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Piotr Kot

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BOUNDARY FUNCTIONS IN $L^2H(\mathbb{B}^n)$

Piotr Kot, Kraków

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Abstract. We solve the Dirichlet problem for line integrals of holomorphic functions in the unit ball:

For a function u which is lower semi-continuous on $\partial\mathbb{B}^n$ we give necessary and sufficient conditions in order that there exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ such that

$$u(z) = \int_{|\lambda|<1} |f(\lambda z)|^2 \, \mathrm{d}\mathfrak{L}^2(\lambda).$$

Keywords: boundary behavior of holomorphic functions, exceptional sets, boundary functions, computed tomography, Dirichlet problem

MSC 2000: 30B30

1. Preface

The main topic of this paper is centered around the following question:

What features of a function f can be recovered from a given collection of line integrals of f? This mathematical problem is encountered in a growing number of diverse settings in medicine, science and technology ranging from the famous application in diagnostic radiology to research in quantum optics. Especially this issue is often discussed in computed tomography (see [2]).

This paper deals with boundary functions. A function u is called a boundary function for a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ if

$$u(z) = \int_{|\lambda|<1} |f(\lambda z)|^2 \, \mathrm{d}\mathfrak{L}^2(\lambda)$$

for $z \in \partial\mathbb{B}^n$ where $\mathrm{d}\mathfrak{L}^2$ denotes the two-dimensional Lebesgue measure on the unit disc $\mathbb{D} := \{|\lambda| < 1\} \subset \mathbb{C}$. Let us observe that the function u can have value ∞ at some points.

The introduction of above definition was inspired by questions posed by Peter Pflug and Jaques Chaumat.

In the 80s Peter Pflug [10] posed the question whether there exists a space $\Omega \subset \mathbb{C}^n$, a complex subspace M in $\mathbb{C}^n$ and a holomorphic function f in Ω integrable with a square such that $f|_{M\cap\Omega}$ is non-integrable with a square.

A similar question was posed by Jaques Chaumat [1] in the late 80s: whether there exists a holomorphic function f in a ball $\mathbb{B}^n$ such that for any linear, complex subspace M in $\mathbb{C}^n$ a holomorphic function $f|_{M\cap\mathbb{B}^n}$ is non-integrable with a square.

The questions just mentioned inspired further investigation among the authors [3], [4], [5], [6], [7], [8], [9]. The papers [4], [5], [6] deal mainly with domains $\Omega \subset \mathbb{C}^{n+m}$ and holomorphic functions $f \in \mathbb{O}(\Omega) \cap L^2(\Omega)$, non-integrable along the directions defined by the formula

$$\Omega_w := \{z \in \mathbb{C}^n : (z, w) \in \Omega\}.$$

Therefore the exceptional set $\widetilde{E}(\Omega, f)$ is defined as

$$\widetilde{E}(\Omega, f) = \{w \in \mathbb{C}^m : f|_{\Omega_w} \notin L^2(\Omega_w)\}.$$

Non-integrable functions along complex lines with the point 0 can also be considered. Papers that consider this problem are [3], [7], [8], [9]. Due to [3], [7] we know that for a convex domain Ω with a boundary of the class C^1 a holomorphic function f non-integrable with a square along any real manifold M of the class C^1 crossing transversally a boundary Ω can be created.

Observe that a set of directions composed of complex lines with the point 0 can be identified with points in the sphere $\partial\mathbb{B}^n$. Therefore the definition of the exceptional set $E(f)$ for a holomorphic function f on $\mathbb{B}^n$ can be presented as follows:

$$E(f) = \left\{z \in \partial\mathbb{B}^n : \int_{|\lambda|<1} |f(\lambda z)|^2 \, \mathrm{d}\mathfrak{L}^2(\lambda) = \infty\right\}.$$

Let E be any circular subset of type G_δ and F_σ in $\partial\mathbb{B}^n$. In the paper [9] we presented a construction of a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ for which $E = E(f)$.

Boundary functions enable us to prove a much stronger result describing the exceptional sets. We also present the solution of the Dirichlet problem for plurisubharmonic functions.

First observe that if u is a boundary function then u is lower semi-continuous and $u(\lambda z) = u(z)$ for $|\lambda| = 1$ and $z \in \partial\mathbb{B}^n$. Our main result describes boundary functions by means of homogeneous polynomials[1].

[1] These polynomials may have degree equal to 0.

Theorem 2.9. *Let u be a lower semi-continuous function on $\partial\mathbb{B}^n$ such that $u(\lambda z) = u(z) \geqslant 0$ when $|\lambda| = 1$ and $z \in \partial\mathbb{B}^n$. The following conditions are equivalent:*

(1) *There exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ for which u is a boundary function.*

(2) *There exist homogeneous polynomials $p_1, \ldots, p_m$ such that*
   (a) $u^{-1}(0) = \{z \in \partial\mathbb{B}^n : \ p_1(z) = p_2(z) = \ldots = p_m(z) = 0\}$,
   (b) $u(z) \geqslant \sum\limits_{j=1}^{m} |p_j(z)|^2$ *for* $z \in \partial\mathbb{B}^n$.

In particular, observe that any continuous and positive function with the same values on circles is a boundary function.

It appears that boundary functions can be used to describe the exceptional sets:

Theorem 3.1. *Let E be a circular subset of $\partial\mathbb{B}^n$. Then there exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ such that $\int_{\mathbb{B}^n \setminus \Lambda(E)} |f|^2 \,\mathrm{d}\mathfrak{L}^{2n} < \infty$ iff E is of type G_δ in $\partial\mathbb{B}^n$. Here $\Lambda(E) = \{\lambda z\colon |\lambda| < 1, z \in E\}$, $E = E(f)$ and $\mathrm{d}\mathfrak{L}^{2n}$ is the $2n$-dimensional Lebesgue measure.*

Moreover, we solve the Dirichlet problem for plurisubharmonic functions:

Theorem 3.2. *If u is a continuous function on $\partial\mathbb{B}^n$ such that $u(\lambda z) = u(z)$, then there exists a constant $c \in \mathbb{R}$ and a sequence of homogeneous polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that p_m is of the degree m and $u(z) = c + \sum\limits_{m\in\mathbb{N}} |p_m(z)|^2$ for $z \in \partial\mathbb{B}^n$. In particular, the function $g(z) = c + \sum\limits_{m\in\mathbb{N}} |p_m(z)|^2$ is continuous on $\overline{\mathbb{B}^n}$, real analytic and plurisubharmonic on $\mathbb{B}^n$.*

2. Boundary functions

Let $\#A$ denote the number of elements in a set A. Let us consider a unitary invariant pseudo-metric ϱ on the boundary of a unit ball $\partial\mathbb{B}^n$:

$$\varrho(z_1, z_2) := \sqrt{1 - |\langle z_1, z_2\rangle|}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard complex scalar product. On a unit ball $\partial\mathbb{B}^n$ there exists a natural, unitary invariant Lebesgue measure. Let us normalize it so that the whole sphere's measure $\partial\mathbb{B}^n$ is equal to 1 and denote this measure by σ. Let

$$K_\varrho(z_0, r) := \{z \in \partial\mathbb{B}^n : \ \varrho(z_0, z) < r\}.$$

Definition 2.1. Every function $u\colon X \to (-\infty,\infty]$ on a metric space X fulfilling one of the equivalent conditions (1), (2), (3) is called lower semicontinuous:

(1) there exists a sequence of functions h_i which are continuous on X such that $h_i \leqslant h_{i+1} \leqslant \ldots \leqslant \lim\limits_{i\to\infty} h_i = u$;

(2) $u^{-1}((a,\infty])$ is an open set for $a \in (-\infty,\infty]$;

(3) $u(x) \leqslant \liminf\limits_{z\to x} u(z)$ for $x \in X$.

