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STAR NUMBER AND STAR ARBORICITY OF A COMPLETE MULTIGRAPH

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Abstract. Let G be a multigraph. The *star number* $\mathrm{s}(G)$ of G is the minimum number of stars needed to decompose the edges of G. The *star arboricity* $\mathrm{sa}(G)$ of G is the minimum number of star forests needed to decompose the edges of G. As usual λK_n denote the λ-fold complete graph on n vertices (i.e., the multigraph on n vertices such that there are λ edges between every pair of vertices). In this paper, we prove that for $n \geqslant 2$

$$\mathrm{s}(\lambda K_n) = \begin{cases} \frac{1}{2}\lambda n & \text{if } \lambda \text{ is even,} \\ \frac{1}{2}(\lambda+1)n - 1 & \text{if } \lambda \text{ is odd,} \end{cases} \tag{1}$$

$$\mathrm{sa}(\lambda K_n) = \begin{cases} \lceil \frac{1}{2}\lambda n \rceil & \text{if } \lambda \text{ is odd, } n = 2, 3 \text{ or } \lambda \text{ is even,} \\ \lceil \frac{1}{2}\lambda n \rceil + 1 & \text{if } \lambda \text{ is odd, } n \geqslant 4. \end{cases} \tag{2}$$

Keywords: decomposition, star arboricity, star forest, complete multigraph

MSC 2000: 05C70

1. Introduction

A *star* is the complete bipartite graph $K_{1,m}$ for some positive integer m. A *star forest* is a forest each component of which is a star. Let G be a multigraph. The *star number* $\mathrm{s}(G)$ of G is the minimum number of stars needed to decompose the edges of G. The *star arboricity* $\mathrm{sa}(G)$ of G is the minimum number of star forests needed to decompose the edges of G. In the literature the star number and the star arboricity were investigated for simple graphs.

For a graph G, the *independence number* $\alpha(G)$ of G is defined to be the maximum size of a set A of vertices in G such that every pair of vertices in A are nonadjacent; the *covering number* $\beta(G)$ of G is defined to be the minimum size of a set B of vertices in G such that every edge of G is incident with at least one vertex in B. It is well known [5] that $\alpha(G)+\beta(G)=|V(G)|$. And it is easy to see that $\beta(G)=\mathrm{s}(G)$ if G is a simple graph. Star numbers, independence numbers and star arboricities were studied for some specific families of graphs. The star number was determined for the power of a cycle [12] (here the power of a cycle is a special case of circulant graphs). The independence numbers were determined for the following graphs: the Cartesian product of two odd cycles [8], the direct product of two paths, or two cycles, or a path and a cycle [10], and some specific family of circulant graphs [13]. The star arboricities were studied for the following graphs: complete bipartite graphs [6], [7], [15], complete regular multipartite graphs [3], cubes [15], crowns [11], and planar graphs [2], [9].

For a graph G and a positive integer λ, we use λG to denote the graph obtained from G by replacing each edge e of G by λ edges with the same ends as e. Hence λK_n is a multigraph on n vertices such that there are λ edges joining every pair of vertices. We call λK_n a *λ-fold complete graph* or a *complete multigraph*. In this paper the star number and the star arboricity of λK_n are determined. To avoid trivialities we assume that $n \geqslant 2$.

2. Star number and star aboricity of a complete multigraph

The *arboricity* $a(G)$ of a multigraph G is the minimum number of forests needed to decompose the edges of G. It is trivial from the definitions that $a(G) \leqslant \mathrm{sa}(G) \leqslant \mathrm{s}(G)$. The arboricity of any nontrivial multigraph is determined by the following well-known formula of Nash-Williams.

Proposition 1 ([4], [14]). *Let G be a nontrivial multigraph. Then*

$$a(G)=\max \lceil |E(H)|/(|V(H)|-1)\rceil$$

where the maximum is taken over all nontrivial induced subgraphs H of G.

