$

Hence, F is an AL function on $[a, b]$.

Theorem 2.6. *Let $f \colon [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (\mathrm{AD})\int_a^x f$ for each $x \in [a, b]$. Then*

(a) *the function F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{\rm ap} = f$ almost everywhere on $[a, b]$; and*

(b) *the functions F and f are measurable.*

Proof. (a) follows from the definition of the ap-Denjoy integral. Since F is approximately continuous almost everywhere on $[a, b]$, F is measurable by [4, Theorem 14.7]. It follows from [4, Theorem 14.12] that f is measurable. □

Theorem 2.7. *Let $F \colon [a, b] \to \mathbb{R}$ be an AL function on $[a, b]$. If F is approximately differentiable almost everywhere on $[a, b]$, then $F'_{\rm ap}$ is ap-Denjoy integrable on $[a, b]$ and $(\mathrm{AD})\int_a^x F'_{\rm ap} = F(x) - F(a)$ for each $x \in [a, b]$.*

Proof. Suppose that F is an AL function on $[a, b]$ and F is approximately differentiable almost everywhere on $[a, b]$. It follows from the definition that $F'_{\rm ap}$ is ap-Denjoy integrable on $[a,b]$. For a constant C, $F+C$ is also an AL function on $[a,b]$, approximately differentiable almost everywhere on $[a,b]$ and $(F+C)'_{\rm ap}=F'_{\rm ap}$ almost everywhere on $[a,b]$. Hence, we have

$$F(x)+C=({\rm AD})\int_a^x F'_{\rm ap} \quad \text{for each } x\in[a,b].$$

Since $F(a)+C=0$, $C=-F(a)$ and

$$({\rm AD})\int_a^x F'_{\rm ap}=F(x)-F(a) \quad \text{for each } x\in[a,b].$$

□

We can easily show that if f is ap-Denjoy integrable on each of intervals $[a,c]$ and $[c,b]$, then f is ap-Denjoy integrable on $[a,b]$ and

$$({\rm AD})\int_a^b f=({\rm AD})\int_a^c f+({\rm AD})\int_c^b f.$$

Theorem 2.8. *Suppose that $f\colon [a,b]\to\mathbb{R}$ is ap-Denjoy integrable on each subinterval $[c,d]\subseteq(a,b)$. If $({\rm AD})\int_c^d f$ converges to a finite limit as $c\to a^+$ and $d\to b^-$, then f is ap-Denjoy integrable on $[a,b]$ and $({\rm AD})\int_a^b f=\lim\limits_{\substack{c\to a^+\\ d\to b^-}}({\rm AD})\int_c^d f$.*

P r o o f. Choose a point $p\in(a,b)$ and fix it. First, we will prove that if f is ap-Denjoy integrable on $[p,d]$ for each $d\in(p,b)$ and $({\rm AD})\int_p^d f$ converges to a finite limit as $d\to b^-$, then f is ap-Denjoy integrable on $[p,b]$ and $({\rm AD})\int_p^b f=\lim\limits_{d\to b^-}({\rm AD})\int_p^d f$.

Let $L=\lim\limits_{d\to b^-}({\rm AD})\int_p^d f$, let $a_0=p$ and let $\{a_k\}$ be an increasing sequence in (p,b) that converges to b. Define a function $F\colon [p,b]\to\mathbb{R}$ by

$$F(x)=F_i(x) \quad \text{if } x\in[a_{i-1},a_i] \quad \text{for each } i=1,2,3,\dots$$

and $F(b)=L$, where F_i is the primitive of f on $[a_{i-1},a_i]$ and $F_i(a_{i-1})=0$ for each i. Since each F_i is an AL function on $[a_{i-1},a_i]$ such that F_i is approximately differentiable almost everywhere on $[a_{i-1},a_i]$ and $(F_i)'_{\rm ap}=f$ almost everywhere on $[a_{i-1},a_i]$, the function F is an AL function on $[p,b]$ such that F is approximately differentiable almost everywhere on $[p,b]$ and $F'_{\rm ap}=f$ almost everywhere on $[p,b]$. Hence, f is ap-Denjoy integrable on $[p,b]$ and

$$({\rm AD})\int_p^b f=F(b)=L=\lim_{d\to b^-}({\rm AD})\int_p^d f.$$

Similarly, we can prove that if f is ap-Denjoy integrable on $[c,p]$ for each $c \in (a,p)$ and $(\mathrm{AD})\int_c^p f$ converges to a finite limit as $c \to a^+$, then f is ap-Denjoy integrable on $[a,p]$ and $(\mathrm{AD})\int_a^p f = \lim\limits_{c\to a^+} (\mathrm{AD})\int_c^p f$.

