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A NOTE ON LOCAL AUTOMORPHISMS

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Abstract. Let H be an infinite-dimensional almost separable Hilbert space. We show that every local automorphism of $\mathscr{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H, is an automorphism.

Keywords: automorphism, local automorphism, algebra of operators on a Hilbert space

MSC 2000: 47B48, 46L40

1. Introduction and statement of the main result

A linear mapping φ of an algebra $\mathscr{A}$ into itself is called a *local automorphism* if for every $a \in \mathscr{A}$ there exists an automorphism φ_a of $\mathscr{A}$ such that $\varphi(a) = \varphi_a(a)$. This notion was introduced by Larson and Sourour in [5]. They have proved that every surjective local automorphism of $\mathscr{B}(X)$, the algebra of all bounded linear operators on an infinite-dimensional Banach space X, is an automorphism [5, Theorem 2.1] (for finite-dimensional spaces X, the result is somewhat different [5, Theorem 2.2]. In [1] Brešar and Šemrl improved this result in the case when X is a separable Hilbert space. They proved that every local automorphism φ of $\mathscr{B}(H)$ (note that here we do not assume surjectivity of φ), where H is an infinite-dimensional separable Hilbert space, is an automorphism [1, Theorem 2]. The aim of this paper is to give a shorter and simpler proof of this result and also to extend it to the most important class of nonseparable Hilbert spaces. Recall that a Hilbert space is *separable* if it has a countable orthonormal basis. We shall say that a Hilbert space is *almost separable* if it has an orthonormal basis of the power less or equal to continuum.

Theorem 1.1. *Let H be an infinite-dimensional almost separable Hilbert space. Then every local automorphism of $\mathscr{B}(H)$ is an automorphism.*

2. Proof of the main result

Throughout, H will be a complex infinite-dimensional Hilbert space and $\mathscr{B}(H)$ the algebra of all bounded linear operators on H. By $\mathscr{F}(H)$ we denote the ideal of all operators in $\mathscr{B}(H)$ of finite rank. For every $T \in \mathscr{B}(H)$ we denote by $\operatorname{Im} T$ the image of T and by $\operatorname{Ker} T$ the kernel of T. Given nonzero $x, y \in H$, by $x \otimes y$ we denote a rank one operator defined by $(x \otimes y)z = \langle z, y\rangle x$, $z \in H$. Note that the spectrum of the operator $x \otimes y$ is equal to the set $\{0, \langle x, y\rangle\}$. Operators $T, S \in \mathscr{B}(H)$ are said to be *similar* if there exists an invertible operator $A \in \mathscr{B}(H)$ such that $S = ATA^{-1}$. Since every automorphism of $\mathscr{B}(H)$ is inner [2], a local automorphism φ of $\mathscr{B}(H)$ can be equivalently defined as a linear mapping with the property that the operators T and $\varphi(T)$ are similar for every $T \in \mathscr{B}(H)$. Note also that any local automorphism φ of an algebra $\mathscr{A}$ preserves idempotents, that is, for any idempotent $p \in \mathscr{A}$, $\varphi(p)$ is again an idempotent.

In order to prove Theorem 1.1, we establish three preliminary results. The first lemma was already proved in [3, Lemma 3]. As its proof is rather short we have included it for the sake of completeness. In the proof of the second lemma we shall basically just follow the arguments from [1]. The core of the paper is the last lemma which is new.

Lemma 2.1. *If X and Y are complex normed linear spaces and $A\colon X \to Y$ is a bijective linear operator such that A^{-1} carries closed hyperplanes to closed hyperplanes, then A is bounded.*

Proof. Let g be a nonzero bounded linear functional on Y. By hypothesis $A^{-1}(\operatorname{Ker} g)$ is a closed hyperplane, so we can choose a bounded linear functional f on X and a vector $u \in X$ such that

$$\operatorname{Ker} f = A^{-1}(\operatorname{Ker} g) \quad \text{and} \quad f(u) = 1.$$

Then any $x \in X$ can be written in the form

$$x = f(x)u + v$$

for some $v \in \operatorname{Ker} f$. Hence

$$g(Ax) = g(A(f(x)u)) + g(Av) = f(x)g(Au).$$

It follows that $g \circ A$ is bounded and thus A is bounded because g is arbitrary. □

Lemma 2.2. *Let H be an infinite-dimensional almost separable Hilbert space and let φ be a local automorphism of $\mathscr{B}(H)$. Then the restriction of φ to $\mathscr{F}(H)$ is either a homomorphism, or an antihomomorphism.*

Proof. Let $P, Q \in \mathscr{B}(H)$ be orthogonal idempotents, that is, $PQ = QP = 0$. Since $P + Q$ is again an idempotent, it follows that $\varphi(P+Q)^2 = \varphi(P+Q)$. Hence $\varphi(P)\varphi(Q) + \varphi(Q)\varphi(P) = 0$, which (by a standard argument) gives $\varphi(P)\varphi(Q) = \varphi(Q)\varphi(P) = 0$. So, we have shown that φ maps any set of pairwise orthogonal idempotents into a set of pairwise orthogonal idempotents.

