Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 4, 1323–1334

Persistent URL: http://dml.cz/dmlcz/128147

Czechoslovak Mathematical Journal, 56 (131) (2006), 1323–1334

ON THE EXISTENCE OF PROLONGATION OF CONNECTIONS

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(Received November 29, 2004)

Dedicated to Professor Ivan Kolář on the occasion of his 70th birthday

Abstract. We classify all bundle functors G admitting natural operators transforming connections on a fibered manifold $Y \to M$ into connections on $GY \to M$. Then we solve a similar problem for natural operators transforming connections on $Y \to M$ into connections on $GY \to Y$.

Keywords: bundle functor, connection, natural operator

MSC 2000: 58A05, 58A20

Introduction

Let G be a bundle functor on the category $\mathscr{F}\mathscr{M}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibres and their local fibered diffeomorphisms. We recall that a connection on a fibered manifold $p\colon Y \to M$ is a smooth section $\Gamma\colon Y \to J^1Y$ of the first jet prolongation of Y, which can also be interpreted as the lifting map (denoted by the same symbol) $\Gamma\colon Y \times_M TM \to TY$. The present paper is devoted to the following problems:

Problem 1. *To classify all bundle functors G on $\mathscr{F}\mathscr{M}_{m,n}$ which admit natural operators transforming connections on $Y \to M$ into connections on $GY \to M$.*

The first author was supported by a grant of the GA ČR No. 201/02/0225.

Problem 2. *To classify all bundle functors G on $\mathscr{F}\mathscr{M}_{m,n}$ which admit natural operators transforming connections on $Y \to M$ into connections on $GY \to Y$.*

If $G = V^F$ is the F-vertical functor determined by a natural bundle F, then I. Kolář and the second author have constructed a connection $\mathscr{V}^F\Gamma$ on $V^FY \to M$, which is called the F-vertical prolongation of Γ, [7]. However, if $G \neq V^F$, then we know no natural operator transforming connections on $Y \to M$ into connections on $GY \to M$. For some particular cases of G it has only been proved that there is no (first order) natural operator of this type, see [1], [5] and [6]. Moreover, the second author has recently proved that under some conditions on the bundle functor G, there are no natural operators transforming connections on $Y \to M$ into connections on $GY \to M$ and also on $GY \to Y$, see [8] and [9].

It turns out that Problems 1 and 2 are closely related to the order of the bundle functor G. That is why we first study some properties of bundle functors on $\mathscr{F}\mathscr{M}_{m,n}$ from a more general point of view. In particular, in Section 2 we classify all bundle functors G on $\mathscr{F}\mathscr{M}_{m,n}$, the base order of which is zero. We show that a bundle functor G on $\mathscr{F}\mathscr{M}_{m,n}$ has base order zero if and only if G is isomorphic to some F-vertical functor V^F. Quite analogously, in Section 3 we characterize all bundle functors G, the fiber order of which is zero. The main result of this paper is the complete solution of Problem 1 and Problem 2, which is described in Section 4 and Section 5.

We remark that the prolongation of connections has motivation e.g. in quantum mechanics and higher order dynamics, see [4] and [10].

Denote by $\mathscr{M}f$ the category of smooth manifolds and all smooth maps, by $\mathscr{M}f_m$ the subcategory of m-dimensional manifolds and their local diffeomorphisms, by $\mathscr{F}\mathscr{M}$ the category of fibered manifolds and fiber respecting mappings and by $\mathscr{F}\mathscr{M}_m$ the subcategory of fibered manifolds with m-dimensional bases and $\mathscr{F}\mathscr{M}$-morphisms with local diffeomorphisms as base maps. In what follows $Y \to M$ stands for $\mathscr{F}\mathscr{M}_{m,n}$-objects and N stands for $\mathscr{M}f_n$-objects. All manifolds and maps are assumed to be infinitely differentiable.

