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THE DISTANCE BETWEEN FIXED POINTS OF SOME PAIRS OF MAPS IN BANACH SPACES AND APPLICATIONS TO DIFFERENTIAL SYSTEMS

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Abstract. Let T be a γ-contraction on a Banach space Y and let S be an almost γ-contraction, *i.e.* sum of an (ε,γ)-contraction with a continuous, bounded function which is less than ε in norm. According to the contraction principle, there is a unique element u in Y for which $u = Tu$. If moreover there exists v in Y with $v = Sv$, then we will give estimates for $\|u-v\|$. Finally, we establish some inequalities related to the Cauchy problem.

Keywords: contraction principle, Cauchy problem

MSC 2000: 34A12, 34L30

Let $(Y,\|\cdot\|)$ be a real Banach space. For a bounded function $\varphi\colon D\subset Y\to Y$ we define the norm

$$\|\varphi\| = \sup_{y\in D}\|\varphi(y)\|.$$

A map $T\colon Y\to Y$ is called γ-contraction if

$$\|Tu-Tv\|\leqslant\gamma\|u-v\|$$

for all $u,v\in Y$. The constant $\gamma\in(0,1)$ is also called the contraction coefficient. According to the contraction principle, there is a unique element u in Y for which $u=Tu$.

Given $\varepsilon>0$, we will say that a continuous, bounded map $S\colon Y\to Y$ is an almost (ε,γ)-contraction if there exists a γ-contraction $T\colon Y\to Y$ for which

$$\|Sy-Ty\|\leqslant\varepsilon,\quad\forall\, y\in Y.$$

It results that an almost (ε,γ)-contraction S can be written as

$$S = T + \varphi, \tag{1}$$

where T is a γ-contraction and φ is continuous and bounded, with

$$\|\varphi\| \leqslant \varepsilon.$$

Proposition 1. *Let $T\colon Y \to Y$ be a γ-contraction $\gamma \in (0,1)$ and let $S\colon Y \to Y$ be an almost (ε,γ)-contraction. Assume that $u \in Y$ is such that $u = Tu$ and there exists $v \in Y$ such that $v = Sv$. Then*

$$\|u - v\| \leqslant \frac{\varepsilon}{1-\gamma}.$$

P r o o f. We have

$$\|u - v\| = \|Tu - Sv\| \leqslant \|Tu - Tv\| + \|Tv - Sv\| \leqslant \gamma\|u - v\| + \varepsilon$$

or

$$\|u - v\| \leqslant \gamma\|u - v\| + \varepsilon.$$

Hence

$$\|u - v\| - \gamma\|u - v\| \leqslant \varepsilon \Leftrightarrow \|u - v\| \leqslant \frac{\varepsilon}{1-\gamma}.$$

□

By taking $\varphi = 0$ in (1), we deduce that every γ-contraction is an (ε,γ)-contraction, so we can prove

Proposition 2. *Let a γ_1-contraction $T_1\colon Y \to Y$ and a γ_2-contraction $T_2\colon Y \to Y$ $(\gamma_1, \gamma_2 \in (0,1))$ with*

$$\|T_1 y - T_2 y\| \leqslant \varepsilon$$

for all y in Y be given. We consider also the corresponding fixed points u and v, i.e.

$$u = T_1 u, \quad v = T_2 v.$$

Then

$$\|u - v\| \leqslant \frac{\varepsilon}{1 - \min\{\gamma_1, \gamma_2\}}.$$

P r o o f. Setting $T = T_1$, $S = T_2$, then $T = T_2$, $S = T_1$, in Proposition 1, we obtain successively

$$\|u - v\| \leqslant \frac{\varepsilon}{1-\gamma_1}, \quad \|u - v\| \leqslant \frac{\varepsilon}{1-\gamma_2},$$

so the inequality is proved. □

We use now these inequalities to establish some estimates in the existence theory of differential systems.

