Jiří Rachůnek; Vladimír Slezák

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NEGATION IN BOUNDED COMMUTATIVE $DR\ell$-MONOIDS

Jiří Rachůnek and Vladimír Slezák, Olomouc

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Abstract. The class of commutative dually residuated lattice ordered monoids ($DR\ell$-monoids) contains among others Abelian lattice ordered groups, algebras of Hájek's Basic fuzzy logic and Brouwerian algebras. In the paper, a unary operation of negation in bounded $DR\ell$-monoids is introduced, its properties are studied and the sets of regular and dense elements of $DR\ell$-monoids are described.

Keywords: $DR\ell$-monoid, MV-algebra, BL-algebra, Brouwerian algebra, negation

MSC 2000: 06D35, 06F05

Commutative dually residuated lattice ordered monoids ($DR\ell$-monoids) were introduced by K. L. N. Swamy in [7] as a common generalization of Brouwerian algebras and Abelian lattice ordered groups. The first author has shown that also algebras of logics behind the theory of fuzzy sets can be considered, changing the signature, as special cases of bounded commutative $DR\ell$-monoids. According to [4] and [5], MV-algebras, i.e. an algebraic counterpart of the Łukasiewicz infinite-valued propositional logic [1], are polynomially equivalent to bounded commutative $DR\ell$-monoids satisfying the double negation law. Further, BL-algebras, i.e. an algebraic semantics of Hájek's basic fuzzy logic [3], are by [6] precisely the duals of bounded commutative $DR\ell$-monoids isomorphic to subdirect products of linearly ordered commutative $DR\ell$-monoids.

In this paper, the concept of a negation in bounded commutative $DR\ell$-monoids is established and some of its properties are studied. It is shown that using identities with negations one can characterize the variety of $DR\ell$-monoids induced by MV-algebras. Further, certain identities which are satisfied by any BL-algebra are founded. Properties of the sets of regular and dense elements are also described,

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especially for $DR\ell$-monoids from the variety defined by the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$. These results generalize some of the ones for BL-algebras from [2].

All the basic concepts and results concerning MV-algebras and BL-algebras can be found in [1] and [3], respectively.

Let us recall now the concept of a commutative $DR\ell$-monoid.

Definition. A *commutative dually residuated lattice ordered monoid* (briefly: *$DR\ell$-monoid*) is an algebra $M = (M, +, 0, \vee, \wedge, -)$ of type $\langle 2, 0, 2, 2, 2\rangle$ satisfying the following conditions:

(i) $(M, +, 0)$ is a commutative monoid;
(ii) $(M, \vee, \wedge)$ is a lattice;
(iii) The operation $+$ distributes over the operations $\vee$ and $\wedge$;
(iv) If $\leqslant$ denotes the order on M induced by the lattice $(M, \vee, \wedge)$, then $x - y$ is the smallest $z \in M$ such that $y + z \geqslant x$ for any $x, y \in M$;
(v) M satisfies the identity

$$((x-y) \vee 0) + y \leqslant x \vee y.$$

Remark. In what follows a *$DR\ell$-monoid* will always mean a commutative $DR\ell$-monoid. Basic properties of $DR\ell$-monoids were shown in [7], [8], [9].

Now we will deal with bounded $DR\ell$-monoids. The least element of such a $DR\ell$-monoid is by [8] always 0. Let us denote the greatest element by 1 and bounded $DR\ell$-monoids then will be considered as algebras $\mathcal{M} = (M, +, 0, 1, \vee, \wedge, -)$ of extended type $\langle 2, 0, 0, 2, 2, 2\rangle$.

Let us define a unary operation $\neg$ on M by

$$\neg x := 1 - x.$$

In the following lemma we show the basic properties of the negation in relation to other operations of $DR\ell$-monoid.

Lemma 1. *Let $M = (M, +, 0, 1, \vee, \wedge, -)$ be a bounded $DR\ell$-monoid and $x, y \in M$. Then*

(1) $\neg\neg 1 = 1$, $\neg\neg 0 = 0$;
(2) $\neg\neg x \leqslant x$, $\neg\neg\neg x = \neg x$;
(3) $\neg\neg x - \neg\neg y = \neg\neg x - y$;
(4) $\neg(x+y) = \neg x - y = \neg y - x = \neg y - \neg\neg x = \neg x - \neg\neg y$;
(5) $\neg(x - \neg\neg x) = 1$;
(6) $\neg\neg(x-y) = \neg\neg x - \neg\neg y$;

(7) $\neg\neg(x+y) \leqslant \neg\neg x + \neg\neg y$;

(8) $\neg(y-x) = \neg\neg(x+\neg y)$;

(9) $(y+x)-y \leqslant x$;

(10) $\neg(x \wedge y) = \neg x \vee \neg y$;

(11) $\neg\neg(x \vee y) = \neg\neg x \vee \neg\neg y$;

(12) $(y-(y-x)) \vee (x-(x-y)) \leqslant x \wedge y$.