1. The foundations

This section contains a survey of some known results which we need in the sequel. Suppose we have two fibered manifolds $p\colon Y \to M$ and $\bar{p}\colon \overline{Y} \to \overline{M}$ and let $s \geqslant r \leqslant q$ be three integers. We say that two $\mathscr{F}\mathscr{M}$-morphisms $f, g\colon Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g}\colon M \to \overline{M}$ determine the same (r, s, q)-jet $j_y^{r,s,q} f = j_y^{r,s,q} g$ at $y \in Y$, $p(y) = x$, if

$$j_y^r f = j_y^r g, \quad j_y^s(f|Y_x) = j_y^s(g|Y_x), \quad j_x^q \underline{f} = j_x^q \underline{g}.$$

By [6], a bundle functor G on $\mathcal{FM}$ is said to be of order r, if from $j^r_y f = j^r_y g$ it follows that $G_y f = G_y g$ for every $\mathcal{FM}$-morphisms $f, g\colon Y \to \overline{Y}$ and every point $y \in Y$. I. Kolář and the second author have recently introduced the following definition of order, which is based on the concept of (r,s,q)-jets. By [7], a bundle functor G on $\mathcal{FM}$ is said to be of order (r,s,q), if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Gf|G_yY = Gg|G_yY$. Then the integer q is called the base order, s is called the fiber order and r is called the total order of G.

It is well known that product preserving functors can be expressed in terms of Weil algebras, [6]. The most important result from this field is that each product preserving functor F on $\mathcal{M}f$ is a Weil functor $F = T^A$ determined by a Weil algebra A. Then the iteration $T^A \circ T^B$ of two Weil functors corresponds to the tensor product $A \otimes B$ of Weil algebras and natural transformations $T^A \to T^B$ are in bijection with algebra homomorphisms $A \to B$.

Given a bundle functor G on $\mathcal{FM}_{m,n}$ and a product fibered manifold $M \times N \to M$, we have three fibered manifold projections $\pi\colon G(M\times N) \to M \times N$, $\pi_1\colon G(M\times N) \to M$ and $\pi_2\colon G(M\times N) \to N$. For $x \in M$, $y \in N$ we will denote by $G_{(x,y)}(M\times N)$, $G_x(M\times N)$ and $G(M\times N)_y$ the fibers with respect to π, π_1 and π_2, respectively.

Let F be a natural bundle on $\mathcal{M}f_n$. The F-vertical functor is a bundle functor V^F on $\mathcal{FM}_{m,n}$ defined by

$$V^F Y = \bigcup_{x\in M} F(Y_x), \quad V^F f = \bigcup_{x \in M} F(f_x)$$

where f_x is the restriction and corestriction of $f\colon Y \to \overline{Y}$ over $\underline{f}\colon M \to \overline{M}$ to the fibers Y_x and $\overline{Y}_{\underline{f}(x)}$, [7]. Clearly, if the order of F is s, then the order of V^F is $(0,s,0)$. For the tangent bundle $F = T$ we obtain the classical vertical bundle, which will be denoted by V instead of V^T. Further, if $F = T^A$ is a Weil functor determined by a Weil algebra A, then V^{T^A} is the vertical Weil functor on $\mathcal{FM}_m \supset \mathcal{FM}_{m,n}$, which will be denoted by V^A.

Let $\Gamma\colon Y \to J^1Y$ be a connection on a fibered manifold $Y \to M$. We recall that a projectable vector field on a fibered manifold $Y \to M$ is an $\mathcal{FM}$-morphism $Z\colon Y \to TY$ over the underlying vector field $M \to TM$ and the flow $\exp tZ$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. Then the flow prolongation of Z with respect to a bundle functor G on $\mathcal{FM}_{m,n}$ is the vector field $\mathcal{G}Z\colon GY \to TGY$ defined by $\mathcal{G}Z = \partial/\partial t|_0 G(\exp tZ)$. By [7], if G has order (r,s,q), then the value of $\mathcal{G}Z$ at each point of G_yY depends on $j^{r,s,q}_y Z$ only. Thus the flow prolongation $\mathcal{G}Z$ can also be interpreted as a map

$$\mathcal{G}_Y\colon GY \times_Y J^{r,s,q}TY \to TGY, \tag{1}$$

where $J^{r,s,q}TY$ denotes the space of all (r,s,q)-jets of projectable vector fields on Y. Further, (1) is linear in the second factor. Given a vector field X on M, its Γ-lift is a projectable vector field ΓX on Y. By (1), the flow prolongation $\mathscr{G}(\Gamma X)$ depends on the q-jets of X only and we obtain a map