It follows easily from Proposition 1 that $a(\lambda K_n)=\lceil \frac{1}{2}\lambda n\rceil$. The inequality that $a(\lambda K_n) \geqslant \lceil \frac{1}{2}\lambda n\rceil$ can also be seen easily, since any forest in λK_n has at most $n-1$ edges. To determine $\mathrm{s}(\lambda K_n)$ and $\mathrm{sa}(\lambda K_n)$, we first consider the easy case of λ even. For a positive integer k, we use S_k to denote the star with k edges.

Lemma 2. *For an even integer λ,* $\mathrm{sa}(\lambda K_n)=\mathrm{s}(\lambda K_n)=\frac{1}{2}\lambda n$.

P r o o f. By the above discussions, we have $\frac{1}{2}\lambda n \leqslant a(\lambda K_n) \leqslant \mathrm{sa}(\lambda K_n) \leqslant \mathrm{s}(\lambda K_n)$. It suffices to show that $\mathrm{s}(\lambda K_n) \leqslant \frac{1}{2}\lambda n$. Trivially the edges of λK_n can be decomposed into $\frac{1}{2}\lambda$ copies of $2K_n$ and the edges of $2K_n$ can be decomposed into n copies of S_{n-1}. Thus the edges of λK_n can be decomposed into $\frac{1}{2}\lambda n$ copies of S_{n-1}, which implies $\mathrm{s}(\lambda K_n) \leqslant \frac{1}{2}\lambda n$. This completes the proof. $\square$

Now we determine $\mathrm{s}(\lambda K_n)$.

Theorem 3.

$$\mathrm{s}(\lambda K_n) = \begin{cases} \frac{1}{2}\lambda n & \textit{if } \lambda \textit{ is even,} \\ \frac{1}{2}(\lambda+1)n - 1 & \textit{if } \lambda \textit{ is odd.} \end{cases}$$

Proof. Due to Lemma 2, we only need to show that for an odd integer λ, $\mathrm{s}(\lambda K_n) = \frac{1}{2}(\lambda+1)n - 1$.

First prove $\mathrm{s}(\lambda K_n) \geqslant \frac{1}{2}(\lambda+1)n - 1$. Let $\mathscr{D}$ be an arbitrary star decomposition of λK_n. We need to show $|\mathscr{D}| \geqslant \frac{1}{2}(\lambda+1)n - 1$. Let $v_1, v_2, \ldots, v_n$ be the vertices of λK_n. For $i = 1, 2, \ldots, n$, let $c(v_i)$ be the number of stars in $\mathscr{D}$ which have centers at v_i (for a star with only one edge, we arbitrarily choose one end of the edge as the center of the star). For $1 \leqslant i < j \leqslant n$, each edge joining v_i and v_j belongs to a star in $\mathscr{D}$ which has center either at v_i or at v_j, and distinct edges joining v_i and v_j belong to distinct stars. Thus $\lambda \leqslant c(v_i) + c(v_j)$. We distinguish two cases.

Case 1. $c(v_i) \geqslant \frac{1}{2}(\lambda+1)$ for $i = 1, 2, \ldots, n$. Then

$$|\mathscr{D}| = c(v_1) + c(v_2) + \ldots + c(v_n) \geqslant \frac{1}{2}n(\lambda+1) > \frac{1}{2}(\lambda+1)n - 1.$$

This completes Case 1.

Case 2. $c(v_i) \leqslant \frac{1}{2}(\lambda-1)$ for some i, say $c(v_1) \leqslant \frac{1}{2}(\lambda-1)$. Then

$$\begin{aligned} |\mathscr{D}| &= c(v_1) + c(v_2) + \ldots + c(v_n) \\ &= \sum_{i=2}^{n} (c(v_1) + c(v_i)) - (n-2)c(v_1) \\ &\geqslant (n-1)\lambda - \tfrac{1}{2}(n-2)(\lambda-1) \\ &= \tfrac{1}{2}(\lambda+1)n - 1. \end{aligned}$$

This completes Case 2.

