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ISOMETRIES OF GENERALIZED MV-ALGEBRAS

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Abstract. In this paper we investigate the relations between isometries and direct product decompositions of generalized MV-algebras.

Keywords: generalized MV-algebra, isometry, direct product decomposition

MSC 2000: 06D35

1. Introduction

The non-commutative generalization of the notion of the MV-algebra was investigated by Georgescu and Iorgulescu [5], [6] under the name of the pseudo MV-algebra and by Rachůnek [12] under the name of the generalized MV-algebra.

For generalized MV-algebras, several equivalent systems of axioms have been used in literature. Below, we will systematically apply the relation between generalized MV-algebras and lattice ordered groups having a strong unit. This relation can be described as follows.

Let G be a lattice ordered group with a strong unit u; denote $A = [0, u]$. For $x, y \in A$ we put

$$x \oplus y = (x + y) \wedge u, \quad \neg x = u - x, \quad \sim x = -x + u, \quad 1 = u.$$

Then $\mathscr{A} = (A; \oplus, \neg, \sim, 1)$ is a generalized MV-algebra. We denote $\mathscr{A} = \Gamma(G, u)$. Dvurečenskij [4] proved that for each generalized MV-algebra $\mathscr{A}_1$ there exists a lattice ordered group G_1 with a strong unit u_1 such that $\mathscr{A}_1 = \Gamma(G_1, u_1)$.

If $\mathscr{A}$ is as above and if the operation $\oplus$ is commutative then $\mathscr{A}$ is an MV-algebra (cf., e.g., the monograph Cignoli, D'Ottaviano and Mundici [2]).

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Let $\mathscr{A}$ be a generalized MV-algebra; under the above notation, let $\mathscr{A} = \Gamma(G, u)$.

For $x, y \in A$ we put

$$\varrho(x, y) = (x \vee y) - (x \wedge y).$$

A bijection $f\colon A \to A$ is defined to be an *isometry* of $\mathscr{A}$ if for each $x, y \in A$ the following conditions are satisfied:

(i) $\varrho(x, y) = \varrho(f(x), f(y))$;

(ii) $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

In this paper we prove that there is a monomorphism of the system of all isometries of $\mathscr{A}$ into the system of all internal direct factors of $\mathscr{A}$.

If $\mathscr{A}$ is an MV-algebra, then (i)$\Rightarrow$(ii); hence in the particular case of MV-algebras, the present definition of isometry coincides with that given in [9].

Consider the following conditions for a bijection $f\colon A \to A$, where A is the underlying set of a generalized MV-algebra $\mathscr{A}$:

(i_1) f is an isometry of $\mathscr{A}$;

(ii_1) there exist $b, c \in A$ with $b \wedge c = 0$, $b \vee c = u$ such that for each $t \in A$,

$$f(t) = (-(t \wedge b) + b) \vee (t \wedge c).$$

The results of [9] and [11] yield that in the case of MV-algebras, the implication

$$(\mathrm{i}_1) \Rightarrow (\mathrm{ii}_1) \tag{1}$$

is satisfied.

We will prove that the implication (1) remains valid for generalized MV-algebras.

Further, in view of [9] (Proposition 5.3), for MV-algebras we have also

$$(\mathrm{ii}_1) \Rightarrow (\mathrm{i}_1). \tag{2}$$

For generalized MV-algebras, the relation (2) does not hold in general.

For related results concerning isometries of lattice ordered groups cf., e.g., Swamy [13], Holland [7] and the author [8]; in [13] it was assumed that the lattice ordered group under consideration is abelian.

2. Preliminaries

For lattices and lattice ordered groups we apply the notation as in Birkhoff [1] and Conrad [3].

Let $\mathscr{A}$ be a generalized MV-algebra with $\mathscr{A} = \Gamma(G, u)$ (under the above notation). Let $\leqslant$ be the partial order on A induced from the partial order in G. We put $(A; \leqslant) = \ell(\mathscr{A})$.

