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BASIC SUBGROUPS IN MODULAR ABELIAN GROUP ALGEBRAS

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Abstract. Suppose F is a perfect field of $\operatorname{char} F = p \neq 0$ and G is an arbitrary abelian multiplicative group with a p-basic subgroup B and p-component G_p. Let FG be the group algebra with normed group of all units $V(FG)$ and its Sylow p-subgroup $S(FG)$, and let $I_p(FG;B)$ be the nilradical of the relative augmentation ideal $I(FG;B)$ of FG with respect to B.

The main results that motivate this article are that $1+I_p(FG;B)$ is basic in $S(FG)$, and $B(1+I_p(FG;B))$ is p-basic in $V(FG)$ provided G is p-mixed. These achievements extend in some way a result of N. Nachev (1996) in Houston J. Math. when G is p-primary. Thus the problem of obtaining a (p-)basic subgroup in FG is completely resolved provided that the field F is perfect.

Moreover, it is shown that $G_p(1+I_p(FG;B))/G_p$ is basic in $S(FG)/G_p$, and $G(1+I_p(FG;B))/G$ is basic in $V(FG)/G$ provided G is p-mixed.

As consequences, $S(FG)$ and $S(FG)/G_p$ are both starred or divisible groups.

All of the listed assertions enlarge in a new aspect affirmations established by us in Czechoslovak Math. J. (2002), Math. Bohemica (2004) and Math. Slovaca (2005) as well.

Keywords: p-basic subgroups, normalized units, group algebras, starred groups

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Introduction

As usual, everywhere in the text FG will designate the group algebra of an abelian group G over a field F with characteristic $p > 0$. For G such a group, B denotes its fixed p-basic subgroup, and G_t denotes its maximal torsion subgroup with p-component G_p and p^n-socle $G[p^n]$. Certainly $G_p = \bigcup\limits_{n=1}^{\infty} G[p^n]$, where $G[p^n] \subseteq G[p^{n+1}]$ and $G_t = \coprod\limits_p G_p = G_p \times \coprod\limits_{q\neq p} G_q$. All other notations and terminology from the abelian group theory not explicitly defined herein are as in [9]. For FG such an F-algebra, $V(FG)$ will denote the group of all normalized invertible elements in FG

with Sylow p-subgroup $S(FG)$. For H a subgroup of G, we define $I(FG;H)$ as the relative augmentation ideal of FG with respect to H with nil-radical $I_p(FG;H)$. The notation and terminology from the group rings theory which are not given here will follow essentially [11].

In 1996, Nachev established in [15] an explicit form of a basic subgroup of $V(RG)$, namely $1+I(RG;B)$, when R is an abelian, unitary, perfect ring of prime char $R=p$ and G is an abelian p-group. Nevertheless, we have obtained in [2] a basic subgroup of $S(RG)$, namely $1+I(RG;B_p)$, when G/G_p is p-divisible and R is as above, thus generalizing the mentioned Nachev's claim. Further developments in this theme we have realized in [4]–[8].

In the present paper and more specifically in the next paragraph, we shall give in an explicit form a basic subgroup of $S(FG)$ and a p-basic subgroup of $V(FG)$ under some minimal restrictions on F and G, namely that F is perfect plus $G_t = G_p$ for the mixed case. Thus, as a culmination of a series of explorations, the question of finding a (p-)basic subgroup in a modular group ring is settled in the case of a perfect coefficient field. The general situation when F is not perfect is also discussed in the epilogue. And so, we begin with

Main results

We start here with a technical claim that gives a satisfactory estimate for the value of heights in group rings.

Lemma (Heights). *For perfect F and any $1+\sum_{i,j} r_{i,j}g_{i,j}(1-c_j) \in 1+I(FG;H)$, it is valid that p-height$(1+\sum_{i,j} r_{i,j}g_{i,j}(1-c_j)) \leqslant \min_s\{p\text{-height}(c_{j_1}^{\pm\varepsilon_1}c_{j_2}^{\pm\varepsilon_2}\dots c_{j_s}^{\pm\varepsilon_s})\}$ where $(j_1, j_2, \dots, j_s)$ is an arbitrary permutation of the indices j, and $\varepsilon_1, \dots, \varepsilon_s \in \mathbb{N}$. In particular, as a corollary, there exists $1 \neq c \in H$ so that p-height$(1+\sum_{i,j} r_{i,j}g_{i,j}(1-c_j)) \leqslant p$-height$(c)$.*

Proof. This is straightforward. $\square$

For our further successful presentation, we need the following extra observation on heights of elements of infinite order.