If $(\mathrm{AD})\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then for any $p \in (a,b)$ the integral $(\mathrm{AD})\int_c^p f$ converges to a finite limit as $c \to a^+$ and $(\mathrm{AD})\int_p^d f$ converges to a finite limit as $d \to b^-$. By the proof of the previous parts, f is ap-Denjoy integrable on $[a,p] \cup [p,b] = [a,b]$ and

$$\begin{aligned}(\mathrm{AD})\int_a^b f &= (\mathrm{AD})\int_a^p f + (\mathrm{AD})\int_p^b f \\ &= \lim_{c\to a^+} (\mathrm{AD})\int_c^p f + \lim_{d\to b^-} (\mathrm{AD})\int_p^d f = \lim_{\substack{c\to a^+\\ d\to b^-}} (\mathrm{AD})\int_c^d f.\end{aligned}$$

□

Recall that a function $f\colon [a,b] \to \mathbb{R}$ is *ap-Henstock integrable* on $[a,b]$ if there exists a real number A with the following property: for each $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that $|(\mathscr{P})\sum f(x)|I| - A| < \varepsilon$ whenever $\mathscr{P} = \{(x,I)\colon x \in [a,b]\}$ is a tagged partition of $[a,b]$ that is subordinate to S. The real number A is called the ap-Henstock integral of f on $[a,b]$ and is denoted by $(\mathrm{AH})\int_a^b f$. If f is ap-Henstock integrable on $[a,b]$, then f is also ap-Henstock integrable on any subinterval I of $[a,b]$. Hence, an interval function F can be defined by $F(I) = (\mathrm{AH})\int_I f$. The function F is called the primitive of f.

The following theorem shows that the ap-Denjoy integral is equivalent to the ap-Henstock integral and the integrals are equal to each other.

Theorem 2.9. *The function $f\colon [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$ if and only if f is ap-Henstock integrable on $[a,b]$ and the integrals are equal to each other.*

Proof. If f is ap-Henstock integrable on $[a,b]$ with the primitive F, then F is ACG_s on $[a,b]$ and $F'_{\mathrm{ap}} = f$ almost everywhere on $[a,b]$ by [4, Theorem 16.18]. By Lemma 2.2, f is ap-Denjoy integrable on $[a,b]$.

Suppose that f is ap-Denjoy integrable on $[a,b]$ with the primitive F. Then F is an AL function on $[a,b]$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{\mathrm{ap}} = f$ almost everywhere on $[a,b]$. Let

$$E = \{x \in [a,b]\colon F'_{\mathrm{ap}}(x) \neq f(x)\}.$$

Then $|E| = 0$. Let $D = [a,b] - E$ and let $\varepsilon > 0$.

For each $x \in D$ there exists a measurable set $D_x \subseteq [a,b]$ such that $x \in D_x^d$ and

$$F'_{\rm ap}(x) = \lim_{\substack{y\to x\\ y\in D_x}} \frac{F(y)-F(x)}{y-x}.$$

Hence, there exists $\delta_x > 0$ such that for every $y \in D_x \cap (x-\delta_x, x+\delta_x) = S_x$

$$|F(y) - F(x) - F'_{\rm ap}(x)(y-x)| \leqslant \varepsilon|y-x|.$$

If $(x,[u,v])$ is a tagged interval that is subordinate to $\{S_x\}$, then

$$\begin{aligned}|F(v) - F(u) - F'_{\rm ap}(x)(v-u)|\\ &\leqslant |F(v) - F(x) - F'_{\rm ap}(x)(v-x)| + |F(x) - F(u) - F'_{\rm ap}(x)(x-u)|\\ &< \varepsilon(v-x) + \varepsilon(x-u) = \varepsilon(v-u).\end{aligned}$$

Hence, there exists a choice S' on D such that $|(\mathscr{P})\sum f(x)|I| - (\mathscr{P})\sum F(I)| < \varepsilon(\mathscr{P})\sum|I|$ whenever $\mathscr{P}$ is a collection of tagged intervals that is subordinate to S'.

By [4, Lemma 9.15] and the fact that F is an AL function on $[a,b]$, there exists a choice S'' on E such that $|(\mathscr{P})\sum f(x)|I|| < \varepsilon$ and $|(\mathscr{P})\sum F(I)| < \varepsilon$ whenever $\mathscr{P}$ is subordinate to S''. Let $S = S' \cup S''$. Then S is a choice on $[a,b]$.

Suppose that $\mathscr{P}$ is a tagged partition of $[a,b]$ that is subordinate to S. Let $\mathscr{P}_E$ be the subset of $\mathscr{P}$ that has tags in E and let $\mathscr{P}_D = \mathscr{P} - \mathscr{P}_E$. Then we have

$$\begin{aligned}&\Big|(\mathscr{P})\sum f(x)|I| - (\mathscr{P})\sum F(I)\Big|\\ &\quad\leqslant \Big|(\mathscr{P}_D)\sum f(x)|I| - (\mathscr{P}_D)\sum F(I)\Big| + \Big|(\mathscr{P}_E)\sum f(x)|I|\Big| + \Big|(\mathscr{P}_E)\sum F(I)\Big|\\ &\quad< \varepsilon(b-a+2).\end{aligned}$$

Hence, f is ap-Henstock integrable on $[a,b]$ and $({\rm AH})\int_a^b f = (\mathscr{P})\sum F(I) = F(b) - F(a) = ({\rm AD})\int_a^b f$. □

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