Let $S \in \mathscr{F}(H)$ be a self-adjoint operator. Then $S = \sum\limits_{i=1}^{n} \lambda_i P_i$, where the P_i's are mutually orthogonal idempotents and the λ_i's are real numbers. Hence $\varphi(S^2) = \varphi(S)^2$ (φ maps orthogonal idempotents into orthogonal idempotents). Replacing in this identity S by $S + T$, where S and T are both self-adjoint, we obtain that $\varphi(ST+TS) = \varphi(S)\varphi(T)+\varphi(T)\varphi(S)$. Since every operator $F \in \mathscr{F}(H)$ can be written in the form $F = S + \mathrm{i}T$ with $S, T \in \mathscr{F}(H)$ self-adjoint, we get $\varphi(F^2) = \varphi(F)^2$. Thus the restriction of φ to $\mathscr{F}(H)$ is a Jordan homomorphism. Since $\mathscr{F}(H)$ is a locally matrix algebra, a result of Jacobson and Rickart [4, Theorem 8] tells us that $\varphi|\mathscr{F}(H) = \varphi + \theta$, where $\varphi\colon \mathscr{F}(H) \to \mathscr{B}(H)$ is a homomorphism and $\theta\colon \mathscr{F}(H) \to \mathscr{B}(H)$ is an antihomomorphism. Pick an idempotent $P \in \mathscr{B}(H)$ of rank one. Then $\varphi(P)$ is the sum of idempotents $\varphi(P)$ and $\theta(P)$. Therefore, as $\varphi(P)$ also has rank one, it follows that either $\varphi(P) = 0$ or $\theta(P) = 0$. Thus, at least one of φ and θ has a nonzero kernel. Since the kernels of homomorphisms and antihomomorphisms are ideals, and since the only nonzero ideal of $\mathscr{F}(H)$ is $\mathscr{F}(H)$ itself, we have $\varphi = 0$ or $\theta = 0$. Thus, the restriction of φ to $\mathscr{F}(H)$ is either a homomorphism or an antihomomorphism. □

Lemma 2.3. *Let H be an infinite-dimensional almost separable Hilbert space and let φ be a local automorphism of $\mathscr{B}(H)$. If the restriction of φ to $\mathscr{F}(H)$ is a homomorphism, then φ is an automorphism.*

Proof. Fix $u \in H$ such that $\|u\| = 1$. As $\varphi(u \otimes u)$ is an idempotent of rank one, we have

$$\varphi(u \otimes u) = v \otimes w,$$

where $\langle v, w\rangle = 1$. Define $A, B\colon H \to H$ by

$$Ax = \varphi(x \otimes u)v, \qquad Bx = \varphi(u \otimes x)^* w.$$

Clearly, A and B are linear operators. Since $\varphi|\mathscr{F}(H)$ is a homomorphism, for all $x, y \in H$ we have

$$\begin{aligned}\varphi(x \otimes y) &= \varphi((x \otimes u)(u \otimes u)(u \otimes y)) \\ &= \varphi(x \otimes u)(v \otimes w)\varphi(u \otimes y) \\ &= (\varphi(x \otimes u)v) \otimes (\varphi(u \otimes y)^* w) = (Ax) \otimes (By).\end{aligned}$$

Moreover,

$$\langle x, y\rangle = \langle Ax, By\rangle,$$

because the spectrum of the operator $x \otimes y$ is equal to the spectrum of the operator $\varphi(x \otimes y) = (Ax) \otimes (By)$. This implies that A and B are injective operators.

Let $P \in \mathscr{B}(H)$ be a nontrivial idempotent and $x \in \operatorname{Ker} P$. Pick an element $y \in H$ such that $\langle x, y\rangle = 1$ and $\langle Pz, y\rangle = 0$ for every $z \in H$. Since $x \otimes y$ and P are orthogonal idempotents and since φ maps orthogonal idempotents into orthogonal idempotents it follows that $\varphi(P)$ and $(Ax) \otimes (By)$ are orthogonal idempotents. In particular, $Ax \in \operatorname{Ker} \varphi(P)$. Now, let $x \in \operatorname{Im} P$. Then $x \in \operatorname{Ker}(I - P)$, which yields (see above) that $Ax \in \operatorname{Ker} \varphi(I - P) = \operatorname{Ker}(I - \varphi(P))$. We use $\operatorname{Ker}(I - \varphi(P)) = \operatorname{Im} \varphi(P)$ to conclude that $Ax \in \operatorname{Im} \varphi(P)$.

