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A NEW APPROACH TO CHORDAL GRAPHS

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Abstract. By a *chordal* graph is meant a graph with no induced cycle of length $\geqslant 4$. By a *ternary system* is meant an ordered pair (W, T), where W is a finite nonempty set, and $T \subseteq W \times W \times W$. Ternary systems satisfying certain axioms (A1)–(A5) are studied in this paper; note that these axioms can be formulated in a language of the first-order logic. For every finite nonempty set W, a bijective mapping from the set of all connected chordal graphs G with $V(G) = W$ onto the set of all ternary systems (W, T) satisfying the axioms (A1)–(A5) is found in this paper.

Keywords: connected chordal graph, ternary system

MSC 2000: 05C75, 05C38

1. Introduction

By a graph we mean here a finite undirected graph with no multiple edge (and no loop). Let G be a graph, and let P (or C) be a path in G (or a cycle in G respectively). We say that P (or C) is an induced path in G (or an induced cycle in G) if no edge of G joins two nonconsecutive vertices of P (or of C respectively).

As usual (cf. [1] and [2]), by a *chordal* (or *triangulated*) graph we mean a graph with no induced cycle of length $\geqslant 4$. Note that chordal graphs are called *rigid circuit* graphs in [3].

Following [4], by a *ternary system* we mean an ordered pair (W, T), where W is a finite nonempty set, and $T \subseteq W \times W \times W$. If $S = (W, T)$ is a ternary system, then we write $V(S) = W$.

Let $S = (W, T)$ be a ternary system, and let $x, y, z \in V(S)$. Similarly as in [4], the following convention will be used: we will write $xySz$ if and only if $(x, y, z) \in T$; otherwise, we will write $\neg(xySz)$.

Let S be a ternary system satisfying the following axioms (A1) and (A2):

(A1) if $uvSw$, then $vuSu$, for all $u, v, w \in V(S)$;
(A2) if $uvSw$, then $w \neq u \neq v$, for all $u, v, w \in V(S)$.

Obviously, the axiom (A1) implies that

$$(1) \qquad xySy \text{ if and only if } yxSx, \text{ for all } x, y \in V(S).$$

By the *underlying* graph of S we mean the graph G defined as follows: $V(G) = V(S)$ and

$$(2) \qquad xy \in E(G) \text{ if and only if } xySy \text{ for all } x, y \in V(S).$$

Let W be an arbitrary finite nonempty set. In the present paper, we will find a bijective mapping from the set of all connected chordal graphs G such that $V(G) = W$ onto the set of all ternary systems S such that $V(S) = W$ and S satisfies the axioms (A1), (A2) and the following axioms (A3), (A4), and (A5):

(A3) if $uvSv$, $vwSx$, $u \neq w$, and $\neg(uwSw)$, then $uvSx$, for all $u, v, w, x \in V(S)$;
(A4) if $uvSw$ and $v \neq w$, then there exists $y \in V(S)$ such that $vySw$, $y \neq u$, and $\neg(uySy)$, for all $u, v, w \in V(G)$;
(A5) if $u \neq v$, then there exists $z \in V(S)$ such that $uzSv$, for all $u, v \in V(G)$.

Note that the axioms (A1)–(A5) can be formulated in a language of the first-order logic.

2. Propositions

In this paper, the letters $h - n$ will be used for denoting integers only.

Let G be a graph, and let $u_0, u_1, \ldots, u_n \in V(G)$, $n \geqslant 1$. Assume that $(u_0, u_1, \ldots, u_n)$ is a path in G. If we put $u = u_0$, $v = u_1$, and $w = u_n$, then we say that $(u_0, u_1, \ldots, u_n)$ is a $uv - w$ path in G.

By the ip-system of G we mean the ternary system S defined as follows: $V(S) = V(G)$ and

$$uvSw \text{ if and only if there exists an induced } uv - w \text{ path in } G,$$

for all $u, v, w \in V(G)$.

Let S be a ternary system satisfying the axioms (A1) and (A2), and let G denote the underlying graph of S. In the next section of this paper, we will prove the following result: G is a connected chordal graph and S is the ip-system of G if and only if S satisfies the axioms (A3), (A4), and (A5).

Propositions 1–4 will be used in the proof of the mentioned result.