$$\mathscr{G}\Gamma\colon\ GY \times_M J^qTM \to TGY, \tag{2}$$

which is linear in the second factor. Moreover, if the base order of G is $q = 0$, then (2) is a connection on $GY \to M$. In the case of the F-vertical bundle $G = V^F$, the connection (2) is called the F-vertical prolongation of Γ and is denoted by $\mathscr{V}^F\Gamma$. For the classical vertical bundle V we obtain the classical vertical prolongation $\mathscr{V}\Gamma\colon\ VY \to J^1VY$, which was also constructed by I. Kolář in [5]. We remark that if $G = V^A$ is the vertical Weil functor, then there is another way to construct the T^A-prolongation $\mathscr{V}^A\Gamma$, see [7]. If the base order q of G is arbitrary (not necessarily zero), then we can construct an induced connection on $GY \to M$ by means of some auxiliary q-th order linear connection $\nabla\colon\ TM \to J^qTM$ on M. Indeed, the composition

$$\mathscr{G}(\Gamma,\nabla) := \mathscr{G}\Gamma \circ (\mathrm{id}_{GY} \times_{\mathrm{id}_M} \nabla)\colon\ GY \times_M TM \to TGY \tag{3}$$

is the lifting map of a connection on $GY \to M$. The second author has recently proved

Proposition 1 ([8]). *Let $G\colon\ \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ be a bundle functor such that the corresponding natural bundle $G^1\colon\ \mathscr{M}f_m \to \mathscr{F}\mathscr{M}$, $G^1M = G(M \times \mathbb{R}^n)$, $G^1\varphi = G(\varphi \times \mathrm{id}_{\mathbb{R}^n})$ is not of order 0. Then there is no $\mathscr{F}\mathscr{M}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to M$.*

Proposition 2 ([9]). *Let $G\colon\ \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ be a bundle functor such that the corresponding natural bundle $G^2\colon\ \mathscr{M}f_n \to \mathscr{F}\mathscr{M}$, $G^2N = G(\mathbb{R}^m \times N)$, $G^2\psi = G(\mathrm{id}_{\mathbb{R}^m} \times \psi)$ is not of order 0. Then there is no $\mathscr{F}\mathscr{M}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to Y$.*

Proposition 3 ([8]). *Let $G\colon\ \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ be a bundle functor such that the corresponding natural bundle $G^1\colon\ \mathscr{M}f_m \to \mathscr{F}\mathscr{M}$, $G^1M = G(M \times \mathbb{R}^n)$, $G^1\varphi = G(\varphi \times \mathrm{id}_{\mathbb{R}^n})$ is not of order 0. Then there is no $\mathscr{F}\mathscr{M}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to Y$.*

Further, in [3] we have proved

Proposition 4. *The F-vertical prolongation $\mathscr{V}^F$ is the only natural operator transforming connections on $Y \to M$ into connections on $V^F Y \to M$.*

2. Classification of bundle functors on $\mathscr{F}\mathscr{M}_{m,n}$ of order $(0,s,0)$

Given a bundle functor $G\colon \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ of order $(0,s,0)$, we can define a bundle functor $F = F^G\colon \mathscr{M}f_n \to \mathscr{F}\mathscr{M}$ by

$$FN = G_0(\mathbb{R}^m \times N), \quad F\psi = G_0(\mathrm{id}_{\mathbb{R}^m} \times \psi), \tag{4}$$

where $\psi\colon N \to \overline{N}$, $0 \in \mathbb{R}^m$. Clearly, F has order s.