We have proved $|\mathscr{D}| \geqslant \frac{1}{2}(\lambda+1)n - 1$ for any star decomposition $\mathscr{D}$ of λK_n. Thus $\mathrm{s}(\lambda K_n) \geqslant \frac{1}{2}(\lambda+1)n - 1$. Now we prove the reverse inequality. Note that λK_n can be decomposed into $\frac{1}{2}(\lambda-1)$ copies of $2K_n$ and one copy of K_n. Since $2K_n$ can be decomposed n copies of S_{n-1}, and K_n can be decomposed into $n-1$ stars, namely $S_{n-1}, S_{n-2}, \ldots, S_1$, we see that λK_n can be decomposed into $\frac{1}{2}(\lambda+1)n - 1$ stars. Thus $\mathrm{s}(\lambda K_n) \leqslant \frac{1}{2}(\lambda+1)n - 1$. This completes the proof. $\square$

Now we determine $\mathrm{sa}(\lambda K_n)$. Due to Lemma 2, we only need to consider the case of λ odd. Note that $\mathrm{sa}(K_n)$ has been determined by J. Akiyama and M. Kano as follows.

Proposition 4 ([1], [9]).

$$\mathrm{sa}(K_n) = \begin{cases} \lceil \frac{1}{2}n \rceil, & n = 2, 3, \\ \lceil \frac{1}{2}n \rceil + 1, & n \geqslant 4. \end{cases}$$

The following lemma is helpful for our discussions.

Lemma 5. *Let λ be any odd integer and n be an integer at least 3. Suppose that $\mathscr{F}$ is a family of edge-disjoint subgraphs of λK_n such that each member in $\mathscr{F}$ is isomorphic to S_{n-1}. Then $|\mathscr{F}| \leqslant \frac{1}{2}(\lambda - 1)n + 1$. Furthermore, if $|\mathscr{F}| = \frac{1}{2}(\lambda - 1)n + 1$, then there are $\frac{1}{2}(\lambda + 1)$ stars in $\mathscr{F}$ with centers at one specific vertex of λK_n, and there are $\frac{1}{2}(\lambda - 1)$ stars in $\mathscr{F}$ with centers at each of the remaining vertices of λK_n.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of λK_n. For $i = 1, 2, \ldots, n$, let $c(v_i)$ denote the number of stars in $\mathscr{F}$ which have centers at v_i. Without loss of generality, assume that $c(v_1) \geqslant c(v_i)$ for $i = 2, 3, \ldots, n$. For each i with $2 \leqslant i \leqslant n$, every S_{n-1} in $\mathscr{F}$ with center at v_1 contributes an edge joining v_1 and v_i; so does every S_{n-1} in $\mathscr{F}$ with center at v_i. Combining these with the fact that there are only λ edges joining v_1 and v_i in λK_n, we have $c(v_1) + c(v_i) \leqslant \lambda$. Now we show that $|\mathscr{F}| \leqslant \frac{1}{2}(\lambda - 1)n + 1$. We distinguish two cases.

Case 1. $c(v_1) \leqslant \frac{1}{2}(\lambda - 1)$. Then

$$\begin{aligned} |\mathscr{F}| &= c(v_1) + c(v_2) + \ldots + c(v_n) \\ &\leqslant \tfrac{1}{2}n(\lambda - 1) < \tfrac{1}{2}(\lambda - 1)n + 1. \end{aligned}$$

This completes Case 1.

Case 2. $c(v_1) \geqslant \frac{1}{2}(\lambda + 1)$.

Let $c(v_1) = \frac{1}{2}(\lambda + 1) + s$ where s is a nonnegative integer. Then $c(v_i) \leqslant \frac{1}{2}(\lambda - 1) - s$, for $2 \leqslant i \leqslant n$. Hence

$$\begin{aligned} |\mathscr{F}| &= c(v_1) + c(v_2) + \ldots + c(v_n) \\ &\leqslant (\tfrac{1}{2}(\lambda + 1) + s) + (n - 1)(\tfrac{1}{2}(\lambda - 1) - s) \\ &= \tfrac{1}{2}(\lambda - 1)n + 1 + (2 - n)s \leqslant \tfrac{1}{2}(\lambda - 1)n + 1. \end{aligned} \tag{1}$$

The last inequality is due to $n \geqslant 3$ and s being nonnegative. This completes Case 2.