P r o o f. (1) By [7] (Lemma 1), $\neg\neg 1 = \neg 0 = 1$, $\neg\neg 0 = \neg 1 = 0$.

(2) By the definition of a $DR\ell$-monoid, $\neg\neg x = 1-(1-x)$ is the least element $z \in M$ such that $(1-x)+z = 1$. We also have $x+(1-x) = 1$, thus $\neg\neg x \leqslant x$.

This also means that $\neg\neg\neg x \leqslant \neg x$ and, by [7] (Lemma 3), $\neg x \leqslant \neg\neg\neg x$.

(3) By [7] (Lemma 6), $\neg\neg x - y = (1-\neg x)-y = (1-y)-\neg x = \neg y - \neg x = \neg\neg\neg y - \neg x = (1-\neg\neg y)-\neg x = (1-\neg x)-\neg\neg y = \neg\neg x - \neg\neg y$.

(4) By [7] (Lemma 6), $\neg(x+y) = 1-(x+y) = (1-x)-y = (1-y)-x = \neg x - y = \neg y - x$. Furthermore, $\neg y - \neg\neg x = (1-y)-(1-(1-x)) = (1-(1-(1-x)))-y = \neg\neg\neg x - y = \neg x - y$.

(5) By [7] (Lemma 3), $x - \neg\neg x \leqslant 1 - \neg\neg x = \neg x$, thus $\neg\neg x \leqslant \neg(x - \neg\neg x)$. Hence, using (2), (4) and [7] (Lemma 2), we have $\neg(x-\neg\neg x) = \neg(x-\neg\neg x) \vee \neg\neg x = (\neg(x-\neg\neg x)-\neg\neg x)+\neg\neg x = \neg((x-\neg\neg x)+\neg\neg x)+\neg\neg x = \neg(x \vee \neg\neg x)+\neg\neg x = \neg x + \neg\neg x = (1-x)+(1-(1-x)) = 1$.

(6) By [7] (Lemma 13), $(1-x)+y \geqslant 1-(x-y) = \neg(x-y)$. Using [7] (Lemmas 6 and 3), we obtain $\neg\neg x - y = (1-\neg x)-y = 1-(\neg x+y) = 1-((1-x)+y) \leqslant 1-\neg(x-y) = \neg\neg(x-y)$.

On the other hand, first observe that by (2) and (4) we have $\neg\neg(\neg\neg x - y) = \neg\neg(\neg(\neg x+y)) = \neg(\neg x+y) = \neg\neg x - y$.

Now, taking into account (5) and [7] (Lemmas 2, 3 and 6) we have $\neg\neg(x-y) - (\neg\neg x - y) = \neg\neg(\neg\neg(x-y)-(\neg\neg x-y)) \leqslant \neg\neg((x-y)-(\neg\neg x-y)) = \neg\neg(x-(y+(\neg\neg x-y))) = \neg\neg(x-(\neg\neg x \vee y)) \leqslant \neg\neg((x-\neg\neg x)\wedge(x-y)) \leqslant \neg\neg(x-\neg\neg x) = \neg 1 = 0$, which implies (using [7] (Lemma 7)) $\neg\neg(x-y) \leqslant \neg\neg x - y$.

The identity (6) now follows from (3).

(7) By (4), $\neg(\neg\neg x + \neg\neg y) = \neg x - \neg\neg y = \neg(x+y)$. Then, by (2), $\neg\neg(x+y) = \neg\neg(\neg\neg x + \neg\neg y) \leqslant \neg\neg x + \neg\neg y$.

(8) By (4), (6) and (2), $\neg\neg(x+\neg y) = \neg(\neg\neg y - \neg\neg x) = \neg\neg\neg(y-x) = \neg(y-x)$.

(9) This follows from $y+((y+x)-y) = y \vee (y+x)$ by [7] (Lemma 2).