Let $x \in H$. Then $x = y + z$, where $y \in \operatorname{Ker} P$ and $z \in \operatorname{Im} P$. Thus $\varphi(P)Ax = \varphi(P)Ay + \varphi(P)Az = \varphi(P)Az = Az$. Therefore, $\operatorname{Im} A$ is invariant under every idempotent $\varphi(P)$, $P \in \mathscr{B}(H)$. Moreover, the restriction of $\varphi(P)$ to $\operatorname{Im} A$ considered as a map from $\operatorname{Im} A$ into itself is equal to CPC^{-1} (here C denotes the bijection $C\colon H \to \operatorname{Im} A$ defined by $Cx = Ax$, $x \in H$). Using the result of Pearcy and Topping [6] which states that every operator in $\mathscr{B}(H)$ is a sum of idempotents we conclude that $\operatorname{Im} A$ is invariant under every $\varphi(T)$, $T \in \mathscr{B}(H)$, and

$$\varphi(T)|\operatorname{Im} A = CTC^{-1}, \quad T \in \mathscr{B}(H). \tag{1}$$

We will prove that C and C^{-1} are bounded operators. Let $K \subseteq H$ be a closed hyperplane. Then $K = \operatorname{Ker} P$ for some idempotent $P \in \mathscr{B}(H)$ and $C(K) = C(\operatorname{Ker} P) = \operatorname{Ker}(\varphi(P)|\operatorname{Im} A)$ (see above). Thus $C(K)$ is a closed hyperplane in $\operatorname{Im} A$. Applying Lemma 2.1 we then conclude that C^{-1} is bounded. Now, suppose that the operator C is not bounded. Let $\{y_n\colon n \in \mathbb{N}\} \subseteq H$ be a set of orthonormal vectors. For every $n \in \mathbb{N}$ we can find $x_n \in \operatorname{Im} A$ such that $C^{-1}x_n = y_n$. Moreover, we can find orthonormal vectors $\{z_n\colon n \in \mathbb{N}\}$ such that $\|Cz_n\| > n\|x_n\|$ for every $n \in \mathbb{N}$. Pick an operator $T \in \mathscr{B}(H)$ such that $Ty_n = z_n$, $n \in \mathbb{N}$. Then $\|CTC^{-1}x_n\| = \|Cz_n\| > n\|x_n\|$, a contradiction ($CTC^{-1}$ is a bounded operator on $\operatorname{Im} A$). So we have proved that C is a bounded operator. Since this is also true for the operator C^{-1} it follows that $\operatorname{Im} A$ is isomorphic to H. In particular, $\operatorname{Im} A$ is closed.

Suppose that H is an infinite-dimensional Hilbert space with an orthonormal basis of the power of the continuum and let $\{e_\lambda\colon \lambda \in [0,1]\}$ be an orthonormal basis in H. Define a linear operator $S\colon H \to H$ by $Se_\lambda = \lambda e_\lambda$, $\lambda \in [0,1]$. Of course, $S \in \mathscr{B}(H)$. Let K be the orthogonal complement of $\operatorname{Im} A$, $H = \operatorname{Im} A \oplus K$. According to this decomposition $\varphi(S)$ has the following matrix representation (see (1))

$$\varphi(S) = \begin{bmatrix} CSC^{-1} & S_1 \\ 0 & S_2 \end{bmatrix} \tag{2}$$

for some operators $S_1\colon K \to \operatorname{Im} A$ and $S_2\colon K \to K$. Since H is equal to the closure of the direct sum of one-dimensional subspaces $\oplus_{\lambda\in[0,1]} \operatorname{Ker}(S - \lambda I)$ and since S and $\varphi(S)$ are similar we have

$$H = \overline{\oplus_{\lambda\in[0,1]} \operatorname{Ker}(\varphi(S) - \lambda I)}, \tag{3}$$

where $\operatorname{Ker}(\varphi(S) - \lambda I)$ are again one-dimensional subspaces. Applying (2) and (3) we get

$$H = \overline{\oplus_{\lambda\in[0,1]} \operatorname{span}\left\{\begin{bmatrix} Ce_\lambda \\ 0 \end{bmatrix}\right\}},$$

where $\operatorname{span}\left\{\begin{bmatrix} Ce_\lambda \\ 0 \end{bmatrix}\right\}$ denotes the linear span of the vector $\begin{bmatrix} Ce_\lambda \\ 0 \end{bmatrix}$. Therefore $H \subseteq \operatorname{Im} A$ and consequently $H = \operatorname{Im} A$. Thus, $A\colon H \to H$ is an invertible bounded linear operator and $\varphi(T) = ATA^{-1}$ for every $T \in \mathscr{B}(H)$. The case when H is an infinite-dimensional Hilbert space with a countable orthonormal basis can be treated similarly, by considering a bounded linear operator $S\colon H \to H$ defined by $Se_n = \frac{1}{n}e_n$, $n \in \mathbb{N}$, where $\{e_n\colon n \in \mathbb{N}\}$ is an orthonormal basis in H. □

P r o o f of Theorem 1.1. By Lemma 2.2 the restriction of φ to $\mathscr{F}(H)$ is either a homomorphism or an antihomomorphism. In view of Lemma 2.3 it suffices to consider the situation when $\varphi|\mathscr{F}(H) = \theta$ is an antihomomorphism. But then, as φ maps $\mathscr{F}(H)$ into itself, $\varphi^2|\mathscr{F}(H) = \theta^2$ is a homomorphism. Observe that φ^2 is also a local automorphism. Applying Lemma 2.3 we then conclude that φ^2 is an automorphism. In particular, φ^2 is onto, which implies that so is φ. Thus, φ satisfies the requirements of the result of Larson and Sourour [5]. Hence φ is an automorphism. □

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