Proposition 1. *Let G be a graph, and let S denote the* ip*-system of G. Then*
(a) *S satisfies the axioms* (A1), (A2) *and* (A4), *and G is the underlying graph of S;*
(b) *if G is connected, then S satisfies the axiom* (A5);
(c) *if G is chordal, then S satisfies the axiom* (A3).

Proof. To prove (a) and (b) is easy. We will prove (c) only.

Consider arbitrary $u, v, w, x \in V(G)$ such that $uvSv$, $vwSx$, $u \neq w$, and $\neg(uwSw)$. Then $uv, vw \in E(G)$, $uw \notin E(G)$, and there exists an induced $vw - x$ path in G. This means that there exist $v_0, v_1, \ldots, v_n \in V(G)$, $n \geqslant 1$, such that $v_0 = v$, $v_1 = w$, $v_n = x$, and $(v_0, v_1, \ldots, v_n)$ is an induced path in G. Obviously, $u \notin \{v_2, \ldots, v_n\}$. Assume that there exists i, $2 \leqslant i \leqslant n$, such that $uv_i \in E(G)$. Then there exists a cycle C of length $i + 2$ in G such that

$$V(C) = \{u, v_0, v_1, \ldots v_i\} \text{ and } E(C) = \{uv_0, v_0v_1, \ldots, v_{i-1}v_i, v_iu\}.$$

Since G is a chordal graph and $(v_0, v_1, \ldots, v_i)$ is an induced path in G, we get $uv_1 \in E(G)$. Hence $uw \in E(G)$, which is a contradiction. This implies that $(u, v_0, v_1, \ldots, v_n)$ is an induced path in G and therefore $uvSx$. Thus S satisfies the axiom (A3). □

Lemma 1. *Let S be a ternary system satisfying the axioms* (A1), (A2) *and* (A3), *and let $u_0, \ldots, u_m \in V(G)$, $m \geqslant 1$. Assume that*

$$u_{i+1}u_iSu_i \text{ for each } i,\ 0 \leqslant i \leqslant m-1, \tag{3}$$

and

$$u_{j+2} \neq u_j \text{ and } \neg(u_{j+2}u_jSu_j) \text{ for each } j,\ 0 \leqslant j \leqslant m-2. \tag{4}$$

Then

$$u_ku_{k-1}Su_0 \text{ for each } k,\ m \geqslant k \geqslant 1. \tag{5}$$

Proof. We proceed by induction on m. If $m = 1$, then $u_1u_0Su_0$. Let now $m \geqslant 2$. By the induction hypothesis,

$$u_lu_{l-1}Su_0 \text{ for each } l,\ m-1 \geqslant l \geqslant 1. \tag{6}$$

Hence $u_{m-1}u_{m-2}Su_0$. By (3), $u_mu_{m-1}Su_{m-1}$. As follows from (4), $u_m \neq u_{m-2}$ and $\neg(u_mu_{m-2}Su_{m-2})$. By the axiom (A3), $u_mu_{m-1}Su_0$. Combining this result with (6), we get (5), which completes the proof. □

Corollary 1. *Let S be a ternary system satisfying the axioms* (A1), (A2) *and* (A3), *and let $u_0, \ldots, u_m \in V(G)$, $m \geqslant 1$. Assume that* (3) *and* (4) *hold. Consider arbitrary h and l, $0 \leqslant h < l \leqslant m$. Then*

(7) $$u_k u_{k-1} S u_h \text{ for each } k,\ \ l \geqslant k \geqslant h+1,$$

and

(8) $$u_k u_{k+1} S u_l \text{ for each } k,\ \ h \leqslant k \leqslant l-1.$$

Proof. The statement (7) immediatelly follows from Lemma 1. Combining Lemma 1 with (1), we get the statement (8). □

Proposition 2. *Let S be a ternary system satisfying the axioms* (A1), (A2) *and* (A3), *and let G denote the underlying graph of S. Then G is chordal.*