For $a \in A$ we denote by $\mathscr{A}_a$ the algebraic structure $([0, a], \oplus_a, \sim_a, a)$, where for each $x, y \in [0, a]$ we have

$$x \oplus_a y = (x + y) \wedge a, \quad \neg_a x = a - x, \quad \sim_a x = -x + a.$$

Then $\mathscr{A}_a$ is a generalized MV-algebra; we call it an *interval subalgebra* of $\mathscr{A}$.

The direct product of generalized MV-algebras is defined in the usual way; we apply the symbol $\prod\limits_{i \in I} \mathscr{A}_i$ or, if $I = \{1, 2, \ldots, n\}$, also the symbol $\mathscr{A}_1 \times \ldots \times \mathscr{A}_n$. For the notion of internal direct product decomposition of a generalized MV-algebra cf. [10].

For our purposes, it suffices to consider here only two-factor internal direct decompositions of a generalized MV-algebra $\mathscr{A}$. These can be defined as follows.

Let $\mathscr{A}_a$ and $\mathscr{A}_b$ be interval subalgebras of $\mathscr{A}$. For each $x \in A$ put $\varphi(x) = (x \wedge a, x \wedge b)$. Assume that φ is an isomorphism of $\mathscr{A}$ onto the direct product $\mathscr{A}_a \times \mathscr{A}_b$. Then we say that $\varphi\colon \mathscr{A} \to \mathscr{A}_a \times \mathscr{A}_b$ is an *internal direct product decomposition* of $\mathscr{A}$ and that $\mathscr{A}_a, \mathscr{A}_b$ are *internal direct factors* of $\mathscr{A}$. The element $x \wedge a$ is the *component* of x in the internal direct factor $\mathscr{A}_a$; we denote it also by $x(\mathscr{A}_a)$.

From each direct product decomposition of $\mathscr{A}$ we obtain by a simple construction an internal direct product decomposition of $\mathscr{A}$ (cf. [10]).

3. Direct product decompositions corresponding to isometries

Below we suppose that $\mathscr{A}$ is a generalized MV-algebra and that, under the above notation, $\mathscr{A} = \Gamma(G, u)$.

Lemma 3.1 (Cf. [8], Lemma 1.1). *Assume that G is abelian. Then for each $a, b, x \in G$, the following conditions are equivalent:*

(α) $\varrho(a, b) = \varrho(a, x) + \varrho(x, b)$;
(β) $x \in [a \wedge b, a \vee b]$.

Proposition 3.2. *Assume that $\mathscr{A}$ is an MV-algebra. Let $f\colon A \to A$ be a bijection. Let the conditions* (i) *and* (ii) *be as in* Section 1. *Suppose that* (i) *is valid for each $x, y \in A$. Then* (ii) *holds for each $x, y \in A$.*

Proof. Since $\mathscr{A}$ is an MV-algebra, G is abelian. Let $x, y, t \in A$. There exists $v \in A$ with $t = f(v)$. The relation

(1) $$t \in f([x \wedge y, x \vee y])$$

is equivalent to

(2) $$v \in [x \wedge y, x \vee y].$$

In view of 3.1, (2) holds iff

(3) $$\varrho(x, y) = \varrho(x, v) + \varrho(v, y).$$

According to (i), (3) is equivalent to

(4) $$\varrho(f(x), f(y)) = \varrho(f(x), f(v)) + \varrho(f(v), f(y)).$$

By applying 3.1 again we conclude that (4) is equivalent to

(5) $$f(v) \in [f(x) \wedge f(y), f(x) \vee f(y)].$$

Hence the relations (1) and (5) are equivalent. Therefore (ii) is valid. □

From 3.2 it follows that in the case of MV-algebras, the definition of isometry given above coincides with the definition of isometry from [9] (where only the condition (i) was imposed).