Definition 2.2. We say that a function u is a boundary function for a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ if

$$u(z) = \int_{|\lambda|<1} |f(\lambda z)|^2 \, \mathrm{d}\mathfrak{L}^2(\lambda)$$

for $z \in \partial\mathbb{B}^n$.

Lemma 2.3. *The inequality $r^{2n-2} \leqslant \sigma(K_\varrho(z_0,r)) \leqslant 2^{n-1}r^{2n-2}$ holds.*

Proof. It is enough to use [11, 1.4.4]. □

Definition 2.4. Let $\alpha > 0$. A subset $A \subset \partial\mathbb{B}^n$ is called α-separated iff $\varrho(z_1,z_2) > \alpha$ for different elements $z_1, z_2 \in A$.

Remark. From the compactness of $\partial\mathbb{B}^n$ it easily follows that for $\alpha > 0$ every α-separated $A \subset \partial\mathbb{B}^n$ is finite.

Lemma 2.5. *Assume that a set A is 2α-separated and $A = \{\xi_1,\ldots,\xi_s\} \subset \partial\mathbb{B}^n$. For $z \in \partial\mathbb{B}^n$ define*

$$A_m(z) := \{\xi \in A\colon\ \alpha m \leqslant \varrho(z,\xi) < \alpha(m+1)\}.$$

Then for $m = 1,2,\ldots$ the set $A_m(z)$ has up to $2^{n-1}(m+2)^{2n-2}$ elements. The set $A_0(z)$ has up to one element. Moreover $s \leqslant \alpha^{2-2n}$.

Proof. It is enough to use the same arguments as in [12, Lemma 1] and [9, Lemma 3.2]. □

Lemma 2.6. *Fix real numbers α, β such that $\beta > \alpha > 0$. There exists a constant $K = K(\alpha,\beta)$ such that $1 \leqslant K \leqslant 2^{n-1}(\beta\alpha^{-1}+1)^{2n-2}$ and for any $t > 0$, we have the following property:*

If $A \subset \partial\mathbb{B}^n$ is an αt-separated set, then A can be divided into up to K disjoint βt-separated subsets.

Proof. It is enough to use the same argument as in [12, Lemma 2]. □

Theorem 2.7. *There exists a positive constant q and a natural number K such that if a function g is continuous on $\partial\mathbb{B}^n$ and $g(z) = g(\lambda z) > 0$ when $|\lambda| = 1$, $z \in \partial\mathbb{B}^n$, then there exists a natural number N_0 and a sequence of homogeneous polynomials $\{p_m\}_{m\in\mathbb{N}}$ of degree m such that*

(1) *$K|p_m(z)|^2 < g(z)$ for $m > N_0$ and $z \in \partial\mathbb{B}^n$,*

(2) *for $m > N_0$ and $z \in \partial\mathbb{B}^n$ there exists $j(m,z) \in \{0,1,\ldots,K-1\}$ such that $qg(z) < |p_{mK+j(m,z)}(z)|^2$,*

(3) *$qg(z) < \sum\limits_{j=0}^{K-1} |p_{mK+j}(z)|^2 < g(z)$ for $m > N_0$ and $z \in \partial\mathbb{B}^n$.*

Proof. There exists $C > 2$ such that

$$\sum_{m=1}^{\infty} 2^n (m+2)^{2n} \exp\Big(-\frac{C^2m^2}{16}\Big) \leqslant \frac{1}{8}\sqrt{\frac{9}{5}}. \tag{2.1}$$

Assume $\alpha = 1$, $\beta = C$ in Lemma 2.6. Hence there exists a natural number K for which the assertion of Lemma 2.6 holds. Further we define

$$q := \frac{1}{16K}$$

and show that for such a choice of K and q the implication required in Theorem 2.7 holds. So let g be a given function continuous on $\partial\mathbb{B}^n$ and fulfilling $g(z) = g(\lambda z)$ for all $z \in \partial\mathbb{B}^n$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then $\inf\limits_{z\in\partial\mathbb{B}^n} g(z) =: g_0 > 0$ and the uniform continuity of g on the compact $\partial\mathbb{B}^n$ implies that there exists $\delta \in (0,1)$ such that $\|\xi - \eta\| < \sqrt{2\delta} \Rightarrow |g(\xi) - g(\eta)| < \frac{1}{9}g_0$ (here $\|\cdot\|$ is the euclidean norm).

Since $|\langle\xi;\eta\rangle| = \langle\xi e^{i\theta};\eta\rangle$, where $\theta = -\arg\langle\xi;\eta\rangle$, we have, as $\|\xi e^{i\alpha}\| = \|\eta\| = 1$, $\|\xi e^{i\theta} - \eta\| = \sqrt{2\,(1 - \operatorname{Re}\langle\xi e^{i\theta};\eta\rangle)} = \sqrt{2}\varrho(\xi,\eta)$ whenever $\eta \neq \lambda\xi$ for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$. So in this case $\varrho(\xi,\eta) < \sqrt{\delta} \Rightarrow \|\xi e^{i\theta} - \eta\| < \sqrt{2\delta}$ and so $|g(\xi e^{i\theta}) - g(\eta)| < \frac{1}{9}g_0 \leqslant \frac{1}{9}g(\eta)$. Finally, the following assertion holds: There exists a $\delta \in (0,1)$ such that for every $\xi,\eta \in \partial\mathbb{B}^n$

$$\varrho(\eta,\xi) < \sqrt{\delta} \Rightarrow |g(\eta) - g(\xi)| < \frac{g_0}{9} = \frac{\inf_{z\in\partial\mathbb{B}^n} g(z)}{9} \leqslant \frac{g(\eta)}{9}, \tag{2.2}$$

since $g(\xi e^{i\theta}) = g(\xi)$ for $\xi \in \partial\mathbb{B}^n$ and $\theta \in \mathbb{R}$.

Now let N be so far an arbitrary positive integer and let $A = \{\xi_1,\ldots,\xi_s\}$ be a maximal $1/\sqrt{4NK}$-separated subset $\partial\mathbb{B}^n$. By Lemma 2.6 there exist K disjoint $C/\sqrt{4NK}$-separated sets $A_0,\ldots,A_{K-1}$ such that $\bigcup\limits_{j=0}^{K-1} A_j = A$. By the second assertion of Lemma 2.5 one has $\#A_j \leqslant \big(2\sqrt{4NK}/C\big)^{2n-2} \leqslant 4^n N^n K^n$, because C was chosen such that $C > 2$.