Proposition 5. *Let $G\colon \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ be a bundle functor of order $(0,s,0)$ and denote by $F = F^G$ its associated bundle functor (4) on $\mathscr{M}f_n$. Then we have a natural equivalence*

$$G \cong V^{(F^G)}.$$

Proof. Let $Y \to M$ be an $\mathscr{F}\mathscr{M}_{m,n}$-object. Define a map $I_Y\colon GY \to V^F Y$ by

$$I_Y(v) = G\Phi(v) \in G_0(\mathbb{R}^m \times Y_{x_0}) = F(Y_{x_0}) = (V^F Y)_{x_0}$$

where $v \in (GY)_{x_0}$, $x_0 \in M$ and $\Phi\colon Y \to \mathbb{R}^m \times Y_{x_0}$ is an $\mathscr{F}\mathscr{M}_{m,n}$-map such that $\Phi|Y_{x_0} = (0, \mathrm{id}_{Y_{x_0}})$. Since G is of order $(0,s,0)$, the definition of $I_Y(v)$ is independent of the choice of Φ. The inverse map is $J_Y\colon V^F Y \to GY$ defined by

$$J_Y(w) = G\Phi^{-1}(w), \quad w \in G_0(\mathbb{R}^m \times Y_{x_0}) = (V^F Y)_{x_0}, \quad x_0 \in M,$$

where Φ is as above. The regularity of G implies the smoothness of I_Y and J_Y, so that I_Y is a diffeomorphism. Finally, from the functoriality of G it follows directly that $I\colon G \to V^F$ is a natural transformation. □

As the order of an arbitrary F-vertical functor V^F is $(0,s,0)$, $s = \mathrm{ord}\,(F)$, we have

Corollary 1. *Let G be a bundle functor on $\mathscr{F}\mathscr{M}_{m,n}$. The following conditions are equivalent:*
(1) *The order of G is $(0,s,0)$ for some s.*
(2) *The base order of G is zero.*
(3) *G is naturally equivalent to some F-vertical functor V^F.*

Proposition 6. *Let $F_1, F_2\colon \mathscr{M}f_n \to \mathscr{F}\mathscr{M}$ be natural bundles. Then $\mathscr{F}\mathscr{M}_{m,n}$-natural transformations $V^{F_1} \to V^{F_2}$ are in bijection with $\mathscr{M}f_n$-natural transformations $F_1 \to F_2$.*

P r o o f. Let $I\colon V^{F_1} \to V^{F_2}$ be a natural transformation. Then we have a natural transformation $J = J^I\colon F_1 \to F_2$, $J_N\colon F_1N \to F_2N$, $J_N(v) = I_{\mathbb{R}^m \times N}(v)$, $v \in (V^{F_1}(\mathbb{R}^m \times N))_0 = F_1N$. Conversely, let $J\colon F_1 \to F_2$ be a natural transformation. We have a natural transformation $I = I^J\colon V^{F_1}Y \to V^{F_2}Y$, $I(v) = J_{Y_{x_o}}(v)$, $v \in (V^{F_1}Y)_{x_o} = F_1(Y_{x_o})$. Obviously, the above correspondences $I \to J^I$ and $J \to I^J$ are mutually inverse. □

Remark 1. Clearly, the F-vertical functor V^F preserves fiber products if and only if the natural bundle F preserves products. By the general theory [6], $F = T^A$ is a Weil functor and the corresponding F-vertical functor V^F is exactly the vertical Weil functor V^A. By [2], every algebra homomorphism $\mu\colon A \to B$ determines a natural transformation $V^\mu\colon V^A \to V^B$ and all natural transformations $V^A \to V^B$ on $\mathscr{F}\mathscr{M}_m$ are of the form V^μ. This corresponds to Proposition 6, which has a more general character.

Remark 2. I. Kolář and the first author have proved that for every fiber product preserving functor G on $\mathscr{F}\mathscr{M}_m$ and every vertical Weil functor V^A there is a canonical natural equivalence $V^AG \cong GV^A$, [2]. Moreover, from the theory of Weil bundles it follows that we have a natural equivalence $V^{A\otimes B} \cong V^A \circ V^B$, where $A \otimes B$ is the tensor product of Weil algebras corresponding to the iterated Weil functor $T^A \circ T^B$. One verifies directly that for F-vertical functors we have the formula

$$V^{F_2 \circ F_1} \cong V^{F_2} \circ V^{F_1}.$$

3. Classification of bundle functors on $\mathscr{F}\mathscr{M}_{m,n}$ of the order $(0,0,q)$

Given a bundle functor $F\colon \mathscr{M}f_m \to \mathscr{F}\mathscr{M}$ of order q, we can define a bundle functor $G^F\colon \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ by

$$G^FY = FM \times_M Y, \quad G^F\Phi = F\underline{\Phi} \times_{\underline{\Phi}} \Phi \tag{5}$$

where $\Phi\colon Y \to \overline{Y}$ is an $\mathscr{F}\mathscr{M}_{m,n}$-morphism over $\underline{\Phi}\colon M \to \overline{M}$. Then G^F is of order $(0,0,q)$.