Lemma 3.3. *Let $a, b \in A$, $a \wedge b = 0$, $a \vee b = u$. Then $\mathscr{A}$ is an internal direct product of generalized MV-algebras $\mathscr{A}_a$ and $\mathscr{A}_b$.*

Proof. For each $x \in A$ we put $\varphi(x) = (x \wedge a, x \wedge b)$. From the fact that the lattice $\ell(\mathscr{A})$ is distributive we conclude that φ is an isomorphism of $\ell(\mathscr{A})$ onto the direct product $\ell(\mathscr{A}_a) \times \ell(\mathscr{A}_b)$. From this and from the results of [10] we obtain that φ is an internal direct product decomposition of $\mathscr{A}$; the corresponding internal direct factors are $\mathscr{A}_b$ and $\mathscr{A}_b$. □

Lemma 3.4. *Let f be an isometry of $\mathscr{A}$. Put $f(0)=a$, $f(u)=b$. Then $a\wedge b=0$ and $a\vee b=u$.*

P r o o f. Denote $a\wedge b=p$, $a\vee b=q$. We have

$$0\leqslant p\leqslant q\leqslant u.$$

Further, $\varrho(0,u)=u$ and $\varrho(a,b)=q-p$. Hence $q-p=u$. If $0<p$ or $q<u$, then $q-p<u$, which is a contradiction. Therefore $p=0$ and $q=u$. □

Lemma 3.5. *Let f,a and b be as in 3.4. Then $\mathscr{A}$ is an internal direct product of $\mathscr{A}_a$ and $\mathscr{A}_b$.*

P r o o f. This is a consequence of 3.3 and 3.4. □

Let us apply the notation as above.

Lemma 3.6. $f(a)=0$ *and* $f(b)=u$.

P r o o f. For $x\in A$ we denote

$$a\wedge x=x_1,\quad b\wedge x=x_2,\quad a\vee x=x_3,\quad b\vee x=x_4.$$

(Cf. Fig. 1.)

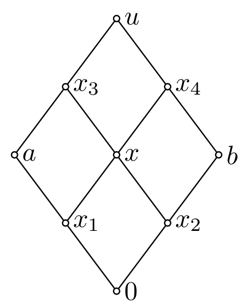

Fig. 1

a) Put $f(a)=x$. Since $\varrho(0,a)=a$ we get

$$a=\varrho(f(0),f(a))=\varrho(a,x)=x_3-x_1=(x_3-a)+(a-x_1).$$

From $x_3-a=x_2$ we obtain $x_2\leqslant a$; but $a\wedge x_2=0$, whence $x_2=0$. Thus $a=a-x_1$, yielding $x_1=0$. Obviously, $x=x_1\vee x_2$, therefore $x=0$ and so $f(a)=0$.

b) Now we put $f(b)=x$. From $\varrho(a,b)=u$ we obtain $\varrho(f(a),f(b))=u$, whence $\varrho(0,x)=u$. Clearly $\varrho(0,x)=x$ and therefore $f(b)=u$. □

Lemma 3.7. *Let x_2 and x_3 be as in* Fig. 1. *Then $f(x_2) = x_3$ and $f(x_3) = x_2$.*

P r o o f. a) We have $x_2 \in [0, b]$, hence $\varrho(x_2, b) = b - x_2$. Further,

$$f(x_2) \in f([0 \wedge b, 0 \vee b]) = [f(0) \wedge f(b), f(0) \vee f(b)] = [a \wedge u, a \vee u] = [a, u],$$
$$\varrho(x_2, b) = \varrho(f(x_2), f(b)) = \varrho(f(x_2), u) = u - f(x_2).$$

Hence we obtain

$$b - x_2 = u - f(x_2),$$
$$f(x_2) = x_2 - b + u.$$

Since $u = a \vee b = a + b = b + a$, we have $f(x) = x_2 + a$. From $x_2 \wedge a = 0$ we now infer $f(x) = x_2 \vee a = x_3$.