(10) By [7] (Lemma 5), $\neg(x \wedge y) = 1-(x \wedge y) = (1-x) \vee (1-y) = \neg x \vee \neg y$.

(11) Since $\neg\neg x \leqslant \neg\neg(x \vee y)$ and $\neg\neg y \leqslant \neg\neg(x \vee y)$, we have $\neg\neg x \vee \neg\neg y \leqslant \neg\neg(x \vee y)$. On the other hand, by (7) and (6), we have $\neg\neg(x \vee y) = \neg\neg((x-y)+y) \leqslant \neg\neg(x-y)+\neg\neg y = (\neg\neg x - \neg\neg y)+\neg\neg y = \neg\neg x \vee \neg\neg y$.

(12) Since $y-(y-x) \leqslant (y-y)+x = x$ and $y-(y-x) \leqslant y$ by [7] (Lemmas 3, 13), we have $y-(y-x) \leqslant x \wedge y$. In a similar way we show that $x-(x-y) \leqslant x \wedge y$, hence $(y-(y-x)) \vee (x-(x-y)) \leqslant x \wedge y$. □

Using negations in bounded $DR\ell$-monoids one can characterize MV-algebras. Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. If we put

$$1 := \neg 0, \quad x+y := x \oplus y, \quad x-y := \neg(\neg x \oplus y),$$
$$x \vee y := \neg(\neg x \oplus y) \oplus y, \quad x \wedge y := \neg(\neg x \vee \neg y),$$

for any $x, y \in A$, then $M = M(\mathcal{A}) = (A, +, 0, 1, \vee, \wedge, -)$ is a bounded $DR\ell$-monoid that will be called *induced* by $\mathcal{A}$. From [5] we know that any bounded $DR\ell$-monoid is induced by some MV-algebra $\mathcal{A}$ if and only if it satisfies the identity

$$\neg\neg x = x.$$

Hence, the class of MV-algebras can be considered as a variety of bounded $DR\ell$-monoids.

Let us show that the class of dual BL-algebras satisfies certain identities containing the negation.

Proposition 2. *If M is a bounded $DR\ell$-monoid and $x, y \in M$, then the following conditions are equivalent:*
(1) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$.
(2) $\neg(x \vee y) = \neg x \wedge \neg y$.
(3) $\neg(x \vee y) + ((x-y) \wedge (y-x)) = \neg(x \vee y)$.

Proof. (1) ⇒ (2) By Lemma 1 (10), (11), $\neg x \wedge \neg y = \neg\neg(\neg x \wedge \neg y) = \neg(\neg\neg x \vee \neg\neg y) = \neg(x \vee y)$.

(2) ⇒ (1) If $\neg(x \vee y) = \neg x \wedge \neg y$, then by Lemma 1 (10), $\neg\neg(x \wedge y) = \neg(\neg x \vee \neg y) = \neg\neg x \wedge \neg\neg y$.

(2) ⇒ (3) By [7] (Lemmas 4, 15), $\neg x = 1 - x = [1-(x \vee y)] + [(x \vee y) - x] = \neg(x \vee y) + (y-x)$. Similarly, $\neg y = \neg(x \vee y) + (x-y)$. It follows that $\neg(x \vee y) = \neg x \wedge \neg y = [\neg(x \vee y) + (y-x)] \wedge [\neg(x \vee y) + (x-y)] = \neg(x \vee y) + [(y-x) \wedge (x-y)]$.

(3) ⇒ (2) Assume that (3) holds. Then $\neg x \wedge \neg y = (\neg(x \vee y) + (y-x)) \wedge (\neg(x \vee y) + (x-y)) = \neg(x \vee y) + ((y-x) \wedge (x-y)) = \neg(x \vee y)$. □

Corollary 3. *If M is a dual BL-algebra, then M fulfils the identities from Proposition 2.*

P r o o f. By [6], Theorem 1, a bounded $DR\ell$-monoid M is a dual BL-algebra if and only if M is representable, i.e. it is isomorphic to a subdirect product of linearly ordered $DR\ell$-monoids. By [10], a $DR\ell$-monoid is representable if and only if it satisfies the identity $(x-y)\wedge(y-x)\leqslant 0$, so in the bounded case the identity $(x-y)\wedge(y-x)=0$. $\square$

