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PROJECTIVE MODULES AND PRIME SUBMODULES

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Abstract. In this paper, we use Zorn's Lemma, multiplicatively closed subsets and saturated closed subsets for the following two topics:

(i) The existence of prime submodules in some cases,

(ii) The proof that submodules with a certain property satisfy the radical formula.

We also give a partial characterization of a submodule of a projective module which satisfies the prime property.

Keywords: prime submodule, primary submodule, $\mathcal{S}$-closed subsets, the radical formula

MSC 2000: 113A10, 13A99, 13C10

0. Introduction

Throughout the paper R will denote a commutative ring with identity. Let M be a unitary module over R. Let B and C be two submodules of M. Then it is clear that the set $\{r \in R\colon\ rC \leqslant B\}$ is an ideal of R, denoted by $(B : C)$. A proper submodule N of M with $\mathfrak{P} = (N : M)$ is said to be $\mathfrak{P}$-*prime* if $rm \in N$ for $r \in R$ and $m \in M$ implies that $r \in \mathfrak{P}$ or $m \in N$. It is well-known that a proper submodule N of M is prime if and only if $\mathfrak{P} = (N : M)$ is a prime ideal in R and the $R/\mathfrak{P}$-module M/N is torsion free. For any submodule N of M, *the radical of* N in M is defined to be the intersection of all prime submodules of M containing N, denoted by M-$\mathrm{rad}_R N$. Also M-$\mathrm{rad}_R 0$ is defined to be the intersection of all prime submodules of M. If there is no prime submodule containing N, then M-$\mathrm{rad}_R N = M$. The radical of submodules has been studied in recent years (see, for example, [6], [8], [8]). In this paper we continue these investigations for a certain case.

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Section 1 is concerned with the existence of prime submodules. We also prove a consequence of the Prime Avoidance Theorem for modules and give its application.

In Section 2, the main aim is to give a necessary and sufficient condition for the equality $M\text{-rad}_R N = \sqrt{(N : M)}M$ where N is a submodule of a projective R-module M. For a submodule N of a finitely generated projective module M, we prove that N is prime if and only if $(N : M)$ is prime and M/N is a projective R/P-module. Moreover, we show that $M\text{-rad}_R N = \sqrt{(N : M)}M + N = RE_M(N)$ for a submodule N of a module M provided M/N is projective. In particular, we show that $M\text{-rad}_R 0 = \sqrt{(0 : M)}M$ for a projective R-module M.

1. $\mathcal{S}$-closed subset of modules

In the first half of this section, we give a consequence of the Prime Avoidance Theorem for modules to which Lu extended the Prime Avoidance Theorem for rings in [4] and we give an application of them. Now we start by recalling the Prime Avoidance Theorem for modules.

Theorem 1.1 (Prime Avoidance Theorem [4]). *Let M be an R-module. Let $N_1, N_2, \ldots, N_n$ be a finite number of submodules of M and let N be a submodule of M such that $N \subseteq N_1 \cup \ldots \cup N_n$. Assume that at most two of the N_i's $(1 \leqslant i \leqslant n)$ are not prime and that $(N_j : M) \nsubseteq (N_k : M)$ whenever $j \neq k$. Then $N \subseteq N_i$ for some i.*

Now we extend [11, Theorem 3.64] to the module case by using Theorem 1.1.

Theorem 1.2. *Let M be an R-module. Suppose that $N_1, \ldots, N_r$ are prime submodules of M such that $(N_i : M) \nsubseteq (N_j : M)$ for $i \neq j$ where $r \geqslant 1$, let N be a submodule of M and let $m \in M$ be such that $mR + N \nsubseteq \bigcup\limits_{i=1}^{r} N_i$. Then there exists $n \in N$ such that $m + n \notin \bigcup\limits_{i=1}^{r} N_i$.*

Proof. Suppose that m lies in each of $N_1, \ldots, N_k$ but in none of $N_{k+1}, \ldots, N_r$. If $k = 0$ then $m = m + 0 \notin \bigcup\limits_{i=1}^{r} N_i$ and so there is nothing to prove. Now we assume that our claim is true for $k \geqslant 1$.