b) Since $x_3 \in [a, u]$ we get

$$f(x_3) \in [f(a) \wedge f(u), f(a) \vee f(u)] = [0, b].$$

Further, $\varrho(a, x_3) = x_3 - a$ and

$$\varrho(a, x_3) = \varrho(f(a), f(x_3)) = \varrho(0, f(x_3)) = f(x_3),$$
$$x_3 - a = f(x_3).$$

But (cf. Fig. 1) $x_3 - a = x_2$, whence $f(x_3) = f(x_2)$. □

Theorem 3.8. *Let f be an isometry of a generalized MV-algebra $\mathscr{A}$. Put $f(0) = a$, $f(u) = b$. Then b is a complement of a in the lattice $\ell(\mathscr{A})$ and for each $x \in A$ the formula*

$$f(x) = (-(x \wedge a) + a) \vee (x \wedge b)$$

is valid.

P r o o f. In view of 3.4, b is a complement of a in $\ell(\mathscr{A})$. Let $x \in A$. We apply the notation as in Fig. 1. We have $x \in [x_2, x_3]$, hence

$$f(x) \in [f(x_2) \wedge f(x_3), f(x_2) \vee f(x_3)].$$

Thus in view of 3.7, $f(x) \in [x_2, x_3]$. Further,

$$\varrho(x, x_2) = x - x_2 = x_1,$$
$$\varrho(x, x_2) = \varrho(f(x), f(x_2)) = \varrho(f(x), x_3) = x_3 - f(x).$$

Hence $x_1 = x_3 - f(x)$ and so $f(x) = -x_1 + x_3$. We have (cf. Fig. 1)

$$x_3 = x \vee a = x_2 \vee a, \quad x_2 \wedge a = 0,$$

thus

$$x_2 \vee a = x_2 + a = a + x_2,$$
$$f(x) = -x_1 + a + x_2.$$

Also, $(-x_1 + a) \wedge x_2 = 0$, thus $(-x_1 + a) + x_2 = (-x_1 + a) \vee x_2$. Therefore

$$f(x) = (-(x \wedge a) + a) \vee (x \wedge b).$$

□

We denote by $F(\mathscr{A})$ the set of all isometries of $\mathscr{A}$. Further, let $D(\mathscr{A})$ be the system of all internal direct factors of $\mathscr{A}$. For $f \in F(\mathscr{A})$ we put $\chi(f) = \mathscr{A}_a$, where a is as in 3.8.

Proposition 3.9. *The mapping χ is a monomorhpism of $F(\mathscr{A})$ into $D(\mathscr{A})$.*

Proof. In view of 3.5, χ is a mapping of $F(\mathscr{A})$ into $D(\mathscr{A})$.

Let a, b and f be as in 3.8. Since the lattice $\ell(\mathscr{A})$ is distributive, each of its elements has at most one complement. Thus b is uniquely determined by a. Therefore, in view of 3.8, f is also uniquely determined by a. Therefore χ is a monomorphism. □

Lemma 3.10. *Let $\mathscr{A}$ be an MV-algebra. Put $f(x) = u - x$ for each $x \in A$. Then f is an isometry of $\mathscr{A}$.*

Proof. This is a consequence of Proposition 5.3 in [9]. □

Lemma 3.11. *Assume that a generalized MV-algebra $\mathscr{A}$ is an internal direct product of generalized MV-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$. For $x \in A$ and $i \in \{1, 2\}$ let x_i be the component of x in $\mathscr{A}_i$; further, let f_i be an isometry of $\mathscr{A}_i$. We put $f(x) = y$ so that $y_i = f_i(x_i)$ $(i = 1, 2)$. Then f is an isometry of $\mathscr{A}$.*

Proof. The assertion follows from the fact that all operations in $\mathscr{A}$ are performed component-wise. □

Proposition 3.12. *Let $\mathscr{A}$ be a generalized MV-algebra. Assume that $a, b \in A$ and that b is a complement of a in the lattice $\ell(\mathscr{A})$. Suppose that the operation $\oplus_a$ in $\mathscr{A}_a$ is commutative. For each $x \in A$ put*

$$(*) \qquad f(x) = (-(x \wedge a) + a) \vee (x \wedge b).$$

Then f is an isometry of $\mathscr{A}$.