Remark. Obviously, if M is a bounded $DR\ell$-monoid and $x,y\in M$ fulfil the conditions from Proposition 2, then $(x-y)\wedge(y-x)\leqslant\neg(x\vee y)$. Simultaneously, for any x,y of an arbitrary bounded $DR\ell$-monoid M the following inequality holds:

$$\neg\neg((x-y)\wedge(y-x))\leqslant\neg(x\vee y)$$

Actually, by Lemma 1 (11), (10), from $y-x\leqslant 1-x$, $x-y\leqslant 1-y$ it follows that $\neg(y-x)\geqslant\neg\neg x$, $\neg(x-y)\geqslant\neg\neg y$, thus $\neg(y-x)\vee\neg(x-y)\geqslant\neg\neg x\vee\neg\neg y=\neg\neg(x\vee y)$. This means that $\neg((y-x)\wedge(x-y))\geqslant\neg\neg(x\vee y)$, which implies $\neg\neg((y-x)\wedge(x-y))\leqslant\neg(x\vee y)$.

However, the stronger inequality $((x-y)\wedge(y-x))\leqslant\neg(x\vee y)$ does not hold in general (and thus the identities from Proposition 2 are not valid generally in bounded $DR\ell$-monoids), as the following example shows.

Example. The lattice L with the Hasse diagram in Figure 1 is a Brouwerian algebra, and thus a bounded commutative $DR\ell$-monoid. But for elements $c,d\in L$ we have: $\neg(c\vee d)=e$, $(c-d)\wedge(d-c)=a$, hence $(c-d)\wedge(d-c)\not\leqslant\neg(c\vee d)$.

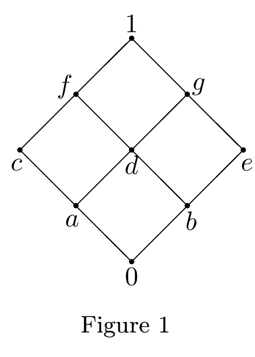

Figure 1

If $M=(M,+,0,1,\vee,\wedge,-)$ is a bounded $DR\ell$-monoid and $x\in M$, then x is said to be a *regular element* if $\neg\neg x=x$. Denote by $R(M)$ the set of all regular elements of M.

Proposition 4. *For any bounded $DR\ell$-monoid $M = (M, +, 0, 1, \vee, \wedge, -)$, $R(M)$ is a subalgebra of the reduct $(M, 0, 1, \vee, -)$.*

P r o o f. This follows from Lemma 1 (1), (6), (11). $\square$

Proposition 5. *If a bounded $DR\ell$-monoid M fulfils some of the conditions*

(a) *M is a dual BL-algebra,*

(b) *M is a Brouwerian algebra,*

then M satisfies the identity

$$\neg\neg(x+y) = \neg\neg x + \neg\neg y.$$

P r o o f. a) Let M be a dual BL-algebra, $x, y \in M$. Then, by Lemma 1,

$$\begin{aligned}
\neg\neg(x+y) &= \neg\neg(x+y) \vee \neg\neg x = \neg(\neg x - y) \vee \neg\neg x \\
&= \neg\neg x + (\neg(\neg x - y) - \neg\neg x) = \neg\neg x + (\neg(\neg x - \neg\neg y) - \neg\neg x) \\
&= \neg\neg x + (\neg x - (\neg y - \neg\neg x)) = \neg\neg x + \neg(\neg\neg x + (\neg y - \neg\neg x)) \\
&= \neg\neg x + \neg(\neg\neg x \vee \neg y), \\
\neg\neg x + \neg\neg y &= 1 \wedge (\neg\neg x + \neg\neg y) = (\neg\neg x + \neg x) \wedge (\neg\neg x + \neg\neg y) \\
&= \neg\neg x + (\neg x \wedge \neg\neg y),
\end{aligned}$$

thus by Corollary 3 we get $\neg\neg(x+y) = \neg\neg x + \neg\neg y$.

b) If M is a Brouwerian algebra and $x, y \in M$, then by Lemma 1 (11) and from the fact that the operations $+$ and $\vee$ coincide in M we obtain

$$\neg\neg(x+y) = \neg\neg(x \vee y) = \neg\neg x \vee \neg\neg y = \neg\neg x + \neg\neg y.$$

$\square$

Remark. The variety of bounded $DR\ell$-monoids satisfying the identity $\neg\neg(x + y) = \neg\neg x + \neg\neg y$ thus contains all algebras of fuzzy logic (or their duals) and Brouwerian algebras, therefore this variety is considerably wide. However, the question whether it is distinct from the variety of all bounded $DR\ell$-monoids still remains open.