Now $N \nsubseteq \bigcup\limits_{i=1}^{k} N_i$, for otherwise by the Prime Avoidance Theorem we would have a contradiction. Thus there exists $d \in N \setminus (N_1 \cup \ldots \cup N_k)$. Hence we have $N_{k+1} \cap \ldots \cap N_r \nsubseteq N_1 \cup \ldots \cup N_k$. Otherwise, since N_j is a prime submodule, by the Prime Avoidance Theorem we get a contradiction. Thus there exists $b \in (N_{k+1} : M) \cap \ldots \cap (N_r : M) \setminus ((N_1 : M) \cup \ldots \cup (N_k : M))$. Let $n = bd \in N$. On the other hand,

$n \in \bigcap_{j=k+1}^{r} N_j$. Then $n = bd \notin N_1 \cup \ldots \cup N_k$. Otherwise, $bd \in N_i$ for $1 \leqslant i \leqslant k$ Since N_i is prime, either $d \in N_i$ or $b \in (N_i : M)$. Then $n \in (N_{k+1} \cap \ldots \cap N_r) \setminus (N_1 \cup \ldots \cup N_k)$. Therefore, since $m \in (N_1 \cup \ldots \cup N_k)$, it follows that $m + n \notin \bigcup_{i=1}^{r} N_i$. □

Now our main aim is to use Zorn's Lemma for the existence of prime submodules under a certain condition. It is concerned with a subset which is closed relative to a multiplicatively closed subset in a commutative ring. Throughout this section, we assume that every multiplicatively closed subset of R contains 1, but does not contain 0. Let $\mathcal{S}$ be a multiplicatively closed subset of a ring R and let M be an R-module. Then following [4], a non-empty subset S^* of M is said to be *$\mathcal{S}$-closed* if $sm \in S^*$ for every $s \in \mathcal{S}$ and $m \in S^*$. Further, an $\mathcal{S}$-closed subset S^* is *saturated* if the following condition is satisfied: whenever $rm \in S^*$ for $r \in R$ and $m \in M$, then $r \in \mathcal{S}$ and $m \in S^*$.

Let N be a prime submodule of an R-module M. Evidently, if $S^* = M \setminus N$ and $\mathcal{S} = R \setminus (N : M)$, then S^* is a saturated $\mathcal{S}$-closed subset of M. Now we give the main theorem of this section.

Theorem 1.3. *Let N be a submodule of an R-module M and let $\mathcal{S}$ be a multiplicatively closed subset of R. Also suppose that S^* is an $\mathcal{S}$-closed subset of M with $N \cap S^* = \emptyset$ and $\mathfrak{P} = (N : M)$ is a maximal ideal in $R \setminus \mathcal{S}$ such that $M/\mathfrak{P}M$ is a finitely generated R-module. Then the set*

$$\Psi = \{K \leqslant M\colon\ N \leqslant K,\ \ K \cap S^* = \emptyset \ \text{ and } \ (K : M) = (N : M)\}$$

of submodules of M has at least one maximal element, and any such maximal element of Ψ is a prime submodule of M. Moreover, it is a maximal submodule in $M \setminus S^*$.

Proof. Clearly the set Ψ is non-empty. Let Δ be a non-empty totally ordered subset of Ψ. Then $Q = \bigcup_{K_i \in \Delta} K_i$ is a submodule of M such that $N \subseteq Q$ and $Q \cap S^* = \emptyset$. Since $M/\mathfrak{P}M$ is finitely generated, we have $(Q : M) = (N : M)$. Thus Q is an upper bound for Δ in Ψ and so it follows from Zorn's Lemma that Ψ has at least one maximal element.

Let U be an arbitrary maximal element of Ψ. Then U is a proper submodule of M. Take $a \in M \setminus U$.Then there exist $s \in S^*$, $r \in R$ and $u \in U$ such that $s = u + ra$. On the other hand, $\mathcal{S} \cap (N : M) = \emptyset$. Take $b \in R \setminus (N : M)$. Then $\mathcal{S} \cap ((N : M) + Rb) \neq \emptyset$ and so there exist $s' \in \mathcal{S}$, $q \in \mathfrak{P}$ and $r' \in R$ such that $s' = q + r'b$. Hence we have $ss' = uq + ur'b + raq + rr'ab$ and so $ab \notin U$. Thus U is prime.