Proof. Denote $\mathscr{A}_1 = \mathscr{A}_a$, $\mathscr{A}_2 = \mathscr{A}_b$. Then $\mathscr{A}$ is an internal direct product of $\mathscr{A}_1$ and $\mathscr{A}_2$ (cf. 3.5). For $x \in A$ and $i \in \{1, 2\}$ let x_i be as in 3.11. Thus

$$x_1 = x \wedge a, \quad x_2 = x \wedge b.$$

If $x \in A_1$ ($x \in A_2$), then $x_1 = x$ ($x_2 = x$).

For $x \in A_1$ we put $f_1(x) = a - x$. According to 3.10 and in view of the fact that $\mathscr{A}_1$ is an MV-algebra, f_1 is an isometry of $\mathscr{A}_1$. Further, let f_2 be the identical mapping on A_2; hence f_2 is an isometry of $\mathscr{A}_2$.

For each $x \in A$ let $f(x)$ be as in $(*)$. Then we have

$$(f(x)_1 = -(x \wedge a) + a = -x_1 + a = f_1(x_1),$$
$$(f(x))_2 = x \wedge b = x_2 = f_2(x_2).$$

Thus in view of 3.11, f is an isometry of $\mathscr{A}$. □

Corollary 3.13. *Let $\mathscr{A}$ be a generalized MV-algebra. Let the mapping $\chi\colon F(\mathscr{A}) \to D(\mathscr{A})$ be as above. Let $a \in A$ be such that the operation $\oplus_a$ in $[0, a]$ is commutative and that a has a complement in the lattice $\ell(\mathscr{A})$. Then $\mathscr{A}_a \in \chi(F(\mathscr{A}))$.*

Consider the following condition for $\mathscr{A}$:

(+) Whenever a_1 and a_2 are comparable elements of A, then $a_1 + a_2 = a_2 + a_1$.

Lemma 3.14. *Assume that $\mathscr{A}$ satisfies the condition (+). Then the operation $\oplus$ in $\mathscr{A}$ is commutative.*

Proof. Let $x, y \in A$. Denote $x \wedge y = q$, $x_1 = -q + x$, $y_1 = -q + y$. Then q, x_1 and y_1 belong to A and $x_1 \wedge y_1 = 0$; hence $x_1 + y_1 = y_1 + x_1$. In view of (+) we have

$$\begin{aligned} x + y &= (q + x_1) + (q + y_1) = q + (q + x_1) + y_1 = q + (q + y_1) + x_1 \\ &= (q + y_1) + q + x_1 = y + x, \\ x \oplus y &= (x + y) \wedge u = (y + x) \wedge u = y \oplus x. \end{aligned}$$

□

Corollary 3.15. *Assume that $\mathscr{A}$ satisfies the condition* $(+)$. *Then the lattice ordered group G is abelian.*

Now suppose that f is an isometry of $\mathscr{A}$. Let a and b be as in 3.8. Consider the generalized MV-algebra $\mathscr{A}_a$.

Lemma 3.16. *For each $x \in \mathscr{A}_a$, $-x + a = a - x$.*

Proof. Let $x \in \mathscr{A}_a$. Then $x \wedge b = 0$ and $x \wedge a = x$, whence in view of 3.8, $f(x) = -x + a$. We have (cf. 3.6)

$$\varrho(x, a) = a - x, \quad \varrho(f(x), f(a)) = \varrho(f(x), 0) = f(x) = -x + a.$$