Conversely, let $G\colon \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ be a bundle functor of order $(0,0,q)$. Define a bundle functor $F = F^G\colon \mathscr{M}f_m \to \mathscr{F}\mathscr{M}$ by

$$FM = G(M \times \mathbb{R}^n)_0, \quad F\varphi = G(\varphi \times \mathrm{id}_{\mathbb{R}^n})_0 \tag{6}$$

where $\varphi\colon M \to \overline{M}$, $0 \in \mathbb{R}^n$. Clearly, $F = F^G$ has order q.

Proposition 7. *Let $G\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor of order $(0,0,q)$ and denote by $F = F^G$ its associated bundle functor (6) on $\mathcal{M}f_m$. Then we have a natural equivalence*

$$G \cong G^{(F^G)}.$$

Proof. Let $Y \to M$ be an $\mathcal{FM}_{m,n}$-object. Define a map $I_Y\colon GY \to G^F Y$ by

$$I_Y(w) = (G\Phi(w), y) \in FM \times_M Y = G^F Y,$$

where $w \in (GY)_y$, $y \in Y_x$, $x \in M$ and $\Phi\colon Y \to M \times \mathbb{R}^n$ is an $\mathcal{FM}_{m,n}$-map such that $\Phi(y) = (x, 0)$, $\underline{\Phi} = \mathrm{id}_M$. Since G is of order $(0,0,q)$ the definition of $I_Y(w)$ is independent of the choice of Φ. The inverse map $J_Y\colon G^F Y \to GY$ is given by

$$J_Y(v, y) = G\Phi^{-1}(v),$$

where $(v, y) \in (G^F Y)_x = (FM \times_M Y)_x$, $x \in M$ and Φ is as above. From the regularity of G follows the smoothness of I_Y and J_Y, so that I_Y is a diffeomorphism. Finally, from the functoriality of G it follows directly that $I\colon G \to G^F$ is a natural transformation. $\square$

Obviously, a bundle functor G on $\mathcal{FM}_{m,n}$ has order $(0,0,q)$ if and only if the fiber order of G is zero.

Proposition 8. *Let $F, \overline{F}\colon \mathcal{M}f_m \to \mathcal{FM}$ be natural bundles of order q. Then $\mathcal{FM}_{m,n}$-natural transformations $G^F \to G^{\overline{F}}$ are in bijection with $\mathcal{M}f_m$-natural transformations $F \to \overline{F}$.*

Proof. Let $F, \overline{F}\colon \mathcal{M}f_m \to \mathcal{FM}$ be natural bundles of order q and let $I\colon F \to \overline{F}$ be a natural transformation. Then we have the induced natural transformation $J = J^I\colon G^F \to G^{\overline{F}}$, $J_Y(v, y) = (I_M(v), y)$, $(v, y) \in G^F Y$, where $Y \to M$ is an $\mathcal{FM}_{m,n}$-object. Conversely, let $G, \overline{G}\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ be bundle functors of order $(0,0,q)$ and let $J\colon G \to \overline{G}$ be a natural transformation. Then we have a natural transformation $I = I^J\colon F^G \to F^{\overline{G}}$, where I_M is the restriction of $J_{M\times\mathbb{R}^n}$. Clearly, the correspondences $I \to J^I$ and $J \to I^J$ are mutually inverse. $\square$

4. The solution of Problem 1

By [6], any bundle functor $G\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ is of finite order. We first prove

Proposition 9. *Let $G\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor of order s. Suppose that the bundle functor $G^1\colon \mathcal{M}f_m \to \mathcal{FM}$ defined by*

$$G^1M = G(M \times \mathbb{R}^n), \quad G^1\varphi = G(\varphi \times \mathrm{id}_{\mathbb{R}^n})$$

is of order zero. Then G is of order $(0, s, 0)$.