For the second claim, let T be a submodule in $M \setminus S^*$ such that $U \subset T$. Then $(U : M)$ is strictly contained in $(T : M)$. Thus there exists an element x in $(T : M) \cap \mathcal{S}$. But this yields that $xs \in S^* \cap T = \emptyset$ for any $s \in S^*$, a contradiction. □

Let M be am R-module and let $\mathfrak{P}$ be a prime ideal of R. Then we recall $M(\mathfrak{P})$, the following subset of M from [8]: $M(\mathfrak{P}) = \{m \in M\colon \mathcal{A}m \subseteq \mathfrak{P}M$ for some ideal $\mathcal{A} \nsubseteq \mathfrak{P}\}$. It is clear that $M(\mathfrak{P})$ is a submodule of M and $\mathfrak{P}M \subseteq M(\mathfrak{P})$.

Corollary 1.4. *Let N be a submodule of an R-module M and let $\mathcal{S}$ be a multiplicatively closed subset of R Also suppose that S^* is an $\mathcal{S}$-closed subset of M with $N \cap S^* = \emptyset$ and $\mathfrak{P} = (N : M)$ is a maximal ideal in $R \setminus \mathcal{S}$ such that $M/\mathfrak{P}M$ is a finitely generated R-module. Then*

(1) *there exists a prime submodule P of M such that $\mathfrak{P} = (P : M)$;*

(2) *$\mathfrak{P} = (\mathfrak{P}M : M)$;*

(3) *$M(\mathfrak{P})$ is a $\mathfrak{P}$-prime submodule of M.*

The following corollary is clear by Proposition 1.8 in [8] but we give it here as an illustration of Corollary 1.4.

Corollary 1.5. *Let M be a finitely generated faithful module and $\mathfrak{P}$ a prime ideal of R. Then there is a prime submodule P of M such that $(P : M) = \mathfrak{P}$.*

Proof. By using the determinant argument, we get that $\mathfrak{P} = (\mathfrak{P}M : M)$. Also we can get a maximal submodule N of M containing $\mathfrak{P}M$. Let $\mathcal{S} = R \setminus \mathfrak{P}$ and $S^* = M \setminus N$. Since N is a prime submodule of M, S^* is an $\mathcal{S}$-closed subset of M. Now the result follows from Corollary 1.4. □

We now turn our attention to the characterization of submodules which satisfy the radical formula by using a saturated closed subset of M. First we recall the following elementary definitions.

Let N be a submodule of an R-module M with $N \neq M$. The *envelope* of N in M is defined by $\{rm\colon r \in R$ and $m \in M$ such that $r^n m \in N$ for $n \in \mathbb{N}\}$ and is denoted by $E_M(N)$. We use $RE_M(N)$ to denote the submodule of M generated by $E_M(N)$. Following [7], we say that N *satisfies the radical formula (s.t.r.f.)* in M provided $M\text{-rad}_R N = RE_M(N)$, and M is said to *s.t.r.f.* if every submodule of M s.t.r.f. in M and analogously a ring R *s.t.r.f.* whenever every R-module s.t.r.f.

Let N be a submodule of an R-module M. Also suppose that $M\text{-rad}_R N$ is generated by the set U. We say that N *satisfies* $(*)$ if $Rm \cap N = 0$ whenever $m \in U \setminus N$. Clearly N satisfies $(*)$ provided that N is a summand submodule of $M\text{-rad}_R N$. Further if M is a $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z} \oplus 0$ then $M\text{-rad}_R N$ is generated by the set $\{(1 + 2\mathbb{Z}, 0 + 8\mathbb{Z}), (0{+}2\mathbb{Z}, 2{+}8\mathbb{Z})\}$ and so N satisfies $(*)$.

Let N be a submodule of an R-module M with $(*)$ and let $\mathcal{Q} = (N : M)$ be a non-zero ideal of R. Then $M\text{-rad}_R N$ is equal to N provided N contains the torsion subset $T(M) = \{m \in M\colon$ there exists $0 \neq r \in R$ such that $rm = 0\}$.