Given that order$(b) = \infty$, we have $\langle b\rangle = \{\dots, b^{-n}, b^{-(n-1)}, \dots, b^{-1}, b^0 = 1, b, \dots, b^{n-1}, b^n, \dots\}$. Therefore $b \in \langle b\rangle \setminus \langle b\rangle^p, \dots, b^{p-1} \in \langle b\rangle \setminus \langle b\rangle^p$; $b^p \in \langle b\rangle^p \setminus \langle b\rangle^{p^2}, \dots, b^{p^2-1} \in \langle b\rangle^p \setminus \langle b\rangle^{p^2}$ etc. This enables us to detect that $\langle b\rangle^{p^\omega} = \bigcap\limits_{n=1}^{\infty} \langle b\rangle^{p^n} = 1$. In fact, for an arbitrary element $x \in \langle b\rangle^{p^\omega}$ we have that $x = b^{n_1p} = b^{n_2p^2} = \dots = b^{n_kp^k} = \dots$ where $n_1, n_2, \dots, n_k \in \mathbb{Z}$ $\forall k \geqslant 1$. Henceforth, $n_1 = n_2p = \dots = n_kp^{k-1} = \dots$ and $|n_1| = |n_kp^{k-1}| = |n_k|p^{k-1} \geqslant p^{k-1}$ $\forall k \geqslant 1$, provided $n_k \neq 0$. But it is easily seen that such an inequality is impossible. Thereby the foregoing chain of equalities is fulfilled precisely when $n_1 = n_2 = \ldots = n_k = \ldots = 0$. This allows us to conclude that $x = 1$, as expected.

Before establishing the basic subgroups, we proceed by proving one important theorem that is the omnibus for our main results.

Theorem 1. *The group $1 + I_p(FG; B)/B_p$ is a direct sum of cyclic groups and so B_p is a direct factor of $1 + I_p(FG; B)$ with a direct sum of cyclics complement.*

Proof. Of course $G_p \neq 1$, otherwise $S(FG) = 1$. Certainly, we write $B = B_p \times M$, where B_p is a direct sum of p-cyclics and M a direct sum of infinite cyclics. Clearly, as it was already observed, B is p-separable or in other terms $B^{p^\omega} = 1$. That is why, according to [2], $1 + I_p(FG; B)$ is separable. Moreover, we may write $B_p = \bigcup\limits_{n=1}^{\infty} B_n$, where $B_n \subseteq B_{n+1}$, all B_n are pure in B_p and $B_n^{p^n} = 1$. Consequently, we see that $B = \bigcup\limits_{n=1}^{\infty} [B_n \times M]$ where $B_n \times M$ is pure in $B_p \times M = B$ whence it is p-pure in G, and $1 + I_p(FG; B) = \bigcup\limits_{n=1}^{\infty} [1 + I_p(FG; B_n \times M)]$ where, in accordance with [4], $1 + I_p(FG; B_n \times M)$ is pure in $1 + I_p(FG; B)$ whence in $S(FG)$.

For the application of the Kulikov criterion [9, p. 110, Theorem 18.1], it is enough to construct an infinite sequence of subgroups $(I_n)_{n=1}^{\infty}$ such that $I = 1 + I_p(FG; B) = \bigcup\limits_{n=1}^{\infty} I_n$, $I_n \subseteq I_{n+1}$ and for every $n \in \mathbb{N}$ the groups I_n are height-finite in I with heights $< n$. Without loss of generality we shall assume that F is perfect since a subgroup of a direct sum of cyclic groups is again a direct sum of cyclic groups.

Indeed, we select I_n in the following manner: $I_n = \langle x^{(n)} = 1 + \sum_{i,j} \alpha_{i,j} g_{i,j} (1 - b_j^{(n)} m_j)$ is a p-element of order $\leqslant p^n | 0 \neq \alpha_{i,j} \in F, g_{i,j} \in G, b_j^{(n)} \in B_n$, $m_{j_1}^{\pm\varepsilon_1} m_{j_2}^{\pm\varepsilon_2} \ldots m_{j_s}^{\pm\varepsilon_s} \in (M \setminus M^{p^n}) \cup \{1\}$ for all possible permutations $(j_1, j_2, \ldots, j_s)$ of the indexes j; $0 \leqslant \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s \leqslant p^n - 1\rangle$.