The following theorem is a direct consequence of Lemma 1 and Propositions 2 and 4.

Theorem 6.

(a) *If a bounded $DR\ell$-monoid M fulfils the condition $\neg\neg(x+y) = \neg\neg x + \neg\neg y$, then $R(M)$ is a subalgebra in $(M, +, 0, 1, \vee, -)$ and the mapping $x \mapsto \neg\neg x$ is a retract of $(M, +, 0, 1, \vee, -)$ onto $(R(M), +, 0, 1, \vee, -)$.*

(b) *If M is a dual BL-algebra, then $R(M)$ is a subalgebra in M.*

Remark. The part (b) of the previous theorem follows also from [2], Theorem 1.2.

Theorem 7. *If a bounded $DR\ell$-monoid M fulfils the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$, then $R(M) = (R(M), +, 0, 1, \vee, \wedge_{R(M)}, -)$, where $y \wedge_{R(M)} z = \neg\neg(y \wedge z)$ for all $y, z \in R(M)$ and the other operations are restrictions of the operations on M, is a $DR\ell$-monoid induced by an MV-algebra.*

P r o o f. From Lemma 1 (2) and from the fact that the operation $-$ is antitone in the second variable it follows that $\neg\neg$ is an interior operator on the lattice $(M, \vee, \wedge)$. Thus $\neg\neg x$ is the greatest element of $R(M)$ included in the element $x \in M$. Further, $(R(M), \leqslant)$ is a lattice and for any $y, z \in R(M)$

$$y \vee_{R(M)} z = y \vee z, \quad y \wedge_{R(M)} z = \neg\neg(y \wedge z).$$

Let $w, y, z \in R(M)$. Then

$$\begin{aligned} w + (y \wedge_{R(M)} z) &= w + \neg\neg(y \wedge z) = \neg\neg w + \neg\neg(y \wedge z) = \neg\neg(w + (y \wedge z)) \\ &= \neg\neg((w+y) \wedge (w+z)) = (w+y) \wedge_{R(M)} (w+z). \end{aligned}$$

Obviously, $y -_{R(M)} z$ exists for any $y, z \in R(M)$ and $y -_{R(M)} z = y - z$.

Thus $(R(M), +, 0, 1, \vee, \wedge_{R(M)}, -)$ is a $DR\ell$-monoid. Furthermore, $\neg\neg\neg\neg x = \neg\neg x$ for any $x \in M$, hence the double negation law holds in $R(M)$, that is $\neg\neg y = y$ for each $y \in R(M)$. According to [5], $R(M)$ is induced by an MV-algebra. $\square$

An element x of a bounded $DR\ell$-monoid M is said to be *dense* if $\neg\neg x = 0$. Denote by $D(M)$ the set of all dense elements of M.

Now, let us recall the concept of an ideal in a $DR\ell$-monoid. For bounded $DR\ell$-monoids it can be defined in the following way:

Definition. Let M be a bounded $DR\ell$-monoid and $\emptyset \neq I \subseteq M$. Then I is called an *ideal* in M if

(a) $x, y \in I \Rightarrow x + y \in I$,

(b) $x \in I, z \in M, z \leqslant x \Rightarrow z \in I$.

Ideals in $DR\ell$-monoids are by [9] in a one-to-one correspondence to congruences. If I is an ideal in M, then for the congruence $\theta(I)$ induced by I we have: If $x, y \in M$, then $\langle x, y\rangle \in \theta(I)$ if and only if $(x-y) \vee (y-x) \in I$. Conversely, if θ is a congruence on M and $I(\theta) = \{x \in M; \langle x, 0\rangle \in \theta\}$, then $I(\theta)$ is an ideal induced by θ.

Theorem 8. *If a bounded $DR\ell$-monoid M fulfils the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$, then $D(M)$ is an ideal in M and $M/D(M)$ is isomorphic to $R(M)$.*

Proof. Let $x, y \in D(M)$. Then $\neg\neg(x+y) = \neg\neg x + \neg\neg y = 0$, thus $x+y \in D(M)$. If $x \in D(M)$, $z \in M$ and $z \leqslant x$, then $\neg\neg z \leqslant \neg\neg x = 0$, thus $z \in D(M)$.