Theorem 1.6. *Let N be a submodule of an R-module M with $(*)$. Also suppose that $\mathcal{Q} = (N : M)$ is a non zero prime ideal of R. If $\mathcal{Q}$ contains the set of all zero divisors on M then M-$\mathrm{rad}_R N = RE_M(N)$. In particular, whenever N is summand M-$\mathrm{rad}_R N = N$.*

Proof. Let M-$\mathrm{rad}_R N$ be generated by the set U. It is enough to show that M-$\mathrm{rad}_R N \subseteq RE_M(N)$. Take $b \in M$-$\mathrm{rad}_R N$ with $b \notin RE_M(N)$ and look for a contradiction. Write $b = r_1m_1 + \ldots + r_nm_n$ for some $r_i \in R$ and $m_i \in U$ $(1 \leqslant i \leqslant n)$. Without loss of generality, we can assume that $r_1m_1 \notin RE_M(N)$. Hence $m_1 \notin N$ and for all $t \in \mathbb{N}$, $r_1^t m_1 \notin N$. Let $\mathcal{S} = R \setminus \mathcal{Q}$ and $S^* = \mathcal{S}m_1$. Then S^* is an $\mathcal{S}$-closed subset of M and clearly, $r_1m_1 \in S^*$ and $S^* \cap N = \emptyset$. By Theorem 1.3, there exists a prime submodule P containing N such that $P \cap S^* = \emptyset$ and $(P : M) = \mathcal{Q}$. It follows that $r_1m_1 \notin P$ and so $r_1m_1 \notin M$-$\mathrm{rad}_R N$. So we get a contradiction and this completes the proof. □

Let M be an R-module. Note that M is said to be *a multiplication module* provided for each submodule N of M there exists an ideal $\mathcal{I}$ of R such that $N = \mathcal{I}M$. In particular, invertible and more generally projective ideals of R are multiplication R-modules. On the other hand, cyclic modules are multiplication modules. (For more details, see for example [1]).

For the rest of this section, we assume R to be a ring in which every ideal is cyclic, M to be a multiplication R-module and S^* to be an $\mathcal{S}$-closed subset of M relative to a multiplicatively closed subset $\mathcal{S}$ of R. Our aim is to prove that every subset of M is contained in a minimal saturated closed subset. In [4, Theorem 4.3] Lu assumes M to be a cyclic R-module. Now we take one more step and assume M to be a multiplication module.

Lemma 1.7. *Let R, M, S^* and $\mathcal{S}$ be as above. Let N be a maximal submodule in $M \setminus S^*$. If S^* is saturated, then the ideal $(N : M)$ is maximal in $R \setminus \mathcal{S}$ so that $(N : M)$ is a prime ideal of R.*

Thus due to Lemma 1.7, [4, Theorem 4.8] can be improved. Hence we have

Lemma 1.8. *Let R, M, S^* and $\mathcal{S}$ be as above. Then S^* is a saturated $\mathcal{S}$-closed subset of M if and only if $S^* = M \setminus \bigcup\limits_{i \in I} P_i$ and $\mathcal{S} = R \setminus \bigcup\limits_{i \in I} \mathfrak{P}_i$ where P_i is a $\mathfrak{P}_i$-prime submodule of M such that $P_i \cap S^* = \emptyset$ for all i.*

Assume that M is a multiplication R-module. For any subset T of M, define $\overline{T} = M \setminus \bigcup\limits_{i \in I} P_i$ and $\mathcal{S} = R \setminus \bigcup\limits_{i \in I} \mathfrak{P}_i$ where P_i is a $\mathfrak{P}_i$-prime submodule of M such that $P_i \cap T = \emptyset$ for all i. Now we have

Theorem 1.9. *Let R, M, $\mathcal{S}$, T and $\overline{T}$ be as above. Then $\overline{T}$ is a minimal saturated $\mathcal{S}$-closed subset of M containing T.*

Proof. Clearly $\overline{T}$ is a saturated $\mathcal{S}$-closed subset of M. Assume that K is a saturated $\mathcal{S}_0$-closed subset of M such that $T \subseteq K \subseteq \overline{T}$. Then by Lemma 1.8, $K = M \setminus \bigcup Q_i$ and $\mathcal{S}_0 = R \setminus \bigcup \mathcal{Q}_i$ where Q_i is a $\mathcal{Q}_i$-prime submodule of M such that $Q_i \cap K = \emptyset$ for all i. Let $x \in \overline{T} = M \setminus \bigcup P_i$. Hence, $x \notin \bigcup P_i$ and so $x \notin \bigcup Q_i$. Therefore, $K = \overline{T}$ and $\mathcal{S} = \mathcal{S}_0$. This completes the proof. □

2. Projective modules

In this section we deal with the radicals of a submodule. In [6], McCasland and Moore proved that $M\text{-rad}_R N = \sqrt{(N:M)}M$ for a finitely generated multiplication R-module M. And in [1], El-Bast and Smith proved the same result for any multiplication R-module. In this section the main aim is to give a necessary and sufficient condition for the equality $M\text{-rad}_R N = \sqrt{(N:M)}M$ for a submodule N of a projective R-module M.