The required inequality that $|\mathscr{F}| \leqslant \frac{1}{2}(\lambda-1)n+1$ has thus been established. Now we prove the "Furthermore" part. Since $|\mathscr{F}| = \frac{1}{2}(\lambda-1)n+1$, only Case 2 in the above discussion is possible and the inequalities in (1) become equalities; from the last inequality, we have $s = 0$ since $n \geqslant 3$, and from the first inequality, we have

$$\begin{aligned} &c(v_2) = c(v_3) = \ldots = c(v_n) = \tfrac{1}{2}(\lambda-1) - s = \tfrac{1}{2}(\lambda-1), \\ &c(v_1) = \tfrac{1}{2}(\lambda+1) + s = \tfrac{1}{2}(\lambda+1). \end{aligned}$$

Thus the required conclusion holds. □

The above lemma is used in the following.

Lemma 6. *For an odd integer* $\lambda \geqslant 3$, $\operatorname{sa}(\lambda K_n) = \frac{1}{2}(\lambda-1)n + \operatorname{sa}(K_n)$.

Proof. It is easy to see that $\operatorname{sa}(\lambda K_2) = \lambda$ for any $\lambda \geqslant 1$. Thus the required equality holds for $n = 2$. So we let $n \geqslant 3$.

By the definition of star arboricity, $\operatorname{sa}(\lambda K_n) \leqslant \operatorname{sa}((\lambda-1)K_n) + \operatorname{sa}(K_n)$ for $\lambda \geqslant 2$. Now λ is odd. By Lemma 2, $\operatorname{sa}((\lambda-1)K_n) = \frac{1}{2}(\lambda-1)n$. Thus $\operatorname{sa}(\lambda K_n) \leqslant \frac{1}{2}(\lambda-1)n + \operatorname{sa}(K_n)$. We now prove the reverse inequality.

Let $\mathscr{D}$ be an arbitrary star forest decomposition of λK_n. We need to show that $|\mathscr{D}| \geqslant \frac{1}{2}(\lambda-1)n + \operatorname{sa}(K_n)$. Let $\mathscr{D}'$ be a subfamily of $\mathscr{D}$ consisting of members which are isomorphic to S_{n-1}. By Lemma 5, $|\mathscr{D}'| \leqslant \frac{1}{2}(\lambda-1)n+1$. Consider two cases: Case 1: $|\mathscr{D}'| = \frac{1}{2}(\lambda-1)n+1$, Case 2: $|\mathscr{D}'| \leqslant \frac{1}{2}(\lambda-1)n$.

Case 1. $|\mathscr{D}'| = \frac{1}{2}(\lambda-1)n+1$.

Let $v_1, v_2, \ldots, v_n$ be the vertices of λK_n. From the "Furthermore" part of Lemma 5, $\mathscr{D}'$ is a family consisting of the following stars: $\frac{1}{2}(\lambda+1)$ S_{n-1}'s with centers at a specific vertex, say v_1, and $\frac{1}{2}(\lambda-1)$ S_{n-1}'s with centers at each of the remaining vertices. Thus $\bigcup_{G\in\mathscr{D}'} E(G)$ is an edge set consisting of the following edges: λ edges joining v_1 and v_i for every i with $2 \leqslant i \leqslant n$, and $\lambda-1$ edges joining v_i and v_j for every pair i, j with $2 \leqslant i < j \leqslant n$. We then see that $\lambda K_n - \bigcup_{G\in\mathscr{D}'} E(G)$ is a disjoint union of K_{n-1} and K_1 (to be specific, the complete graph on the vertices $v_2, v_3, \ldots, v_n$ and the trivial graph on the vertex v_1). Thus $\mathscr{D} - \mathscr{D}'$ is a star forest decomposition of K_{n-1}, which implies $|\mathscr{D} - \mathscr{D}'| \geqslant \operatorname{sa}(K_{n-1})$. Hence

$$\begin{aligned} |\mathscr{D}| &= |\mathscr{D}'| + |\mathscr{D} - \mathscr{D}'| \\ &\geqslant \tfrac{1}{2}(\lambda-1)n + 1 + \operatorname{sa}(K_{n-1}) \geqslant \tfrac{1}{2}(\lambda-1)n + \operatorname{sa}(K_n). \end{aligned}$$

The last inequality is due to the fact $\operatorname{sa}(K_n) \leqslant \operatorname{sa}(K_{n-1}) + 1$, which follows from the definition of star arboricity. This completes Case 1.