Now we are in position to define the polynomials $p_m(z)$ for $m = 1, 2, \ldots$. We define for $j = 0, \ldots, K-1$:

$$p_{NK+j}(z) = \sum_{\xi \in A_j} \sqrt{\frac{g(\xi)}{2K}} \langle z; \xi \rangle^{NK+j}.$$

Let

$$B_{j,m}(z) := \Big\{ \xi \in A_j \colon \frac{Cm}{2\sqrt{4NK}} \leqslant \varrho(z, \xi) < \frac{C(m+1)}{2\sqrt{4NK}} \Big\}$$

for $z \in \partial \mathbb{B}^n$ and define, for $j = 0, \ldots, K-1$:

$$h_{NK+j}(z) = \sum_{\substack{m=1 \\ \xi \in B_{j,m}(z)}}^{\infty} \sqrt{\frac{g(\xi)}{2K}} \left| \langle z; \xi \rangle \right|^{NK+j}.$$

By the assertion of Lemma 2.5

$$\# B_{j,m}(z) \leqslant 2^{n-1}(m+2)^{2n-2} \leqslant 2^n (m+2)^{2n}$$

and so

$$\sum_{m=1}^{\infty} \# B_{j,m}(z) \exp\Big(- \frac{C^2 m^2}{16} \Big) \leqslant \sum_{m=1}^{\infty} 2^n (m+2)^{2n} \exp\Big(- \frac{C^2 m^2}{16} \Big) \leqslant \frac{1}{8} \sqrt{\frac{9}{5}} \tag{2.3}$$

by (2.1).

Now, we give an upper estimate of $h_{NK+j}(z)$. We have

$$h_{NK+j}(z) \leqslant \sum_{\substack{m=1 \\ \xi \in B_{j,m}(z) \\ \varrho(z,\xi) < \sqrt{\delta}}}^{\infty} \sqrt{\frac{g(\xi)}{2K}} \left| \langle z; \xi \rangle \right|^{NK} + \sum_{\substack{m=1 \\ \xi \in B_{j,m}(z) \\ \varrho(z,\xi) \geqslant \sqrt{\delta}}}^{\infty} \sqrt{\frac{g(\xi)}{2K}} \left| \langle z; \xi \rangle \right|^{NK}.$$

By (2.2) the first sum is less than $\sqrt{\frac{5}{9} g(z) K^{-1}} \sum_{m=1, \xi \in B_{j,m}(z)}^{\infty} \left| \langle z; \xi \rangle \right|^{NK}$. But $\xi \in B_{j,m}(z)$ implies $\varrho(z, \xi) \geqslant Cm/2\sqrt{4NK}$ and so $|\langle z; \xi \rangle| \leqslant 1 - C^2 m^2 / 16NK \leqslant \exp(-C^2 m^2 / 16NK)$ because $1 - C^2 m^2 / 16NK < 1$.

Hence by (2.3) we have

$$\sum_{\substack{m=1 \\ \xi \in B_{j,m}(z) \\ \varrho(z,\xi) < \sqrt{\delta}}}^{\infty} \sqrt{\frac{g(\xi)}{2K}} \left| \langle z; \xi \rangle \right|^{NK} < \sqrt{\frac{5g(z)}{9K}} \sum_{m=1}^{\infty} \# B_{j,m}(z) \exp\Big(- \frac{C^2 m^2}{16} \Big)$$

$$\leqslant \sqrt{\frac{5g(z)}{9K}} \frac{1}{8} \sqrt{\frac{9}{5}} \leqslant \frac{1}{8} \sqrt{\frac{g(z)}{K}}.$$

If $\varrho(z,\xi) \geqslant \sqrt{\delta}$, then $|\langle z;\xi\rangle| \leqslant 1-\delta$. Moreover, $\sum\limits_{m=1}^{\infty} \#B_{j,m}(z) \leqslant \#A_j \leqslant 4^n N^n K^n$ and so

$$\sum_{\substack{m=1 \\ \xi\in B_{j,m}(z) \\ \varrho(z,\xi)\geqslant\sqrt{\delta}}}^{\infty} \sqrt{\frac{g(\xi)}{2K}}\, |\langle z;\xi\rangle|^{NK} \leqslant 4^n N^n K^n \sqrt{\frac{\|g\|_\infty}{2K}}\, (1-\delta)^{NK},$$

where $\|g\|_\infty = \max\limits_{z\in\partial\mathbb{B}^n} |g(z)|$. Since $0 < 1-\delta < 1$, there exists $N_1 \in \mathbb{N}$ such that for every $N > N_1$and $z \in \partial\mathbb{B}^n$ we have $4^n N^n K^n \sqrt{\|g\|_\infty/2K}\,(1-\delta)^{NK} < \frac{1}{8}\sqrt{g_0/K} \leqslant \frac{1}{8}\sqrt{g(z)/K}$. So we have proved that for every $z \in \partial\mathbb{B}^n$ and every $N > N_1$

$$h_{NK+j}(z) \leqslant \frac{1}{8}\sqrt{\frac{g(z)}{K}} + \frac{1}{8}\sqrt{\frac{g(z)}{K}} = \frac{1}{4}\sqrt{\frac{g(z)}{K}}.$$

Let N_2 be a natural number such that $N_2 > N_1$ and $C/2\sqrt{4NK} < \sqrt{\delta}$ for $N > N_2$. As $B_{j,0}(z)$ contains at most one element and $\varrho(\xi,z) < C/2\sqrt{4NK} < \sqrt{\delta}$ for $\xi \in B_{j,0}(z)$ and $N > N_2$, we have for $N > N_2 > N_1$ and every $z \in \partial\mathbb{B}^n$

$$\begin{aligned} |p_{NK+j}(z)| &\leqslant \sum_{\xi\in B_{j,0}(z)} \sqrt{\frac{g(\xi)}{2K}}\, |\langle z;\xi\rangle|^{NK+j} + h_{NK+j}(z) \\ &\leqslant \sqrt{\frac{5g(z)}{9K}} + \frac{1}{4}\sqrt{\frac{g(z)}{K}} < \frac{3}{4}\sqrt{\frac{g(z)}{K}} + \frac{1}{4}\sqrt{\frac{g(z)}{K}} = \sqrt{\frac{g(z)}{K}} \end{aligned}$$

by (2.2), so (1) is proved. Moreover, for any $N > N_2$ we have

$$\sum_{j=0}^{K-1} |p_{NK+j}(z)|^2 < \sum_{j=0}^{K-1} \frac{g(z)}{K} = g(z).$$

The right inequality in (3) is proved.