Proof. Suppose to the contrary, that G contains an induced cycle C of length $m \geqslant 4$. There exist pairwise distinct $u_0, u_1, \ldots, u_m \in V(G)$ such that $V(C) = \{u_0, u_1 \ldots, u_m\}$ and $E(C) = \{u_0u_1, u_1u_2, \ldots, u_{m-1}u_m, u_mu_0\}$. Since C is an induced cycle of G, we have $u_0u_2, u_1u_3, \ldots, u_{m-2}u_m \notin E(G)$. Since G is the underlying graph of S, (3) and (4) hold. By Lemma 1, $u_mu_{m-1}Su_0$. Since $u_mu_0 \in E(G)$ and $u_{m-1}u_0 \notin E(G)$, we have $u_0u_mSu_m$ and $\neg(u_0u_{m-1}Su_{m-1})$. Recall that $u_0 \neq u_{m-1}$. The axiom (A3) implies that $u_0u_mSu_0$, which contradicts the axiom (A2). Thus G is chordal. □

Lemma 2. *Let S be a ternary system satisfying the axioms* (A1), (A2) *and* (A3), *let G denote the underlying graph of S, and let $u_0, u_1, \ldots, u_m \in V(S)$, $m \geqslant 1$. Assume that* (3) *and* (4) *hold. Then $(u_0, u_1, \ldots, u_m)$ is an induced path in G.*

Proof. By Proposition 2, G is chordal. As follows from (3), $(u_0, u_1, \ldots, u_m)$ is a walk in G. Consider arbitrary h and i, $0 \leqslant h < i \leqslant m$. By Corollary 1, $u_hu_{h+1}Su_i$. As follows from the axiom (A2), $u_h \neq u_i$. Hence the vertices $u_0, u_1, \ldots, u_m$ are pairwise distinct and therefore $(u_0, u_1, \ldots u_m)$ is a path in G. Suppose, to the contrary, that $(u_0, u_1, \ldots, u_m)$ is not an induced path in G. Then there exist k and l, $0 \leqslant k < l \leqslant m$, such that $l - k \geqslant 2$, $u_ku_l \in E(G)$, and

both $(u_{k+1}, \ldots, u_l)$ and $(u_k, \ldots, u_{l-1})$ are induced paths in G.

By virtue of (4), $u_ku_{k+2}, u_{l-2}u_l \notin E(G)$. Hence $l - k \geqslant 3$. Let C denote the cycle in G obtained from the path $(u_k, u_{k+1}, \ldots u_l)$ by adding the edge u_ku_l. Since G is chordal and $(u_{k+1}, \ldots, u_l)$ is an induced path in G, we have $u_ku_{l-1} \in E(G)$. This implies that $(u_k, \ldots, u_{l-1})$ is not an induced path in G, which is a contradiction. Thus the lemma is proved. □

Recall that if S is a ternary system, then $V(S)$ is finite. This fact will be used in the proof of the following lemma.

Lemma 3. *Let S be a ternary system satisfying the axioms* (A1)–(A4), *let G denote the underlying graph of S, and let $u, v, w \in V(S)$. Assume that $uvSw$. Then there exist $u_0, u_1, \ldots, u_n \in V(S)$, $n \geqslant 1$, such that $u_0 = u$, $u_1 = v$, $u_n = w$,* (4) *holds,*

$$u_i u_{i+1} S u_n \text{ for each } i,\ 0 \leqslant i \leqslant n-1,$$

and $(u_0, \ldots, u_n)$ is an induced path in G.

Proof. We will construct an infinite sequence

$$\sigma = (u_0, u_1, u_2, \ldots)$$

of elements in $V(S)$ with the following properties:

$u_0 = u$ and $u_1 = v$;

if $k \geqslant 2$ and $u_{k-1} = w$, then $u_k = w$;

if $k \geqslant 2$ and $u_{k-1} \neq w$, then $u_{k-1}u_kSw$, $u_k \neq u_{k-2}$, and $\neg(u_{k-2}u_kSu_k)$.

As follows from the axiom (A4), σ is well-defined.

Consider an arbitrary $m \geqslant 1$ such that $u_{m-1} \neq w$. We have

$$u_0u_1Sw, \ldots, u_{m-1}u_mSw$$

and thus, by the axiom (A1), (3) holds. Combining the definition of σ with (1), we see that (4) holds, too. By Lemma 2, $(u_0, u_1, \ldots u_m)$ is an induced path in G.

Since $V(S)$ is finite, it is clear that there exists $n \geqslant 1$ such that $u_{n-1} \neq w$ and $u_n = w$, which completes the proof. □

Proposition 3. *Let S be a ternary system satisfying the axioms* (A1)–(A4), *let G denote the underlying graph of S, and let S^* denote the* ip*-system of G. Then*

$$uvSw \text{ implies } uvS^*w$$

for all $u, v, w \in V(S)$.