Proof. Let $\Phi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ be a $(0,0)$-preserving $\mathcal{FM}_{m,n}$-map satisfying $j^{0,s,0}_{(0,0)}\Phi = j^{0,s,0}_{(0,0)}\,\mathrm{id}$ and let $v \in G_{(0,0)}(\mathbb{R}^m \times \mathbb{R}^n)$. It remains to show that $G\Phi(v) = v$. In general, Φ is of the form $\Phi(x,y) = (\underline{\Phi}(x), \varphi(x,y))$. Because of the zero order of G^1, replacing Φ by $(\underline{\Phi}^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ \Phi$ we can assume that $\Phi(x,y) = (x, \varphi(x,y))$. Further, as G^1 is of order zero we have

$$G\Phi(v) = G^1\Big(\frac{1}{t}\,\mathrm{id}_{\mathbb{R}^m}\Big) \circ G\Phi \circ G^1(t\,\mathrm{id}_{\mathbb{R}^m})(v) = G(\mathrm{pr}_{\mathbb{R}^m}, \varphi \circ (t\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n}))(v),$$

where $\mathrm{pr}_{\mathbb{R}^m}\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the projection. Using the regularity of G and putting $t \to 0$ we get $G\Phi(v) = G(\mathrm{id}_{\mathbb{R}^m} \times \varphi_0)(v)$, where $\varphi_0 = \varphi(0,\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$. Then the assumption $j^{0,s,0}_{(0,0)}\Phi = j^{0,s,0}_{(0,0)}\,\mathrm{id}$ gives $j^s_{(0,0)}(\mathrm{id}_{\mathbb{R}^m} \times \varphi_0) = j^s_{(0,0)}\,\mathrm{id}$. Finally, from the fact that G is of order s we get $G\Phi(v) = v$. □

By Corollary 1, a bundle functor $G\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ is of order $(0,s,0)$ if and only if G is isomorphic to some F-vertical functor V^F. In Proposition 4 we have proved that there is one and only one natural operator transforming connections on $Y \to M$ into connections on $V^FY \to M$. On the other hand, from Proposition 1 it follows that if G^1 is not of order zero, then G does not admit natural operators transforming connections on $Y \to M$ into connections on $GY \to M$. Finally, taking into account Proposition 9 and summing up we have proved

Theorem 1. *A bundle functor $G\colon \mathcal{FM}_{m,n} \to \mathcal{FM}$ admits an $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to M$ if and only if G is isomorphic to some F-vertical bundle functor V^F. For V^F such natural operator is unique.*

Using Corollary 1 we have

Corollary 2. *Let G be a bundle functor on $\mathscr{F}\mathscr{M}_{m,n}$. The following conditions are equivalent:*

(1) *The order of G is $(0, s, 0)$ for some s.*
(2) *The base order of G is zero.*
(3) *G is naturally equivalent to some F-vertical functor V^F.*
(4) *There is an $\mathscr{F}\mathscr{M}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to M$.*

By formula (3), an arbitrary bundle functor G on $\mathscr{F}\mathscr{M}_{m,n}$ admits a natural operator transforming connections on $Y \to M$ into connections on $GY \to M$ by means of an auxiliary higher order linear connection ∇ on M. By Corollary 2, if the base order of G is not zero, then the use of a linear connection ∇ is unavoidable.