Let N_0 be a natural number such that $N_0 > \max\{N_2, N_1\}$ and

$$\Big(1 - \frac{1}{4NK}\Big)^{NK+K} > \frac{97}{100}\exp\Big(-\frac{1}{4}\Big) \geqslant \frac{3}{4}$$

for $N \geqslant N_0$. Let $z \in \partial\mathbb{B}^n$ and $N \geqslant N_0$. Since A is a maximal $1/\sqrt{4NK}$-separated subset of $\partial\mathbb{B}^n$ there exists $\xi_z \in A$ such that $\varrho(z,\xi_z) \leqslant 1/\sqrt{4NK}$. Due to $A = \bigcup\limits_{j=0}^{K-1} A_j$ there exists $j_z \in \{0,1,\ldots,K-1\}$ such that $\xi_z \in A_{j_z}$. As $B_{j_z,0}(z)$ contains at most one element and due to $C > 2$ we have $\varrho(z,\xi_z) \leqslant 1/\sqrt{4NK} < C/2\sqrt{4NK} < \sqrt{\delta}$, therefore $B_{j_z,0}(z)=\{\xi_z\}$ and $g(\xi_z)>\frac{8}{9}g(z)$ by (2.2). Now we may estimate

$$\begin{aligned}|p_{NK+j_z}(z)| &\geqslant \sqrt{\frac{g(\xi_z)}{2K}}\,|\langle z;\xi_z\rangle|^{NK+j_z}-h_{NK+j_z}(z)\\ &\geqslant \sqrt{\frac{g(\xi_z)}{2K}}\Big(1-\frac{1}{4NK}\Big)^{NK+K}-h_{NK+j_z}(z)\\ &> \sqrt{\frac{4g(z)}{9K}}\frac{3}{4}-\frac{1}{4}\sqrt{\frac{g(z)}{K}}=\frac{1}{4}\sqrt{\frac{g(z)}{K}}=\sqrt{qg(z)},\end{aligned}$$

so (2) is proved. Moreover,

$$\sum_{j=0}^{K-1}|p_{NK+j}(z)|^2\geqslant |p_{NK+j_z}(z)|^2>qg(z).$$

The left inequality in (3) is proved, which completes the proof. □

Lemma 2.8. *Let h be a function continuous on $\partial\mathbb{B}^n$ and such that $h(z)=h(\lambda z)>0$ when $|\lambda|=1$, $z\in\partial\mathbb{B}^n$. For any $\varepsilon>0$ and any natural number $a\in\mathbb{N}$ there exists a number $b\in\mathbb{N}$ and a sequence of homogeneous polynomials p_m of degree m such that*

$$h(z)-\varepsilon<\sum_{m=a}^{b}|p_m(z)|^2<h(z)$$

for $z\in\partial\mathbb{B}^n$.

Proof. Select numbers $q\in(0,1)$ and $K\in\mathbb{N}$ from Theorem 2.7. We construct a sequence of natural numbers b_j and a sequence of homogeneous polynomials p_m of degree m such that $b_j+K-1<b_{j+1}$ and

$$0<h(z)-\sum_{m=a}^{b_{j+1}+K-1}|p_m(z)|^2<(1-q)\Big(h(z)-\sum_{m=a}^{b_j+K-1}|p_m(z)|^2\Big). \tag{2.4}$$

Let $b_1=a$ and $p_a=\ldots=p_{b_1+K-1}=0$. Therefore b_2 and the polynomials p_m for $m\in\{b_2,\ldots,b_2+K-1\}$ can be created on the basis of Theorem 2.7. If now we have the numbers $b_1,\ldots,b_t$ and the polynomials p_m for $m\in\{a,\ldots,b_t+K-1\}$ then we create a number b_{t+1} and the polynomials p_m for $m\in\{b_{t+1},\ldots,b_{t+1}+K-1\}$. It suffices to use Theorem 2.7 assuming that $g=h-\sum\limits_{m=a}^{b_t+K-1}|p_m|^2$.

Observe that the property (2.4) implies the inequality

$$0<h(z)-\sum_{m=a}^{b_{j+1}+K-1}|p_m(z)|^2<(1-q)^j\Big(h(z)-\sum_{m=a}^{b_1+K-1}|p_m(z)|^2\Big)=(1-q)^jh(z)$$

for $z \in \partial\mathbb{B}^n$. Let now j_0 be such that

$$(1-q)^{j_0} \sup_{z\in\partial\mathbb{B}^n} h(z) < \varepsilon.$$

Then

$$h(z) - \varepsilon < h(z) - (1-q)^{j_0} h(z) < \sum_{m=a}^{b_{j_0+1}+K-1} |p_m(z)|^2 < h(z).$$

Therefore it suffices to assume that $b = b_{j_0+1} + K - 1$. □

Theorem 2.9. *Let u be a lower semi-continuous function on $\partial\mathbb{B}^n$ such that $u(\lambda z) = u(z) \geqslant 0$ when $|\lambda| = 1$ and $z \in \partial\mathbb{B}^n$. The following conditions are equivalent:*

(1) *There exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ for which u is a boundary function.*

(2) *There exist homogeneous polynomials $p_1, \ldots, p_k$ such that*
   (a) $u^{-1}(0) = \bigcap\limits_{j=1}^{k} \{z \in \partial\mathbb{B}^n : \ p_j(z) = 0\}$,
   (b) $u(z) \geqslant \sum\limits_{j=1}^{k} |p_j(z)|^2$ *for* $z \in \partial\mathbb{B}^n$.

P r o o f. (1) ⇒ (2): There exist homogeneous polynomials p_j of degree j such that

$$f(z) = \sum_{j=0}^{\infty} \frac{\sqrt{j+1}}{\pi} p_j(z).$$

In this case we have

$$u(z) = \int_{|\lambda|<1} |f(\lambda z)|^2 \, \mathrm{d}\mathfrak{L}^2(\lambda) = \sum_{j=0}^{\infty} |p_j(z)|^2$$

for $z \in \partial\mathbb{B}^n$. There exist indices $j_1, \ldots, j_k$ such that

$$u^{-1}(0) = \partial\mathbb{B}^n \cap \bigcap_{j=0}^{\infty} p_j^{-1}(0) = \partial\mathbb{B}^n \cap \bigcap_{m=1}^{k} p_{j_m}^{-1}(0).$$

Obviously $u(z) \geqslant \sum\limits_{m=1}^{k} |p_{j_m}(z)|^2$ for $z \in \partial\mathbb{B}^n$, which completes the proof.

(2) ⇒ (1): Let $A := \{z \in \partial\mathbb{B}^n : \ p_1(z) = p_2(z) = \ldots = p_k(z) = 0\}$. Define

$$g(z) := \frac{u(z)}{\left(\sum\limits_{j=1}^{k} |p_j(z)|^2\right)^2}.$$

Due to the property (b) we have $\liminf_{z\to x} g(z) = g(x) = +\infty$ for $x \in A$. Moreover,

$$\liminf_{z\to x} g(z) = \frac{\liminf_{z\to x} u(z)}{\left(\sum_{j=1}^{k} |p_j(x)|^2\right)^2} \geqslant \frac{u(x)}{\left(\sum_{j=1}^{k} |p_j(x)|^2\right)^2} = g(x) > 0$$

for $x \in \partial\mathbb{B}^n \setminus A$. In particular, due to Definition 2.1 the function g is lower semi-continuous with positive values.