Let $f\colon D \subset \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function defined on a rectangle

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R}^m;\ |x - x_0| \leqslant a, \quad \|y - y_0\| \leqslant b\}$$

where $a, x_0 \in \mathbb{R}$ and $b, y_0 \in \mathbb{R}^m$. Here $\|\cdot\|$ denotes a norm on the m-dimensional space $\mathbb{R}^m$. Let us consider the Cauchy problem

$$\text{(PC)} \qquad \begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

This problem is uniquely solvable (at least locally) if f is Lipschitz with respect to the second argument, *i.e.*,

$$\|f(x, y_1) - f(x, y_2)\| \leqslant L\|y_1 - y_2\|, \quad \forall\ (x, y_1),\ (x, y_2) \in D,$$

for some positive real constant L. According to a well-known result, the solution of the Cauchy problem (PC) is defined at least on

$$y\colon\ (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$$

where

$$\delta = \min\left\{a, \frac{b}{M}\right\}.$$

The constant M satisfies

$$\|f(x, y)\| \leqslant M \quad \forall\ (x, y) \in D,$$

possibly

$$M = \sup_{(x,y)\in D} \|f(x, y)\|.$$

Moreover, a well-known theorem due to Peano says that the continuity condition on f ensures the existence of a solution of the Cauchy problem (PC). For proof and other details, see [5], [6]. We introduce

Theorem 1. *Assume that continuous functions* $f, g\colon D \subset \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ *satisfy the following conditions:*

a) *f is Lipschitz with respect to the second argument, i.e.*

$$\|f(x, y_1) - f(x, y_2)\| \leqslant L\|y_1 - y_2\| \quad \forall\ (x, y_1),\ (x, y_2) \in D,$$

for some $L > 0$.

b) *There exists* $\varepsilon > 0$ *such that* $\|f - g\| \leqslant \varepsilon$.

Let u and v be solutions of the Cauchy problems

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0, \end{cases} \qquad \begin{cases} y' = g(x, y), \\ y(x_0) = y_0 \end{cases}$$

respectively and denote $M = \max\{\|f\|, \|g\|\}$.

Then for every $0 < \delta < \min\{a, b/M, 1/L\}$ *we have*

$$\|u(x) - v(x)\| \leqslant \frac{\varepsilon}{\delta^{-1} - L} \qquad \forall\ x \in [x_0 - \delta, x_0 + \delta].$$

P r o o f. The given Cauchy problems are equivalent to the integral equations

$$u(x) = y_0 + \int_{x_0}^{x} f(s, u(s))\,\mathrm{d}s,\ \ v(x) = y_0 + \int_{x_0}^{x} g(s, v(s))\,\mathrm{d}s,$$

so we naturally define operators

$$T, S\colon\ C(I) \to C(I)$$

by the formulas

$$Tu(x) = y_0 + \int_{x_0}^{x} f(s, u(s))\,\mathrm{d}s,\ \ Sv(x) = y_0 + \int_{x_0}^{x} g(s, v(s))\,\mathrm{d}s.$$

By $C(I)$ we mean the Banach space of all continuous functions

$$y\colon\ I \to \mathbb{R}^m,\ \ I = [x_0 - \delta, x_0 + \delta],\ \ \delta < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\},$$

endowed with the norm of uniform convergence,

$$\|y\| = \max_{x \in I} \|y(x)\|.$$

Now the given Cauchy problems can be written as fixed point problems

$$u = Tu,\ v = Sv,\ u, v \in C(I).$$

We will use Proposition 1 to prove Theorem 1. In $Y = C(I)$ we have

$$\begin{aligned}\|Ty_1(x) - Ty_2(x)\| &= \left\| \int_{x_0}^{x} [f(s, y_1(s)) - f(s, y_2(s))]\,\mathrm{d}s \right\| \\ &\leqslant \left| \int_{x_0}^{x} \|f(s, y_1(s)) - f(s, y_2(s))\|\,\mathrm{d}s \right| \\ &\leqslant L \left| \int_{x_0}^{x} \|y_1(s) - y_2(s)\|\,\mathrm{d}s \right| \\ &\leqslant L\delta \|y_1 - y_2\|.\end{aligned}$$

Hence

$$\|Ty_1 - Ty_2\| \leqslant \gamma \|y_1 - y_2\|$$

with $\gamma = L\delta < 1$. Further,

$$\begin{aligned}\|Ty - Sy\| &= \left\| \int_{x_0}^{x} [f(s, y(s)) - g(s, y(s))]\,\mathrm{d}s \right\| \\ &\leqslant \left| \int_{x_0}^{x} \|f(s, y(s)) - g(s, y(s))\|\,\mathrm{d}s \right| \leqslant \varepsilon \left| \int_{x_0}^{x} \mathrm{d}s \right| \leqslant \varepsilon\delta.\end{aligned}$$

Hence

$$\|Ty - Sy\| \leqslant \delta\varepsilon \quad \forall\ y \in C(I),$$

so S is an almost $(\delta\varepsilon, \gamma)$-contraction. The hypotheses of Proposition 1 are fulfilled, so

$$\|u - v\| \leqslant \frac{\varepsilon\delta}{1 - \delta L}.$$

$\square$

Further, we give a uniqueness result for a class of Cauchy problems.