Since $1 + I_p(FG; B) \subseteq S(FG) = 1 + I(FG; G_p)$, there follows at once the simple but however crucial fact that every $x \in 1 + I_p(FG; B)$ is of the form $x = 1 + \sum_{i,j} f_{i,j} g_{i,j} (1 - b_{j,p}) + \sum_{k,l,m} r_{k,l,m} a_{k,l,m} (1 - g_{l,p})(1 - b_m)$, where $f_{i,j}, r_{k,l,m} \in F$; $g_{i,j}, a_{k,l,m} \in G$; $b_{j,p} \in B_p, g_{l,p} \in G_p$; $b_m \in B$. Therefore, we redefine the wanted generating subgroups like this:

$$I_n = \Big\langle x^{(n)} = 1 + \sum_{i,j} f_{i,j}^{(n)} g_{i,j}^{(n)} (1 - b_{j,p}^{(n)}) + \sum_{k,l,m} r_{k,l,m}^{(n)} a_{k,l,m}^{(n)} (1 - g_{l,p}^{(n)})(1 - b_m^{(n)}) | f_{i,j}^{(n)} \in F,\ g_{i,j}^{(n)} \in G,\ b_{j,p}^{(n)} \in B_n;$$

$r_{k,l,m}^{(n)} \in F, a_{k,l,m}^{(n)} \in G, g_{l,p}^{(n)} \in G[p^n],\ b_{m_1}^{(n)^{\pm\varepsilon_1}} \ldots b_{m_s}^{(n)^{\pm\varepsilon_s}} \in (M \setminus M^{p^n}) \cup \{1\}$
for all possible permutations $(m_1, \ldots, m_s)$ of the indices m;
$0 \leqslant \varepsilon_1, \ldots, \varepsilon_s \leqslant p^n - 1 \Big\rangle$,

by using the elementary argument that $I(FG; B_n \times M) = I(FG; B_n) + I(FG; M)$. We also emphasize that we may fix $g_{l,p}^{(n)} \in (B_n G^p[p^n] \setminus G^{p^n}[p^n]) \cup G^{p^\omega}[p^n]$, although it is not necessary for our final conclusions.

Apparently, $I_n \subseteq 1 + I_p(FG; B_n \times M)$.

We first of all observe that $b_{m_1}^{(n)^{\pm\varepsilon_1}} \ldots b_{m_s}^{(n)^{\pm\varepsilon_s}} \notin G^{p^n}$ whenever $\varepsilon_1, \ldots, \varepsilon_s \in \mathbb{N}$ because of the fact that M is pure in B, hence it is p-pure in G. Moreover, a useful observation is that every non-trivial element of I_n may be not eventually a generator of I_n but however it does not belong to I^{p^n}. In order to verify this claim, we can restrict our attention to two generators since the general case holds either by virtue of an ordinary mathematical induction or by copying directly the idea described below.

Indeed, write $x_1^{(n)} = 1 + \sum_{i,j} \alpha_{i,j}^{(n)} g_{i,j}^{(n)} (1 - b_{j,p}^{(n)}) + \sum_{k,l,m} \beta_{k,l,m}^{(n)} a_{k,l,m}^{(n)} (1 - g_{l,p}^{(n)}) \times (1 - b_m^{(n)})$ and $x_2^{(n)} = 1 + \sum_{i,j} \alpha'^{(n)}_{i,j} g'^{(n)}_{i,j} (1 - b'^{(n)}_{j,p}) + \sum_{k,l,m} \beta'^{(n)}_{k,l,m} a'^{(n)}_{k,l,m} (1 - g'^{(n)}_{l,p}) \times (1 - b'^{(n)}_m)$. Further, their multiplying gives $x_1^{(n)} x_2^{(n)} \in 1 + I_p(FG; B_n \times [(M \setminus M^{p^n}) \cup \{1\}])$ where the direct product is taken in $G_p \times M$. Applying Lemma (Heights) for appropriate members in $G_p \times M$, or in more precise words in the estimation of heights substituting the elements $0 \neq 1 - b'^{(n)}_m \in FM \leqslant FG$ for $1 - g'^{(n)}_{l,p} \in FG_p$ when the expression does not depend on $1 - b_{j,p}^{(n)}$ or $1 - b_m^{(n)}$, by what we have shown above, it is clear that $x_1^{(n)} x_2^{(n)}$ has height as computed in I less than or equal to the height of some nonidentity element from $B_n \times [(M \setminus M^{p^n}) \cup \{1\}]$ or from $G_p \times (M \setminus M^{p^n})$ where $G_p \times M = BG_p$ is p-pure in G. This substantiates our claim. So, $I_n \cap I^{p^n} = 1$ as desired, and we are done.