5. The solution of Problem 2

Let $G\colon \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}$ be a bundle functor. Suppose first that there exists a natural operator D transforming connections Γ on $Y \to M$ into connections $D(\Gamma)$ on $GY \to Y$. Composing $D(\Gamma)$ with Γ we obtain a connection $\widetilde{D}(\Gamma)$ on $GY \to M$. Clearly, if $\Gamma\colon Y \times_M TM \to TY$, then $\widetilde{D}(\Gamma)\colon GY \times_M TM \to TGY$ is defined by

$$\widetilde{D}(\Gamma)(u, v) = D(\Gamma)(u, \Gamma(y, v)), \quad (u, v) \in GY \times_M TM, \quad u \in (GY)_y.$$

By Theorem 1, $G \cong V^F$ and the order of G is $(0, s, 0)$, $s = \operatorname{ord}(F)$. From Proposition 2 it follows that the functor $G^2\colon \mathscr{M}f_n \to \mathscr{F}\mathscr{M}$ defined by

$$G^2 N = G(\mathbb{R}^m \times N), \quad G^2\psi = G(\mathrm{id}_{\mathbb{R}^m} \times \psi)$$

is of order zero. Therefore $F\colon \mathscr{M}f_n \to \mathscr{F}\mathscr{M}$ is of order zero as well, i.e. F is isomorphic to a trivial bundle functor

$$F^W\colon \mathscr{M}f_n \to \mathscr{F}\mathscr{M}, \quad F^W N = N \times W, \quad F^W\psi = \psi \times \mathrm{id}_W$$

for some manifold W. Then the corresponding F-vertical functor $G = V^{(F^W)}$ is also isomorphic to a trivial bundle functor

$$G^W\colon \mathscr{F}\mathscr{M}_{m,n} \to \mathscr{F}\mathscr{M}, \quad G^W Y = Y \times W, \quad G^W\Phi = \Phi \times \mathrm{id}_W$$

for some W. So we have proved

Proposition 10. *If there is a natural operator transforming connections on $Y \to M$ into connections on $GY \to Y$, then G is isomorphic to a trivial bundle functor G^W for some manifold W.*

On the other hand, if $G = G^W$ is a trivial bundle functor, then we have a trivial connection on $Y \times W \to Y$. This defines a natural operator transforming connections on $Y \to M$ into connections on $G^W Y \to Y$. We have

Theorem 2. *A bundle functor G on $\mathcal{FM}_{m,n}$ admits an $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to Y$ if and only if G is isomorphic to a trivial bundle functor G^W for some manifold W. For G^W such natural operator is unique.*

Proof. Because of the existence of a trivial connection on $G^W Y \to Y$, it suffices to prove only the uniqueness part. Clearly, the difference of two connections on $Y \times W \to Y$ is a map $(Y \times W) \times_Y TY \to V(Y \times W)$. So it remains to show that any $\mathcal{FM}_{m,n}$-natural vector bundle map

$$\Delta(\Gamma)\colon\ (Y \times W) \times_Y TY \to V(Y \times W)$$

over $Y \times W$ is zero. First, the $\mathcal{FM}_{m,n}$-invariance implies that the map Δ is determined by the values

$$\Delta(\Gamma)\Big((0,0), w, \frac{\partial}{\partial x^1}_{(0,0)}\Big) \in V_{((0,0),w)}(\mathbb{R}^{m,n} \times W) \tag{7}$$

for all connections Γ on $\mathbb{R}^{m,n} \to \mathbb{R}^m$ and all $w \in W$. In local coordinates (x^i, y^j) on $\mathbb{R}^{m,n}$ a connection Γ has the coordinate expression

$$\Gamma = \sum_{i=1}^{m} \mathrm{d}x^i \otimes \frac{\partial}{\partial x^i} + \sum_{k=1}^{m} \sum_{l=1}^{n} \Gamma^l_k \,\mathrm{d}x^k \otimes \frac{\partial}{\partial y^l}.$$

By the corollary of non-linear Peetre theorem (Corollary 19.8 in [6]), it suffices to restrict ourselves to connections Γ on $\mathbb{R}^{m,n} \to \mathbb{R}^m$ with coefficients of the form

$$\Gamma^l_k = \sum_{|\alpha|+|\beta| \leqslant K} \Gamma^l_{k\alpha\beta} x^\alpha y^\beta$$