Theorem 2. *Let $\varphi, \psi_n \colon D \subset \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be continuous and consider the Cauchy problems*

$$(\mathrm{PC}_n) \qquad \begin{cases} y' = \varphi(x, y) + \psi_n(x, y), \\ y(x_0) = y_0, \end{cases} \qquad n \geqslant 1.$$

Assume that

a) φ *is Lipschitz with respect to the second argument, i.e.*

$$\|\varphi(x,y_1)-\varphi(x,y_2)\| \leqslant L\|y_1-y_2\| \quad \forall\ (x,y_1),\ (x,y_2)\in D.$$

b) ψ_n *are Lipschitz with respect to the second argument, i.e.*

$$\|\psi_n(x,y_1)-\psi_n(x,y_2)\| \leqslant L'\|y_1-y_2\| \ \ \forall\ (x,y_1),\ (x,y_2)\in D,\ \ n\geqslant 1.$$

c) *The sequence* $(\psi_n)_{n\geqslant 1}$ *converges to* ψ *uniformly on* D.
Then the Cauchy problem

$$(\mathrm{PC}_\infty) \qquad \begin{cases} y'=\varphi(x,y)+\psi(x,y),\\ y(x_0)=y_0 \end{cases}$$

has (locally) a unique solution.

Proof. According to the Peano theorem, the problem (PC_∞) has at least one solution. From c) it follows that the sequence $(\psi_n)_{n\geqslant 1}$ is uniformly bounded, *i.e.*

$$\|\psi_n\| \leqslant M,\ \ \|\psi\| \leqslant M,\quad \forall\ n\geqslant 1,$$

for some $M>0$. Then each problem (PC_n) has a unique solution u_n defined at least on

$$u_n\colon\ (x_0-\delta, x_0+\delta)\to\mathbb{R},$$

where $\delta>0$ is chosen in

$$0<\delta<\min\Big\{a, \frac{b}{M}, \frac{1}{L+L'}\Big\}.$$

We apply Theorem 1 to the Cauchy problems

$$\begin{cases} y'=\varphi(x,y)+\psi_n(x,y),\\ y(x_0)=y_0, \end{cases} \qquad \begin{cases} y'=\varphi(x,y)+\psi(x,y),\\ y(x_0)=y_0. \end{cases}$$

In order to respect the notation from Theorem 1, let us put

$$f(x,y)=\varphi(x,y)+\psi_n(x,y),\ \ g(x,y)=\varphi(x,y)+\psi(x,y).$$

Then evidently f is Lipschitz with respect to the second argument,

$$\|f(x,y_1)-f(x,y_2)\| \leqslant (L+L')\|y_1-y_2\| \ \ \forall\ (x,y_1),\ (x,y_2)\in D$$

and

$$\|f-g\| = \|\psi_n - \psi\|.$$

From Theorem 1 we obtain that for every solution u of the problem (PC_∞), we have

$$\|u_n(x) - u(x)\| \leqslant \frac{\|\psi_n - \psi\|}{\delta^{-1} - L - L'} \quad \forall\, x \in (x_0 - \delta, x_0 + \delta)\,.$$

Finally, by taking the limit for $n \to \infty$, we obtain

$$u = \lim_{n\to\infty} u_n \text{ (uniformly)},$$

which proves the uniqueness of u. □

References

[1] *C. Mortici*: Approximate methods for solving the Cauchy problem. Czechoslovak Math. J. *55* (2005), 709–718. Zbl 1081.34009

[2] *C. Mortici and S. Sburlan*: A coincidence degree for bifurcation problems. Nonlinear Analysis, TMA *53* (2003), 715–721. Zbl 1028.47046

[3] *C. Mortici*: Operators of monotone type and periodic solutions for some semilinear problems. Mathematical Reports *54* (1/2002), 109–121. Zbl 1062.47062

[4] *C. Mortici*: Semilinear equations in Hilbert spaces with quasi-positive nonlinearity. Studia Cluj. *4* (2001), 89–94. Zbl 1027.47044

[5] *D. Pascali and S. Sburlan*: Nonlinear Mappings of Monotone Type. Alphen aan den Rijn, Sijthoff & Noordhoff International Publishers, The Netherlands, 1978. Zbl 0423.47021

[6] *S. Sburlan, L. Barbu and C. Mortici*: Ecuaţii Diferenţiale. Integrale şi Sisteme Dinamice. Editura Ex Ponto, Constanţa, Romania, 1999. Zbl 0951.34003

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