There exist homogeneous polynomials q_j such that

$$\left(\sum_{j=1}^{k} |p_j(z)|^2\right)^2 = \sum_{j=1}^{k^2} |q_j(z)|^2 .$$

Let $d = \max_{j=1,\dots,k^2} \deg(q_j)$.

There exists a sequence of continuous functions $\{g_j\}_{j\in\mathbb{N}}$ on $\partial\mathbb{B}^n$ such that

- $g_j(z) = g_j(\lambda z) > 0$ when $|\lambda| = 1$,
- $g_j \leqslant g_{j+1} \leqslant \ldots \leqslant g(z)$,
- $\lim_{j\to\infty} g_j(z) = g(z)$.

We create sequences of natural numbers $\{a_{i,j}\}_{i\in\mathbb{N}}^{j=1,\dots,k^2}$, $\{b_{i,j}\}_{i\in\mathbb{N}}^{j=1,\dots,k^2}$ and a sequence of homogeneous polynomials r_m of degree m such that

(1) $a_{i,j} < b_{i,j} + d < a_{i,j+1}$ for $j = 1, \dots, k^2 - 1$,
(2) $a_{i,k^2} < b_{i,k^2} + d < a_{i+1,1}$,
(3) $\sum_{m=a_{i,j}}^{b_{i,j}} |r_m(z)| < 2^{-i}$ for $z \in (1 - 2^{-i+1})\mathbb{B}^n$ and $j = 1, \dots, k^2$,
(4) $g_s(z) - (s+1)^{-1} < \sum_{i=1}^{s} \sum_{m=a_{i,j}}^{b_{i,j}} |r_m(z)|^2 < g_s(z)$ for $s \in \mathbb{N}$ and $j = 1, \dots, k^2$.

For $s = 1$ it is possible to select the numbers $a_{1,1}, \dots, a_{1,k^2}$, $b_{1,1}, \dots, b_{1,k^2}$ and polynomials r_m for $m = 1, \dots, b_{1,k^2}$ on the basis of Lemma 2.8. Assume that we have already created $a_{i,1}, \dots, a_{i,k^2}$, $b_{i,1}, \dots, b_{i,k^2}$ and polynomials r_m for $m = 1, \dots, b_{i,k^2}$ and $i = 1, \dots, s$. We define now proper data for Lemma 2.8. Let

$$h := g_{s+1} - \sum_{i=1}^{s} \sum_{m=a_{i,j}}^{b_{i,j}} |r_m|^2 .$$

As

$$\sum_{i=1}^{s} \sum_{m=a_{i,j}}^{b_{i,j}} |r_m|^2 < g_s \leqslant g_{s+1},$$

the function h has been properly defined. Let a be so large that $a > b_{s,k^2} + 1$ and

$$\sqrt{\|h\|_\infty}\frac{\|z\|^a}{1-\|z\|} \leqslant 2^{-s-1}$$

for $z \in (1-2^{-s})\mathbb{B}^n$. For the number a selected in this way as well as for the function h we find—on the basis of Lemma 2.8—proper numbers $a_{s+1,1}, \ldots, a_{s+1,k^2}$, $b_{s+1,1}, \ldots, b_{s+1,k^2}$ ($a < a_{s+1,i} < b_{s+1,i}$) and polynomials r_m for $a_{s+1,1} \leqslant m \leqslant b_{s+1,k^2}$. It suffices now to check the condition (3). As

$$\sum_{m=a_{s+1,j}}^{b_{s+1,j}} |r_m(z)|^2 < h(z)$$

for $z \in \partial\mathbb{B}^n$, we have $|r_m(z)| \leqslant \sqrt{\|h\|_\infty}\,\|z\|^m$. Therefore

$$\sum_{m=a_{s+1,j}}^{b_{s+1,j}} |r_m(z)| \leqslant \sum_{m=a_{s+1,j}}^{b_{s+1,j}} \sqrt{\|h\|_\infty}\,\|z\|^m \leqslant \sqrt{\|h\|_\infty}\frac{\|z\|^a}{1-\|z\|} \leqslant 2^{-s-1}$$

for $z \in (1-2^{-s})\mathbb{B}^n$.

We now define

$$f(z) = \sum_{i=1}^{\infty}\sum_{j=1}^{t^2}\sum_{m=a_{i,j}}^{b_{i,j}} \frac{\sqrt{m+\deg(q_j)+1}}{\pi} q_j r_m.$$

Due to the property (3) of the polynomials r_m the function f is holomorphic. Moreover, due to the properties (1)–(2) of the numbers $a_{i,j}, b_{i,j}$ we have the equality:

$$\int_{|\lambda|\leqslant 1} |f(\lambda z)|^2 \,\mathrm{d}\mathfrak{L}^2(\lambda) = \sum_{i=1}^{\infty}\sum_{j=1}^{k^2}\sum_{m=a_{i,j}}^{b_{i,j}} |q_j(z)|^2\,|r_m(z)|^2$$

for $z \in \partial\mathbb{B}^n$.

If $z \in A$, then $u(z) = 0$ and $q_j(z) = 0$ for $j = 1, \ldots, k^2$ and

$$\int_{|\lambda|\leqslant 1} |f(\lambda z)|^2 \,\mathrm{d}\mathfrak{L}^2(\lambda) = \sum_{i=1}^{\infty}\sum_{j=1}^{k^2}\sum_{m=a_{i,j}}^{b_{i,j}} 0 = 0 = u(z).$$

Let now $z \in \partial\mathbb{B}^n \setminus A$. We define $\mathfrak{J} = \{j\colon\ q_j(z) \neq 0\}$. Obviously $\mathfrak{J} \neq \emptyset$. In this case we have:

$$\begin{aligned}\int_{|\lambda|\leqslant 1} |f(\lambda z)|^2 \,\mathrm{d}\mathfrak{L}^2(\lambda) &= \lim_{N\to\infty} \sum_{i=1}^{N} \sum_{j\in\mathfrak{J}} \sum_{m=a_{i,j}}^{b_{i,j}} |q_j(z)|^2 \,|r_m(z)|^2 \\ &\leqslant \lim_{N\to\infty} \sum_{j\in\mathfrak{J}} |q_j(z)|^2\, g_N(z) = \sum_{j\in\mathfrak{J}} |q_j(z)|^2\, g(z) \\ &= \sum_{j\in\mathfrak{J}} |q_j(z)|^2 \,\frac{u(z)}{\left(\sum\limits_{j=1}^{k} |p_j(z)|^2\right)^2} = u(z).\end{aligned}$$

Similarly

$$\begin{aligned}\int_{|\lambda|\leqslant 1} |f(\lambda z)|^2 \,\mathrm{d}\mathfrak{L}^2(\lambda) &= \lim_{N\to\infty} \sum_{i=1}^{N} \sum_{j\in\mathfrak{J}} \sum_{m=a_{i,j}}^{b_{i,j}} |q_j(z)|^2 \,|r_m(z)|^2 \\ &\geqslant \lim_{N\to\infty} \sum_{j\in\mathfrak{J}} |q_j(z)|^2 \left(g_N(z) - \frac{1}{N+1}\right) = u(z).\end{aligned}$$

□

We show now that in Theorem 2.9 it is impossible to weaken the conditions. Observe that Example 2.10 implies that there exists a continuous function u on $\partial\mathbb{B}^n$ such that $u(\lambda z) = u(z) \geqslant 0$ when $|\lambda| = 1$, $z \in \partial\mathbb{B}^n$, for which condition 2a in Theorem 2.9 is fulfilled and which is not a boundary function.