After this, we shall establish in the sequel that $[I_n B_p] \cap I^{p^n} \subseteq B_p$, whence $I/B_p = \bigcup_{n=1}^{\infty} [I_n B_p / B_p]$ will be a direct sum of cyclics utilizing the above cited Kulikov criterion. Well, take an arbitrary element x in the left hand-side. Thus $x = b_p y = b_p(f_1 c_1 + \ldots + f_s c_s)$, where $b_p \in B_p$; $f_1 c_1 + \ldots + f_s c_s \in I_n$; $s \in \mathbb{N}$. It is not difficult to see that the sum $f_1 c_1 + \ldots + f_s c_s$ contains a member that belongs to B, say for instance $c_1 = b \in B$, and on the other hand it contains a member of the form $g_p b$ where $g_p \in G_p$, say $c_2 = g_p b$. We work by analogy for the remaining members. Consequently, we write $b_p c_1 \in G^{p^n}$, …, $b_p c_s \in G^{p^n}$, hence we may derive $c_2 c_1^{-1} \in G^{p^n}$, …, $c_s c_1^{-1} \in G^{p^n}$, i.e. $c_i c_j^{-1} \in G^{p^n}$ for all $1 \leqslant i, j \leqslant s$. That is why $g_p = c_2 c_1^{-1} \in G^{p^n}$ etc. But as we have shown above, the elements of I_n have heights $< n$, hence we deduce that $x \in B_p$, as required. Besides, it is a routine matter to see that B_p is pure in $1 + I_p(FG; B)$, so a theorem due to L. Kulikov

([9], p. 143, Theorem 28.2) guarantees that B_p must be itself a direct factor of I, as claimed.

The theorem is proved. $\square$

We continue with two preliminary assertions of technical character, starting with

Lemma. *B_p is basic in G_p.*

P r o o f. This is routine and so we omit the details. $\square$

The following intersection ratios play a key role for our good presentation.

Proposition (Intersection)**.** *Suppose $A, B \leqslant G$ and $1 \in P \leqslant F$. Then*
($*$) $[B(1 + I_p(FG; B))] \cap S(PA) = 1 + I_p(PA; A \cap B)$;
($**$) $[G_p(1 + I_p(FG; B))] \cap S(PA) = A_p(1 + I_p(PA; A \cap B))$.

P r o o f. ($*$) Certainly $[B(1+I_p(FG;B))] \cap S(PA) = (1+I_p(FG;B)) \cap S(PA) = 1 + I_p(PA; A \cap B)$, owing to relations in [2], [3].

($**$) Take x in the left hand-side. Then $x = g_p \sum\limits_{g \in G} f_g g = \sum\limits_{a \in A} \alpha_a a$, where $g_p \in G_p$, $f_g \in F$, $\alpha_a \in P$ and $\sum\limits_{g \in \bar{g}B} f_g = \begin{cases} 0, & \bar{g} \notin B, \\ 1, & \bar{g} \in B \end{cases}$ for each $\bar{g} \in G$. On the other hand, for every $g \in G$ and $a \in A$ from the sums we have $g_p g = a$ and $f_g = \alpha_a$. That is why, since in the support of $\sum\limits_{g \in G} f_g g$ there is an element $b \in B$ and an element from G_p, while in the support of $\sum\limits_{a \in A} \alpha_a a$ there exists an element of A_p, it is almost apparent that $g_p = a_p b_p$ for some $a_p \in A_p$ and $b_p \in B_p$. This follows because of the fact that $b \in AB_p$; $b^{p^t} = b'^{p^t}$ for some $t \in \mathbb{N}$ so that $b' \in A \cap B$ since $g_p h' \in A$ and $g_p h' b' \in A$ for some $h' \in G$. Thus x may be written as $x = a_p \sum\limits_{g \in G} f_g g b_p$. Finally $x \in A_p((1 + I_p(FG; B)) \cap S(PA)) = A_p(1 + I_p(PA; A \cap B))$, as well. This finishes the proof in general. $\square$

The next fact is valuable.