Case 2. $|\mathscr{D}'| \leqslant \frac{1}{2}(\lambda-1)n$.

Note that each member in $\mathscr{D}'$ has exactly $n-1$ edges and each member in $\mathscr{D}-\mathscr{D}'$ has at most $n-2$ edges. Hence

$$\begin{aligned}|E(\lambda K_n)| &\leqslant |\mathscr{D}'|(n-1)+|\mathscr{D}-\mathscr{D}'|(n-2)\\ &= |\mathscr{D}|(n-2)+|\mathscr{D}'| \leqslant |\mathscr{D}|(n-2)+\tfrac{1}{2}(\lambda-1)n.\end{aligned}$$

Thus $\lambda\binom{n}{2} \leqslant |\mathscr{D}|(n-2)+\frac{1}{2}(\lambda-1)n$, which implies that

$$(n-2)|\mathscr{D}| \geqslant \lambda\binom{n}{2}-\tfrac{1}{2}(\lambda-1)n=\tfrac{1}{2}\lambda n(n-2)+\tfrac{1}{2}n.$$

Hence

$$\begin{aligned}|\mathscr{D}| &\geqslant \tfrac{1}{2}\lambda n+\tfrac{n}{2(n-2)}=\tfrac{1}{2}(\lambda-1)n+\tfrac{n(n-1)}{2(n-2)}\\ &> \tfrac{1}{2}(\lambda-1)n+\tfrac{1}{2}(n+1) \geqslant \tfrac{1}{2}(\lambda-1)n+\lceil\tfrac{1}{2}n\rceil.\end{aligned}$$

Thus $|\mathscr{D}| \geqslant \frac{1}{2}(\lambda-1)n+\lceil\frac{1}{2}n\rceil+1$.

Combining this with Proposition 4, we obtain

$$|\mathscr{D}| \geqslant \tfrac{1}{2}(\lambda-1)n+\operatorname{sa}(K_n).$$

This completes Case 2.

Since we have proved that $|\mathscr{D}| \geqslant \frac{1}{2}(\lambda-1)n+\operatorname{sa}(K_n)$ for any star forest decomposition $\mathscr{D}$ of λK_n, we obtain $\operatorname{sa}(\lambda K_n) \geqslant \frac{1}{2}(\lambda-1)n+\operatorname{sa}(K_n)$. This completes the proof. □

Now we have the star arboricity of λK_n as follows.

Theorem 7.

$$\operatorname{sa}(\lambda K_n)=\begin{cases}\lceil\frac{1}{2}\lambda n\rceil & \textit{if } \lambda \textit{ is odd, } n=2,3 \textit{ or } \lambda \textit{ is even,}\\ \lceil\frac{1}{2}\lambda n\rceil+1 & \textit{if } \lambda \textit{ is odd, } n\geqslant 4.\end{cases}$$

P r o o f. By Lemma 2, the formula holds for even λ. By Proposition 4, the formula holds for $\lambda=1$. As to odd $\lambda\geqslant 3$, by Lemma 6 and Proposition 4,

$$\begin{aligned}\operatorname{sa}(\lambda K_n)=\tfrac{1}{2}(\lambda-1)n+\operatorname{sa}(K_n)&=\begin{cases}\frac{1}{2}(\lambda-1)n+\lceil\frac{1}{2}n\rceil, & n=2,3,\\ \frac{1}{2}(\lambda-1)n+\lceil\frac{1}{2}n\rceil+1, & n\geqslant 4\end{cases}\\ &=\begin{cases}\lceil\frac{1}{2}\lambda n\rceil, & n=2,3,\\ \lceil\frac{1}{2}\lambda n\rceil+1, & n\geqslant 4.\end{cases}\end{aligned}$$

This completes the proof. □

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