Example 2.10. Fix homogeneous polynomials $p_1, \ldots, p_m$. Assume that there exists $z_0 \in \mathbb{C}^n \setminus \{0\}$ such that $p_i(z_0) = 0$ for $i = 1, \ldots, m$. The function

$$u = \exp\left(-1\Big/\sum_{i=1}^{m} |p_i|^2\right)$$

is not a boundary function.

P r o o f. Assume that u is a boundary function. Therefore on the basis of Theorem 2.9 there exist homogeneous polynomials $q_1, \ldots, q_s$ such that

(1) $u^{-1}(0) = \{z \in \partial\mathbb{B}^n : \ q_1(z) = q_2(z) = \ldots = q_s(z) = 0\}$,

(2) $u(z) \geqslant \sum\limits_{j=1}^{s} |q_j(z)|^2$ for $z \in \partial\mathbb{B}^n$.

Without loss of generality we can assume that $q_1 \neq 0$. There exists a holomorphic function $g : \mathbb{D} \to \mathbb{C}^n$ on the one dimensional unit disc $\mathbb{D}$ such that $g(0) = z_0$ and $(q_1 \circ g)^{-1}(0) = \{0\}$. There exists an index j_0 such that $p_{j_0} \circ g \neq 0$. Moreover, we can assume that $j_0 = 1$ and $(p_1 \circ g)^{-1}(0) = \{0\}$. It follows that there exist natural numbers $n_1, n_2 \geqslant 1$ and functions h_1, h_2 which are continuous, non-negative on $\mathbb{D}$ and such that

$$|\lambda|^{n_1} h_1(\lambda) = \sum_{i=1}^{m} |p_i \circ g(\lambda)|^2,$$
$$|\lambda|^{n_2} h_2(\lambda) = \sum_{i=1}^{s} |q_i \circ g(\lambda)|^2$$

for $\lambda \in \mathbb{D}$ and $h_1(0), h_2(0) > 0$. In particular, the property (2) implies the inequality

$$\begin{aligned} \exp\Big(\frac{-1}{|\lambda|^{n_1} h_1(\lambda)}\Big) &= \exp\Big(-1\Big/\sum_{i=1}^{m} |p_i \circ g(\lambda)|^2\Big) \\ &\geqslant \sum_{j=1}^{s} |q_j \circ g(\lambda)|^2 = |\lambda|^{n_2} h_2(\lambda). \end{aligned}$$

Therefore

$$0 < h_2(0) \leqslant \lim_{\lambda \to 0} \frac{\exp(-1/|\lambda|^{n_1} h_1(\lambda))}{|\lambda|^{n_2}} = 0,$$

which is impossible. □

Example 2.11 implies that there exists a continuous function u on $\partial\mathbb{B}^n$ such that $u(\lambda z) = u(z) \geqslant 0$ when $|\lambda| = 1$, $z \in \partial\mathbb{B}^n$, which is not a boundary function but for which condition 2b in Theorem 2.9 is fulfilled.

Example 2.11. Let $p(z_1, z_2, z_3) = z_1$. There exists a continuous function u such that $u(\lambda z) = u(z) \geqslant |p(z)|^2 \geqslant 0$ when $|\lambda| = 1$, $z \in \partial\mathbb{B}^3$, which is not a boundary function.

P r o o f. Let $w_1 := (0, 0, 1)$ and $w_m := (0, \sin\frac{1}{2}\pi/m, \cos\frac{1}{2}\pi/m)$ for $m \geqslant 2$. Then $w_j \notin \mathbb{C}w_i$ for $i \neq j$ and $\lim_{m\to\infty} w_m = w_1 = (0, 0, 1)$. We define a function

$$h(z) := \exp\Big(\sum_{m\in\mathbb{N}} 2^{-m} \ln(1 - |\langle z, w_m\rangle|)\Big).$$

Observe that h is a continuous function on $\partial\mathbb{B}^3$ and

$$h^{-1}(0) = p^{-1}(0) \cap \partial\mathbb{B}^3 \cap \bigcup_{m\in\mathbb{N}} \mathbb{C}w_m.$$

Moreover, $h(\lambda z) = h(z) \geqslant 0$ when $|\lambda| = 1$, $z \in \partial\mathbb{B}^3$.

Let $u(z) = h(z) + |p(z)|^2$. Obviously $u(\lambda z) = u(z) \geqslant |p(z)|^2 \geqslant 0$ when $|\lambda| = 1$, $z \in \partial\mathbb{B}^3$. If u is a boundary function then on the basis of Theorem 2.9 there exist homogeneous polynomials $q_1, \ldots, q_s$ such that

$$u^{-1}(0) = \{z \in \partial\mathbb{B}^n : \ q_1(z) = q_2(z) = \ldots = q_s(z) = 0\} .$$

Therefore

$$\bigcup_{m\in\mathbb{N}} \mathbb{C}w_m = \{z \in \mathbb{C}^n : \ q_1(z) = q_2(z) = \ldots = q_s(z) = 0\} ,$$

which obviously is impossible. If it were true, we would obtain an analytic set with infinitely many irreducible components.

In the next example we show that in Theorem 2.9, conditions 2a and 2b have to be fulfilled at the same time by the same polynomials $p_1, \ldots, p_k$.

Example 2.12. Let $p_1(z_1, z_2, z_3) = z_1 z_3$, $p_2(z_1, z_2, z_3) = z_1$, $p_3(z_1, z_2, z_3) = z_3$ and $u := |p_1|^2 + \exp\bigl(-(|p_2|^2 + |p_3|^2)^{-1}\bigr)$. The function u fulfils the properties

(1) $u^{-1}(0) = \bigl\{z \in \partial\mathbb{B}^3 : \ p_2(z) = p_3(z) = 0\bigr\}$,
(2) $u(z) \geqslant |p_1(z)|^2$ for $z \in \partial\mathbb{B}^3$.

Moreover, the function u is not a boundary function.

Proof. If u is a boundary function, then $v(z_1, z_2) := u(z_1, z_2, 0)$ is also a boundary function, which is impossible on the basis of Example 2.10. □

We present how Theorem 2.9 can be used.