Therefore $a - x = -x + a$. □

Lemma 3.17. *Let $x, y \in \mathscr{A}_a$. Then*

$$(x \vee y) - (x \wedge y) = -(x \wedge y) + (x \vee y).$$

Proof. We have $f(x) = -x + a$, $f(y) = -y + a$. Further,

$$\varrho(x, y) = (x \vee y) - (x \wedge y),$$
$$\varrho(f(x), f(y)) = ((-x + a) \vee (-y + a)) - ((-x + a) \wedge (-y + a)).$$

Since

$$(-x + a) \vee (-y + a) = ((-x) \vee (-y)) + a = -(x \wedge y) + a,$$
$$(-x + a) \wedge (-y + a) = -(x \vee y) + a,$$

we get

$$\varrho(f(x), f(y)) = (-(x \wedge y) + a) + (-a + (x \vee y)) = -(x \wedge y) + (x \vee y).$$

Therefore we have $(x \vee y) - (x \wedge y) = -(x \wedge y) + (x \wedge y)$. □

Corollary 3.18. *$\mathscr{A}_a$ satisfies the condition* $(+)$.

Proposition 3.19. *Let $\mathscr{A}$ be a generalized MV-algebra. Let f, a and b be as in 3.8. Then the operation $\oplus_a$ in $\mathscr{A}_a$ is commutative.*

Proof. This is a consequence of 3.18 and 3.14. □

For a generalized MV-algebra $\mathscr{A}$ we denote by $D_c(\mathscr{A})$ the set of all internal direct factors X of $\mathscr{A}$ such that the operation $\oplus$ in X is commutative. Let $\chi\colon F(\mathscr{A}) \to D(A)$ be as above.

From 3.13 and 3.19 we obtain

Proposition 3.20. $\chi(F(\mathscr{A})) = D_c(\mathscr{A})$.

Thus there exists a one-to-one correspondence between isometries of $\mathscr{A}$ and elements of $D_c(\mathscr{A})$.

In connection with 3.19 let us consider the following example. Let G_1 be a lattice ordered group which fails to be abelian. Let $G = Z \circ G_1$, where $\circ$ denotes the operation of the lexicographic product. Put $u = (1, 0)$, $\mathscr{A} = \Gamma(G, u)$, $a = u$ and $b = 0$. Then a is a complement of b in $\ell(\mathscr{A})$ and $\mathscr{A}_a = \mathscr{A}$. Hence the operation $\oplus_a$ coincides with $\oplus$ and it is clear that this operation fails to be commutative. For each $x \in A$ let $f(x)$ be as in 3.8. Then in view of 3.19, f fails to be an isometry on $\mathscr{A}$. Hence, by applying the notation from Section 2, we conclude that the implication $(\mathrm{ii}_1)\Rightarrow(\mathrm{i}_1)$ is not valid, in general, for generalized MV-algebras.

The following theorem generalizes the result of [11].

Theorem 3.21. *Let f be an isometry of a generalized MV-algebra $\mathscr{A}$. Then $f(f(x)) = x$ for each $x \in A$.*

Proof. Let $x \in A$ and let a, b be as in 3.8. Hence we have

$$f(x) = (-(x \wedge a) + a) \vee (x \wedge b).$$

Put $f(x) = y$. Then

$$f(y) = (-(y \wedge a) + a) \vee (y \wedge b).$$

Since $-(x \wedge a) + a \leqslant a$, we get

$$(-(x \wedge a) + a) \wedge b = 0$$

and thus

$$y \wedge b = (x \wedge b) \wedge b = x \wedge b. \tag{1}$$

Further, in view of $(x \wedge b) \wedge a = 0$ we obtain

$$y \wedge a = (-(x \wedge a) + a) \wedge a = -(x \wedge a) + a.$$

In view of 3.16,

$$-(x \wedge a) + a = a - (x \wedge a).$$

This yields

$$(-(y \wedge a) + a = -(a - (x \wedge a)) + a = x \wedge a.$$

We get

$$f(y) = (x \wedge a) \vee (x \wedge b) = x \wedge (a \vee b) = x \wedge u = x,$$

whence $f(f(x)) = x$. □

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