Remark. In some instances $(B \cap A)(1 + I(PA; A \cap B)) \supseteq B(1 + I_p(FG; B)) \cap \begin{cases} V(PA) \\ AS(PA) \end{cases} \supset (B \cap A)(1 + I_p(PA; A \cap B))$.

Now, we are in a position to present the following announcement in [8].

Theorem 2. *Assume that F is perfect. Then $1+I_p(FG;B)$ is a basic subgroup of $S(FG)$, and $B(1+I_p(FG;B))$ is a p-basic subgroup of $V(FG)$ provided G is p-mixed.*

P r o o f. We shall show below that the three necessary conditions from the definition of a basic and a p-basic subgroup are satisfied, respectively.

1. "*direct sums of cyclics*". And so, according to Theorem 1, $1+I_p(FG;B)/B_p$ is direct sum of cyclics. Clearly B_p is pure in G_p whence in $S(FG)$, and so in $1+I_p(FG;B)$ as a subgroup of $S(FG)$. Consequently, a theorem of L. Kulikov [9, p. 143, Theorem 28.2] ensures that $1+I_p(FG;B) \cong B_p \times (1+I_p(FG;B))/B_p$. But by the Lemma, B_p is a direct sum of cyclics, thus finishing the first step.

2. "*purity*". In view of the fact that B is p-pure in G and some preliminaries from [2], [3], we get $(1+I_p(FG;B)) \cap S^{p^n}(FG) = (1+I_p(FG;B)) \cap S(FG^{p^n}) = 1+I_p(FG^{p^n};G^{p^n} \cap B) = 1+I_p(FG^{p^n};B^{p^n}) = (1+I_p(FG;B))^{p^n}$, completing the second step.

3. "*divisibility*". Consider an element of the type $1+\alpha g(1-g_p)$, where $\alpha \in F$, $g \in G$ and $g_p \in G_p$. But $G = BG^p$, whence $g = ba^p$ where $b \in B$ and $a \in G$. Besides, the Lemma yields that $G_p = B_p G_p^p$, hence $g_p = cd^p$ where $c \in B_p$ and $d \in G_p$. On the other hand, $1-cd^p = (1-c)d^p + (1-d^p)$, so furthermore $\alpha g(1-d^p) = \alpha ba^p(1-d^p) = \alpha a^p(1-d^p) - \alpha(1-b)a^p(1-d^p)$ and $1+\alpha g(1-g_p) = [1+\alpha a^p(1-d^p)](1-\alpha(1+\alpha a^p(1-d^p))^{-1} \cdot a^p(1-b)(1-d^p) + \alpha(1+\alpha a^p(1-d^p))^{-1} \cdot gd^p(1-c)) \in S^p(FG)(1+I_p(FG;B))$. Because by [3] each element in $S(FG) = 1+I(FG;G_p)$ is a finite sum of members of the above kind, then $S(FG) = (1+I_p(FG;B))S^p(FG)$, thus proving the third step. Finally, we conclude that the first part is verified.

Next, the second half will be supplied.

The point 1. follows like this: By what we have just shown above $B(1+I_p(FG;B))/B \cong 1+I_p(FG;B)/B_p$ is a direct sum of cyclics. But B is pure in $B(1+I_p(FG;B))$. In fact, B is p-pure in G hence in $V(FG)$, and so in $B(1+I_p(FG;B))$ as a subgroup of $V(FG)$. On the other hand, using the modular law [9], for each prime $q \neq p$ we deduce that $B \cap [B(1+I_p(FG;B))]^q = B \cap [B^q(1+I_p(FG;B))] = B^q(B \cap (1+I_p(FG;B))) = B^q \cdot B_p = B^q$. Thereby, as we have just seen from [9], $B(1+I_p(FG;B)) \cong B \times (1+I_p(FG;B))/B_p$, which guarantees our claim.