Example 2.13. If u is a lower semi-continuous function on $\partial\mathbb{B}^n$ and $u(z\lambda) = u(z) > 0$ when $|\lambda| = 1$, then u is a boundary function.

Proof. Let $p(z) = \inf\limits_{w\in\partial\mathbb{B}^n} u(w)$. Obviously it is a homogeneous polynomial of degree 0. Now it suffices to use Theorem 2.9. □

Example 2.14. Let $p_1, \ldots, p_m$ be any homogeneous polynomials. We will show that

$$u(z) = \left(\sum_{i=1}^m |p_i|\right)^\alpha + M \exp\left(-1\Big/\sum_{i=1}^m |p_i|\right)$$

for $\alpha > 0$, $M \geqslant 0$ is a boundary function.

Proof. It suffices to assume that $u \neq 0$. Let $c := \frac{1}{2} \max\limits_{z\in\partial\mathbb{B}^n} \sum\limits_{i=1}^m |p_i(z)|$. Obviously $c > 0$. Let N be any natural number such that $N > \alpha$. We can estimate

$$1 > \sum_{i=1}^m \frac{|p_i(z)|}{c} \geqslant \sum_{i=1}^m \frac{|p_i(z)|^2}{c^2}$$

for $z \in \partial\mathbb{B}^n$. In particular,

$$u(z) \geqslant \left(\sum_{i=1}^{m} |p_i(z)|\right)^{\alpha} \geqslant \frac{c^{\alpha}}{c^{2N}}\left(\sum_{i=1}^{m} |p_i(z)|^2\right)^{N}$$

for $z \in \partial\mathbb{B}^n$ and $u^{-1}(0) = \{z \in \partial\mathbb{B}^n : \ p_1(z) = \ldots = p_m(z) = 0\}$. Therefore on the basis of Theorem 2.9 the function u is a boundary function. □

Example 2.15. If u_1, u_2 are boundary functions, then the following functions are also boundary functions:[2] $u_1 + u_2$, $u_1 u_2$, $\max\{u_1, u_2\}$, $\min\{u_1, u_2\}$.

P r o o f. Due to Definition 2.1 there exist sequences $f_{m,1}$ and $f_{m,2}$ of continuous, non-negative functions such that $f_{m,i} \leqslant f_{m+1,i}$ and $u_i = \lim\limits_{m\to\infty} f_{m,i}$. In particular,

- $u_1 + u_2 = \lim\limits_{m\to\infty} f_{m,1} + f_{m,2}$,
- $u_1 u_2 = \lim\limits_{m\to\infty} f_{m,1} f_{m,2}$,
- $\max\{u_1, u_2\} = \lim\limits_{m\to\infty} \max\{f_{m,1}, f_{m,2}\}$,
- $\min\{u_1, u_2\} = \lim\limits_{m\to\infty} \min\{f_{m,1}, f_{m,2}\}$.

Therefore, on the basis of Definition 2.1 we can see that the functions $u_1 + u_2$, $u_1 u_2$, $\max\{u_1, u_2\}$, $\min\{u_1, u_2\}$ are lower semi-continuous functions. As these functions have also the same values on the circles, therefore on the basis of Theorem 2.9 there exist homogeneous polynomials $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ such that

- $u_1^{-1}(0) = \{z \in \partial\mathbb{B}^n : \ p_1(z) = p_2(z) = \ldots = p_r(z) = 0\}$,
- $u_1(z) \geqslant \sum\limits_{j=1}^{r} |p_j(z)|^2$ for $z \in \partial\mathbb{B}^n$,
- $u_2^{-1}(0) = \{z \in \partial\mathbb{B}^n : \ q_1(z) = \ldots = q_s(z) = 0\}$,
- $u_2(z) \geqslant \sum\limits_{j=1}^{s} |q_j(z)|^2$ for $z \in \partial\mathbb{B}^n$.

We can additionally assume that

$$\max\left\{\sum_{j=1}^{s} |q_j(z)|^2, \sum_{j=1}^{r} |p_j(z)|^2\right\} \leqslant 1 \tag{2.5}$$

for $z \in \partial\mathbb{B}^n$. In particular,

(1) $(u_1 + u_2)^{-1}(0) = \bigcap\limits_{i=1}^{r} \bigcap\limits_{j=1}^{s} \{z \in \partial\mathbb{B}^n : \ p_i(z) = q_j(z) = 0\}$,

(2) $(u_1 + u_2)(z) \geqslant \sum\limits_{j=1}^{r} |p_j(z)|^2 + \sum\limits_{j=1}^{s} |q_j(z)|^2$ for $z \in \partial\mathbb{B}^n$,

(3) $(u_1 u_2)^{-1}(0) = \bigcap\limits_{i=1}^{r} \bigcap\limits_{j=1}^{s} \{z \in \partial\mathbb{B}^n : \ p_i(z) q_j(z) = 0\}$,

[2] We assume $0 * \infty = 0$.

(4) $(u_1u_2)(z) \geqslant \sum_{i=1}^{r}\sum_{j=1}^{s} |p_i(z)q_j(z)|^2$ for $z \in \partial\mathbb{B}^n$,

(5) $\max\{u_1(z), u_2(z)\}^{-1}(0) = \bigcap_{i=1}^{r}\bigcap_{j=1}^{s}\{z \in \partial\mathbb{B}^n :\ \frac{1}{2}p_i(z) = \frac{1}{2}q_j(z) = 0\}$,

(6) $\max\{u_1(z), u_2(z)\} \geqslant \frac{1}{2}(u_1(z) + u_2(z)) \geqslant \sum_{j=1}^{r} |\frac{1}{2}p_j(z)|^2 + \sum_{j=1}^{s} |\frac{1}{2}q_j(z)|^2$ for $z \in \partial\mathbb{B}^n$,

(7) $\min\{u_1(z), u_2(z)\}^{-1}(0) = \bigcap_{i=1}^{r}\bigcap_{j=1}^{s}\{z \in \partial\mathbb{B}^n :\ p_i(z)q_j(z) = 0\}$,

(8) $\min\{u_1(z), u_2(z)\} \geqslant \sum_{i=1}^{r}\sum_{j=1}^{s} |p_i(z)q_j(z)|^2$ for $z \in \partial\mathbb{B}^n$.

In the last inequality it is necessary to use (2.5).

Due to Theorem 2.9 the functions $u_1 + u_2$, u_1u_2, $\max\{u_1, u_2\}$ and $\min\{u_1, u_2\}$ are boundary functions. □

Example 2.16. If u_m is a sequence of boundary functions then $\sum_{m\in\mathbb{N}} u_m$ is a boundary function. If $\{u_m\}_{m\in\mathbb{I}}$ is a set of boundary functions then $\sup_{m\in\mathbb{I}} u_m$ is a boundary function.