Denote by $\theta(D(M))$ the congruence induced on M by $D(M)$. It means that $\langle x, y\rangle \in \theta(D(M))$ iff $(x-y) \vee (y-x) \in D(M)$. Hence $\langle x, y\rangle \in \theta(D(M))$ iff $\neg\neg((x-y) \vee (y-x)) = 0$, thus by Lemma 1 (11), (6) iff $(\neg\neg x - \neg\neg y) \vee (\neg\neg y - \neg\neg x) = 0$. This is equivalent to $(\neg\neg x - \neg\neg y) = 0$ and $(\neg\neg y - \neg\neg x) = 0$, i.e. $\neg\neg x \leqslant \neg\neg y$ and $\neg\neg y \leqslant \neg\neg x$ which yields $\neg\neg x = \neg\neg y$. Therefore, $M/D(M) \cong R(M)$. $\square$

A $DR\ell$-monoid M is said to be *simple* if M is non-trivial and has no proper congruences distinct from the identity. In the following theorem which generalizes Theorem 1.7 in [2] we characterize simple $DR\ell$-monoids from the variety studied just now.

Theorem 9. *Let M be a bounded $DR\ell$-monoid satisfying the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$. Then M is simple if and only if it is induced by a simple MV-algebra.*

Proof. If M satisfies the assumptions, then by Theorem 8 $D(M)$ is its ideal. In addition, let M be a simple $DR\ell$-monoid. Then M has a unique proper ideal, thus $D(M) = \{0\}$. Hence, by Theorem 7, M is induced by an MV-algebra. $\square$

Definition. If M is a bounded $DR\ell$-monoid and I is an ideal in M, then I is called an *MV-ideal* if a $DR\ell$-monoid $M/\theta(I)$ is induced by an MV-algebra.

In the following theorem MV-ideals in $DR\ell$-monoids satisfying the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$ are characterized.

Theorem 10. *Let a bounded $DR\ell$-monoid M satisfy the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$ and let I be an ideal in M. Then the following conditions are equivalent.*
(1) *I is an MV-ideal.*
(2) *$x - \neg\neg x \in I$ for any $x \in M$.*
(3) *$\neg\neg x \in I \Rightarrow x \in I$ for any $x \in M$.*
(4) *$D(M) \subseteq I$.*

Proof. (1) $\Leftrightarrow$ (2) $M/\theta(I)$ is an MV-algebra $\Leftrightarrow \forall x \in M\colon \langle x, \neg\neg x\rangle \in \theta(I) \Leftrightarrow \forall x \in M\colon (x - \neg\neg x) \vee (\neg\neg x - x) \in I \Leftrightarrow \forall x \in M\colon x - \neg\neg x \in I$.

(2) $\Rightarrow$ (3) Let $x - \neg\neg x \in I$, $\neg\neg x \in I$. Then $x = x \vee \neg\neg x = (x - \neg\neg x) + \neg\neg x \in I$.

(3) $\Rightarrow$ (4) Let $\neg\neg x \in I \Rightarrow x \in I$. Then $y \in D(M) \Rightarrow \neg\neg y = 0 \in I \Rightarrow y \in I$. Thus $D(M) \subseteq I$.

(4) $\Rightarrow$ (1) Let $D(M) \subseteq I$. Then $M/\theta(I)$ is isomorphic to a subalgebra in $M/\theta(D(M))$, which is an MV-algebra. □

If M is a bounded $DR\ell$-monoid, we denote by $\mathrm{Rad}(M)$ the intersection of all proper maximal ideals in M. ($\mathrm{Rad}(M)$ is called the *radical* of M.)

Theorem 11. *Let M be a bounded $DR\ell$-monoid M satisfying the identity $\neg\neg(x+y) = \neg\neg x + \neg\neg y$. Then*
(a) $D(M) \subseteq \mathrm{Rad}(M)$*;*
(b) $D(M) = \mathrm{Rad}(M)$ *if and only if $R(M)$ is induced by a semi-simple MV-algebra.*

P r o o f. (a) Let I be a maximal proper ideal in M. Then M/I is a simple $DR\ell$-monoid, thus by Theorem 9 and Theorem 10, $D(M) \subseteq I$. Hence $D(M) \subseteq \mathrm{Rad}(M)$.

(b) Now it is obvious that $D(M) = \mathrm{Rad}(M)$ if and only if $R(M)$ is isomorphic to a subdirect product of simple $DR\ell$-monoids, thus by Theorem 8 and [1] if and only if $R(M)$ is induced by a semi-simple MV-algebra. □

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Author's address: Department of Algebra and Geometry, Faculty of Sciences, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `rachunek@inf.upol.cz`, `slezak@prfnw.upol.cz`.