for any $K \in \mathbb{N}$. Using the invariance with respect to the homotheties $t\,\mathrm{id}_{\mathbb{R}^{m,n}}$, $t \neq 0$ and then the homogeneous function theorem from [6] we see that Δ is determined by the values (7) for all connections Γ on $\mathbb{R}^{m,n} \to \mathbb{R}^m$ whose coefficients are polynomials of degree $\leqslant 1$ and all $w \in W$. Further, taking into account the invariance of Δ with respect to the base homotheties $t\,\mathrm{id}_{\mathbb{R}^m}\times\mathbb{R}^n$ and then using the homogeneous function theorem we deduce that Δ is determined by the values

$$\Delta\Big(\sum \mathrm{d}x^i\otimes\frac{\partial}{\partial x^i}+y^j\,\mathrm{d}x^k\otimes\frac{\partial}{\partial y^l}\Big)\Big((0,0),w,\frac{\partial}{\partial x^1}_{(0,0)}\Big)\in V_{((0,0),w)}(\mathbb{R}^{m,n}\times W)$$

and

$$\Delta\Big(\sum \mathrm{d}x^i\otimes\frac{\partial}{\partial x^i}+\mathrm{d}x^k\otimes\frac{\partial}{\partial y^l}\Big)\Big((0,0),w,\frac{\partial}{\partial x^1}_{(0,0)}\Big)\in V_{(0,0),w)}(\mathbb{R}^{m,n}\times W)$$

for all $w\in W$ and all $k=1,\dots,m$ and $j,l=1,\dots,n$. Hence Δ is uniquely determined by the values

$$\Delta\Big(\sum \mathrm{d}x^i\otimes\frac{\partial}{\partial x^i}+\mathrm{d}x^k\otimes Y\Big)((0,0),w,v)\in V_{((0,0),w)}(\mathbb{R}^{m,n}\times W)$$

for all $k=1,\dots,m$, all vector fields Y on $\mathbb{R}^n$, all $w\in W$ and all $v\in T_{(0,0)}\mathbb{R}^{m,n}$. Clearly, any non-vanishing vertical vector field Y on $\mathbb{R}^{m,n}$ not depending on x^i can be transformed locally into $\partial/\partial y^1$ by means of a fibered isomorphism of the form $(\mathrm{id}_{\mathbb{R}^m}\times\psi)$. Using the regularity and the invariance of Δ with respect to $\mathscr{F}\mathscr{M}_{m,n}$-maps of the form $\mathrm{id}_{\mathbb{R}^m}\times\psi$ we see that Δ is determined by the values

$$\Delta\Big(\sum \mathrm{d}x^i\otimes\frac{\partial}{\partial x^i}+\mathrm{d}x^k\otimes\frac{\partial}{\partial y^1}\Big)((0,0),w,v)\in V_{((0,0),w)}(\mathbb{R}^{m,n}\times W)$$

for all k,w,v as above. Because of the invariance of Δ with respect to the $\mathscr{F}\mathscr{M}_{m,n}$-map

$$(x^1,\dots,x^m,y^1-x^k,y^2,\dots,y^n),$$

Δ is uniquely determined by the values

$$\Delta\Big(\sum \mathrm{d}x^i\otimes\frac{\partial}{\partial x^i}\Big)((0,0),w,v))\in V_{((0,0),w)}(\mathbb{R}^{m,n}\times W)$$

for v,w as above. Finally, using the invariance of Δ with respect to the homotheties $t\,\mathrm{id}_{\mathbb{R}^{m,n}}$, we get $\Delta(\sum \mathrm{d}x^i\otimes\partial/\partial x^i)((0,0),w,v)=0$. Thus we have proved that $\Delta=0$, which completes the proof. □

Remark 3. By Theorem 2, if G is not isomorphic to a trivial bundle functor, then there is no natural operator transforming connections on $Y\to M$ into connections on $GY\to Y$. However, if we restrict ourselves to some aditional structure on GY, then natural operators may exist. For example, in 46.10 of [6] there are constructed first order operators, natural on the local isomorphisms of affine bundles, which transform connections on $Y\to M$ into connections on $VY\to Y$.

Remark 4. There is another approach to the prolongation of connections. The second author has recently proved that a vector bundle functor H on $\mathcal{M}f$ with the point property admits natural operators transforming connections on a fibered manifold $p\colon Y \to M$ into connections on $Hp\colon HY \to HM$ if and only if H preserves products, see [8].

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