P r o o f. On the basis of Theorem 2.9 there exist sequences of numbers $\{j_m\}_{m\in\mathbb{N}}$ ($\{j_m\}_{m\in\mathbb{I}}$) and homogeneous polynomials $\{p_{k,m}\}_{m\in\mathbb{N}}^{k\in\{1,\dots,j_m\}}$ ($\{p_{k,m}\}_{m\in\mathbb{I}}^{k\in\{1,\dots,j_m\}}$) such that

- $u_m^{-1}(0) = \bigcap_{i=1}^{j_m}\{z \in \partial\mathbb{B}^n :\ p_{i,m}(z) = 0\}$,
- $u_m(z) \geqslant \sum_{i=1}^{j_m} |p_{i,m}(z)|^2$ for $z \in \partial\mathbb{B}^n$.

Let $u = \sum_{m\in\mathbb{N}} u_m$ ($u = \sup_{m\in\mathbb{I}} u_m$). Obviously u is a lower semi-continuous function and $u(\lambda z) = u(z)$ when $|\lambda| = 1$. If $u^{-1}(0)$ is an empty set, then on the basis of Theorem 2.9 it is a boundary function.

Otherwise, there exists[3] $k \in \mathbb{N}$ ($k \in \mathbb{I}$) such that $u^{-1}(0) = u_k^{-1}(0)$. Therefore

(1) $u^{-1}(0) = \bigcap_{i=1}^{j_k}\{z \in \partial\mathbb{B}^n :\ p_{i,k}(z) = 0\}$,

(2) $u(z) \geqslant \sum_{i=1}^{j_k} |p_{i,k}(z)|^2$ for $z \in \partial\mathbb{B}^n$.

Due to Theorem 2.9 u is a boundary function. □

[3] This follows from the fact that $u_k^{-1}(0)$ is a remainder of an analytical set.

3. Applications of boundary functions

Let E be any circular subset of type G_δ and F_σ in $\partial\mathbb{B}^n$. In the paper [9] we presented a construction of a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ for which E is the exceptional set. In this paper we have used some properties of Wojtaszczyk polynomials [12]. Now we present a much stronger result as an application of boundary functions.

We say that $E \subset \partial\mathbb{B}^n$ is a circular if $z \in E \Longrightarrow \lambda z \in E$ for every $\lambda \in \mathbb{C}, |\lambda| = 1$. By σ we denote a natural measure on $\partial\mathbb{B}^n$ so that $\sigma(\partial\mathbb{B}^n) = 1$.

Theorem 3.1. *Let E be a circular subset of $\partial\mathbb{B}^n$. Then there exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ such that $\int_{\mathbb{B}^n\setminus\Lambda(E)} |f|^2\,\mathrm{d}\mathfrak{L}^{2n} < \infty$ iff E is of type G_δ in $\partial\mathbb{B}^n$. Here $\Lambda(E) = \{\lambda z\colon |\lambda| < 1, z \in E\}$, $E = E(f)$ and $\mathrm{d}\mathfrak{L}^{2n}$ is the $2n$-dimensional Lebesgue measure.*

Proof. Note that the easy "if"-part of the proof can be found in [9, Proposition 2.2].

There exists a sequence $\{U_m\}_{m\in\mathbb{N}}$ in $\partial\mathbb{B}^n$ of open circular sets such that

$$\sum_{m\in\mathbb{N}} \sigma\left(U_m \setminus E\right) \leqslant 1$$

and $E = \bigcap_{m\in\mathbb{N}} U_m$. Let

$$\chi_m(z) := \begin{cases} 1 & \text{for} \quad z \in U_m, \\ 0 & \text{for} \quad z \in \partial\mathbb{B}^n \setminus U_m. \end{cases}$$

Obviously χ_m is a lower semi-continuous function. Let $u = 1 + \sum_{m\in\mathbb{N}} \chi_m$. The function u is also a lower semi-continuous function such that $u(\lambda z) = u(z) > 0$ for $|\lambda| = 1$ and $z \in \partial\mathbb{B}^n$. On the basis of Theorem 2.9 (for $k = 1$, $p_1 = 1$) there exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ for which u is a boundary function. Observe now that

$$\int_{\partial\mathbb{B}^n\setminus E} u\,\mathrm{d}\sigma = 1 + \sum_{m\in\mathbb{N}} \sigma\left(U_m \setminus E\right) \leqslant 2.$$

There exists a constant $C > 0$ such that

$$\begin{aligned} C\int_{\mathbb{B}^n\setminus\Lambda(E)} |f|^2\,\mathrm{d}\mathfrak{L}^{2n} &\leqslant \int_{\partial\mathbb{B}^n\setminus E}\int_{|\lambda|\leqslant 1} |f(\lambda z)|^2 d\mathfrak{L}^2(\lambda)\,\mathrm{d}\sigma(z) \\ &= \int_{\partial\mathbb{B}^n\setminus E} u\,\mathrm{d}\sigma \leqslant 2, \end{aligned}$$

which completes the proof. □

We give a nontrivial solution of the Dirichlet problem for plurisubharmonic functions as another application of boundary functions.

Theorem 3.2. *If u is a continuous function on $\partial\mathbb{B}^n$ such that $u(\lambda z)=u(z)$, then there exists a constant $c\in\mathbb{R}$ and a sequence of homogeneous polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that p_m is of the degree m and $u(z)=c+\sum\limits_{m\in\mathbb{N}}|p_m(z)|^2$ for $z\in\partial\mathbb{B}^n$. In particular, the function $g(z)=c+\sum\limits_{m\in\mathbb{N}}|p_m(z)|^2$ is continuous on $\overline{\mathbb{B}^n}$, real analytic and plurisubharmonic on $\mathbb{B}^n$.*

P r o o f. Let $c:=\inf\limits_{z\in\partial\mathbb{B}^n}u(z)-1$. Then $u(z)-c\geqslant 1$. Therefore due to Theorem 2.9 there exists a holomorphic function f such that

$$u(z)-c=\int_{|\lambda|<1}|f(\lambda z)|^2\,\mathrm{d}\mathfrak{L}^2(\lambda).$$

In particular, there exists a sequence of homogeneous polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that p_m is of the degree m and $u(z)=c+\sum\limits_{m\in\mathbb{N}}|p_m(z)|^2$ for $z\in\partial\mathbb{B}^n$. Let $g(z)=c+\sum\limits_{m\in\mathbb{N}}|p_m(z)|^2$. Because the series $\sum\limits_{m\in\mathbb{N}}|p_m(z)|^2$ is uniformly convergent to a continuous function $u(z)-c$ on $\partial\mathbb{B}^n$ and $\sum\limits_{m=N_1}^{N_2}|p_m(z)|^2$ is a plurisubharmonic function on $\mathbb{C}^n$, therefore g is a continuous function on $\overline{\mathbb{B}^n}$ and it is a plurisubharmonic and real analytic function on $\mathbb{B}^n$. $\square$

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Author's address: Piotr Kot, Politechnika Krakowska, Instytut Matematyki, ul. Warszawska 24, 31-155 Kraków, Poland, e-mail: `pkot@usk.pk.edu.pl`.