Proof. Consider arbirary $u, v, w \in V(S)$ such that $uvSw$. By Lemma 3, there exist $u_0, u_1, \ldots, u_n \in V(S)$, $n \geqslant 1$, such that $u_0 = u, u_1 = v, u_n = w$ and $(u_0, u_1, \ldots, u_n)$ is an induced path in G. Thus $u_0u_1S^*u_n$; we have uvS^*w. □

Proposition 4. *Let S be a ternary system satisfying the axioms* (A1)–(A5), *and let G denote the underlying graph of S. Then G is connected.*

P r o o f. Consider arbitrary $u, v \in V(G)$, $u \neq v$. By the axiom (A5), there exists $z \in V(G)$ such that $uzSv$. By Lemma 3, there exist $u_0, u_1, \ldots, u_n \in V(G)$, $n \geqslant 1$, such that $u_0 = u$, $u_1 = z$, $u_n = v$, and $(u_0, u_1, \ldots, u_n)$ is an induced path in G. This implies that G is connected. □

3. The main result

Let G be a graph. Consider $x, y, z \in V(G)$. If there exists at least one induced $xy - z$ path in G, then we denote by $d_G(xy - z)$ the minimum length of an induced $xy - z$ path in G.

The following theorem is the main result of this paper.

Theorem 1. *Let S be a ternary system satisfying the axioms* (A1) *and* (A2), *and let G denote the underlying graph of S. Then G is a connected chordal graph and S is the* ip*-system of G if and only if S satisfies the axioms* (A3), (A4), *and* (A5).

P r o o f. Assume that G is a connected chordal graph and S is the ip-system of G. By Proposition 1, S satisfies the axioms (A3), (A4), and (A5).

Conversely, assume that S satisfies the axioms (A3), (A4), and (A5). By Proposition 2, G is chordal; by Proposition 4, G is connected. Let S^* denote the ip-system of G. According to Proposition 1, G is the underlying graph of S^*. We wish to prove that

$$uvSw \text{ if and only if } uvS^*w \text{ for all } u, v, w \in V(S).$$

The "if" part of this statement immediately follows from Proposition 3. It remains to prove the "only if" part.

Consider arbitrary $u, v, w \in V(S)$ such that uvS^*w. Put $n = d_G(uv - w)$. Obviously, $n \geqslant 1$. We want to prove that $uvSw$. We proceed by induction on n. Let first $n = 1$; then $w = v$ and uvS^*v; hence $uvSw$. Let now $n \geqslant 2$. Clearly, there exist $u_0, u_1, \ldots, u_n \in V(G)$ such that $(u_0, u_1, \ldots, u_n)$ is an induced path in G, $u_0 = u$, $u_1 = v$, and $u_n = w$. Obviously, $(u_1, u_2, \ldots, u_n)$ is also an induced path in G. Since $d_G(u_1u_2 - u_n) \leqslant n - 1$, the induction hypothesis implies that $u_1u_2Su_n$. Since (u_0, u_1, u_2) is an induced path in G, we have $u_0u_1 \in E(G)$, $u_0 \neq u_2$, and $u_0u_2 \notin E(G)$. Hence $u_0u_1Su_1$ and $\neg(u_0u_2Su_2)$. The axiom (A3) implies that $u_0u_1Su_n$; we have $uvSw$, which completes the proof. □

The next corollary is an immediate consequence of Theorem 1.

Corollary 2. *A graph G is a connected chordal graph if and only if there exists a ternary system S satisfying the axioms* (A1)–(A5) *such that G is the underlying graph of S.*

For every finite nonempty set W, we denote by $\mathcal{G}_W$ the set of all connected chordal graphs G such that $V(G) = W$ and by $\mathcal{T}_W$ the set of all ternary systems S satisfying the axioms (A1)–(A5) such that $V(S) = W$.

If W is a finite nonempty set, then for every $G \in \mathcal{G}_W$, we denote by $\sigma_W(G)$ the ip-system of G.

The next theorem is a reformulation of Theorem 1:

Theorem 2. *For every finite nonempty set W, σ_W is a bijective mapping from $\mathcal{G}_W$ onto $\mathcal{T}_W$.*

Thus, roughly speaking, connected chordal graphs can be considered as ternary systems satisfying the axioms (A1)–(A5) and vice versa.

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