The point 2. holds for the following reason: It is well-known that ([13], [11]) $V(FG) = G(1+I(FG;G_p))$. But by [2], [3], $1+I(FG;G_p) = S(FG)$. Hence, $V(FG) = GS(FG)$. By making use of the intersection Proposition plus the modular law we compute $[B(1+I_p(FG;B))] \cap (BS(FG))^{p^n} = [B(1+I_p(FG;B))] \cap (B^{p^n}S(FG^{p^n})) = B^{p^n}([B(1+I_p(FG;B))] \cap S(FG^{p^n})) = B^{p^n}(1+I_p(FG^{p^n};G^{p^n} \cap B)) = B^{p^n}(1+I_p(FG^{p^n};B^{p^n})) = [B(1+I_p(FG;B))]^{p^n}$. In this way, the modular law and the fact from the Lemma that $G_p = B_pG_p^{p^n}$ imply that $[BS(FG)] \cap [GS(FG)]^{p^n} = [BS(FG)] \cap [G^{p^n}S^{p^n}(FG)] = S^{p^n}(FG)[(BS(FG)) \cap G^{p^n}] = [(BG_p) \cap G^{p^n}]S^{p^n}(FG) = [(BG_p^{p^n}) \cap G^{p^n}]S^{p^n}(FG) = (B \cap G^{p^n})S^{p^n}(FG) = B^{p^n}S^{p^n}(FG) = [BS(FG)]^{p^n}$. Owing to the transitivity of p-purity, we derive at once that $B(1 + I_p(FG; B))$ is p-pure in $V(FG)$, as required.

The point 3. can be justified as follows: We have $G = BG^p$ and so by what we have proved above $S(FG) = S^p(FG)(1 + I_p(FG; B))$. Hence, $V(FG) = BG^pS^p(FG)(1 + I_p(FG; B)) = B(1 + I_p(FG; B))[GS(FG)]^p = B(1 + I_p(FG; B))V^p(FG)$, as desired. This completes the proof. □

Commentary. Inspired by the Intersection Remark, we note that the condition stated in the second half of Theorem 2 on G being p-mixed is necessary.

We now come to the construction of proper basic subgroups of $S(FG)/G_p$ and $V(FG)/G$. In other words, we concentrate on

Theorem 3. *Let F be perfect. Then $G_p(1 + I_p(FG; B))/G_p$ is basic in $S(FG)/G_p$, and $G(1 + I_p(FG; B))/G$ is basic in $V(FG)/G$ presuming G_t is p-primary.*

Proof. By virtue of Theorem 1, $1 + I_p(FG; B)/B_p$ is a direct sum of cyclic groups, hence so is $G_p(1 + I_p(FG; B))/G_p$ as its isomorphic group. Moreover, since from Lemma $G_p = B_pG_p^{p^n}$, with this in hand or directly, with the aid of the intersection Proposition we calculate $[G_p(1 + I_p(FG; B))] \cap S^{p^n}(FG) = [G_p(1 + I_p(FG; B))] \cap S(FG^{p^n}) = G_p^{p^n}(1 + I_p(FG^{p^n}; G^{p^n} \cap B)) = G_p^{p^n}(1 + I_p(FG^{p^n}; B^{p^n})) = [G_p(1 + I_p(FG; B))]^{p^n}$, which shows that $G_p(1+I_p(FG; B))$ is pure in $S(FG)$. Therefore by [9] we infer that so is $G_p(1 + I_p(FG; B))/G_p$ in $S(FG)/G_p$. On the other hand, $S(FG)/G_p/G_p(1 + I_p(FG; B))/G_p \cong S(FG)/G_p(1 + I_p(FG; B))$, which is divisible as an epimorphic image of the divisible group $S(FG)/(1 + I_p(FG; B))$, which has been examined above. These conclusions mean that the first half is completed.

The equality $V(FG) = GS(FG)$ leads us to $V(FG)/G \cong S(FG)/G_p$, and thus the proof of the second part is similar to the above demonstration. The theorem is proved. □

We recall that any abelian p-group is said to be starred if it has the same cardinality as its basic subgroup.

As an application of our main achievement, we formulate:

Theorem 4. *Suppose F is perfect. Then $S(FG)$ and $S(FG)/G_p$ are direct sums of divisible and starred groups.*

P r o o f. Taking into account Theorems 2 and 3, $1+I_p(FG;B)$ is basic in $S(FG)$, and $G_p(1+I_p(FG;B))/G_p$ is basic in $S(FG)/G_p$.

First of all, let $B=1$, i.e. let G be p-divisible. It is obviously true that then $S(FG)$ is divisible, hence so is $S(FG)/G_p$.

Let now $B \neq 1$. We shall argue first that $S(FG)$ must be a starred group. In order to show this, it suffices to estimate the power of $1+I_p(FG;B)$. If both F and G are finite, $S(FG)$ is finite whence starred. In the remaining case when F or G is infinite, $\aleph_0 \leqslant \max(|F|,|G|)$. Besides, we may assume that $G_p \neq 1$, otherwise $S(FG)=1$. Next, when $|G| \geqslant \aleph_0$, consider the set of elements $1+fg(1-g_p)(1-b) = 1+fg-fgg_p-fgb+fgg_pb$ where $0 \neq f \in F$, $1 \neq g \in G \setminus h[\langle g_p \rangle \times \{1,b,b^{-1}\}]$ whenever $h \in G$ and $h \neq g$, $g_p \in G_p \setminus B$, $b \in B \setminus G_p$ when $G_p \neq B_p \neq B$; or $1+fg(1-b_p)=1+fg-fgb_p$ where $0 \neq f \in F$, $1 \neq g \in G \setminus h\langle b_p \rangle$ whenever $h \in G$ and $h \neq g$, $1 \neq b_p \in B_p$ when $G_p = B_p \neq 1$ or $B = B_p \neq 1$. If $G=G_p=B$ we are done. If $|G| < \aleph_0$ but $|R| \geqslant \aleph_0$, we consider $1+f(1-g_p)(1-b)$ or $1+f(1-b_p)$, respectively, where $\{0,\pm 1\} \neq f \in F$.

In all of these situations, g_p, b and b_p are fixed elements while f and g are not. Since the above constructed elements from the group algebra are in canonical form, we obviously see that $1+fg-fgg_p-fgb+fgg_pb = 1+rh-rhg_p-rhb+rhg_pb$, respectively, $1+fg-fgb_p = 1+rh-rhb_p$, if and only if $f=r$ and $g=h$. So, $\max(|F|,|G|) = |F| \cdot |G| \leqslant |1+I_p(FG;B)|$. Finally, we infer that $|1+I_p(FG;B)| = |S(FG)|$, as promised.

For the other case, bearing in mind that $G_p(1+I_p(FG;B))/G_p \cong (1+I_p(FG;B))/B_p$, we examine the elements $1+fg(1-g_p)(1-b)$ when $B_p=1$ or $(1+fg(1-b_p))B_p$ otherwise, where f, g, g_p, b and b_p are defined as above. We treat the variant of infinite G since in the remaining one B_p must be finite, whence $|(1+I_p(FG;B))/B_p| = |1+I_p(FG;B)| \geqslant \aleph_0$ and thus the first step is applicable. The further conclusions, that we pursue, follow by the same arguments as the one given above. So, $|(1+I_p(FG;B))/B_p| = \max(|F|,|G|) = |S(FG)/G_p|$. The proof is complete. $\square$

Remark. The last result strengthens in a certain way a similar fact established by us in [6] and [7].

The following example is of an independent interest.

Example. There exists a p-primary starred summable C_λ-group of length Ω which is not totally projective.

Indeed, as is well-known by Hill-Cutler ([10]–[1]), there is a p-group G of length Ω which is C_λ and summable but not totally projective; it it not obvious whether or not this group is starred. In view of ([4], [8, Theorem 8]) along with the previous

Theorem 4, $V(FG)$ is a p-torsion summable starred C_λ-group whenever F is a perfect field with $\operatorname{char} F = p \neq 0$. But it is not totally projective, because otherwise a result of May (see, for instance, [14] or [11]) would imply that so is G, against our construction.

Problems. Does it follow that the same conclusion as in the example holds true if the starred p-groups are replaced by fully starred p-groups, that is, a more restricted class of groups?

We only mention that the same arguments as in the example are not valid because of the fact that $V(FG)$ is not fully starred since its subgroup G is not even starred. However, we conjecture that $V(FG)/G$ is fully starred under the considered circumstances.

We close the study with

Concluding discussion

When F is not perfect, it is a difficult matter to determine whether or not $1 + I_p(FG; B)$ is a basic subgroup of $S(FG)$. However, some advance in this theme is the following. Adapting the central result in [12], the countable union of all $S^{p^n}(FG)$-high subgroups of $S(FG)$ will definitely be a basic subgroup of $S(FG)$. A lot of information is contained in a high subgroup $\mathcal{H}$ of $S(FG)$, too. Indeed, $\mathcal{H}$ should be pure in $S(FG)$ and $S(FG)/\mathcal{H}$ should also be divisible; notice that $\mathcal{H}$ contains the basic subgroup of $S(FG)$ as it is well-known. Of central interest is then the problem of finding these special subgroups of $S(FG)$ (see, for instance, [8]). This, however, is a work for a future occasion.

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