Let $\mathfrak{P}$ be a prime ideal of R. Recall that a submodule N of an R-module M is said to be $\mathfrak{P}$-primary if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in \sqrt{(N:M)} = \mathfrak{P}$. It is well known that $\mathfrak{P}F$ is a prime submodule of F such that $(\mathfrak{P}F : F) = \mathfrak{P}$ for a free R-module F. Hence we have the following known lemma

Lemma 2.1. *Let F be a free R-module and $\mathcal{P}$ a $\mathfrak{P}$-primary ideal of R. Then $\mathcal{P}F$ is a $\mathfrak{P}$-primary submodule of F.*

Theorem 2.2. *Let M be a projective R-module. Then either $\mathcal{P}M = M$ or $\mathcal{P}M$ is a $\mathfrak{P}$-primary submodule of M for every $\mathfrak{P}$-primary ideal $\mathcal{P}$ of R.*

Proof. Let M be a projective R-module. Thus $F = M \oplus A$ where F is a free module and A is an R-module. Let $\{f_i = m_i + a_i\}_{i \in I}$ be a basis for F where $m_i \in M$ and $a_i \in A$. Assume that $\mathcal{P}M \neq M$ for a $\mathfrak{P}$-primary ideal $\mathcal{P}$ of R. First, we show that $\sqrt{(\mathcal{P}M : M)} = \mathfrak{P}$. Take a non-zero element $r \in \sqrt{(\mathcal{P}M : M)}$ but not in $\mathfrak{P}$. Then for some integer n we have $r^n M \leqslant \mathcal{P}M \leqslant \mathcal{P}F$. Then by Lemma 2.1 we get $M \leqslant \mathcal{P}F$. Let $x \in M$ and so $x = \sum r_i f_i$ where $r_i \in \mathcal{P}$. Then $x - \sum r_i m_i = \sum r_i a_i \in M \cap A = 0$ and hence $x \in \mathcal{P}M$. It follows that $\mathcal{P}M = M$, a contradiction. Therefore, we get $\sqrt{(\mathcal{P}M : M)} = \mathfrak{P}$.

Let $r \in R$ and $m \in M$ be such that $rm \in \mathcal{P}M$ with $r \notin \mathfrak{P}$. Then $m \in \mathcal{P}F$ and so $m \in \mathcal{P}M$. This completes the proof. □

As corollaries to Theorem 2.2 we have

Corollary 2.3. *Let M be a projective R-module and let $\mathfrak{P}$ be a prime ideal of R. Then the following statements are equivalent.*

(1) *$\mathfrak{P}M$ is a prime submodule of M.*

(2) *There exists a prime submodule U of M such that $\mathfrak{P} = (U : M)$.*

(3) *$(\mathfrak{P}M : M) = \mathfrak{P}$.*

Corollary 2.4. *Let M be a projective R-module. Then either $M(\mathfrak{P}) = \mathfrak{P}M$ or $M(\mathfrak{P}) = M$ for every prime ideal $\mathfrak{P}$ of R.*

Lemma 2.5. *Let N be a submodule of a projective R-module M. Then*

$$M\text{-rad}_R[(N : M)M] = \sqrt{(N : M)}M.$$

Proof. Let U be a prime submodule of M containing $(N : M)M$. Then $(N : M) \subseteq (U : M)$ and so $\sqrt{(N : M)}M \subseteq (U : M)M \subseteq U$. This means that $\sqrt{(N : M)}M \subseteq M\text{-rad}_R[(N : M)M]$. For the converse, let $\mathfrak{P}$ be a prime ideal of R such that $(N : M) \subseteq \mathfrak{P}$. Then $\mathfrak{P}M$ is a prime submodule of M or $\mathfrak{P}M = M$. So we have $M\text{-rad}_R[(N : M)M] \subseteq \bigcap(\mathfrak{P}M)$. Since M is a projective R-module, $M\text{-rad}_R[(N : M)M] \subseteq \bigcap(\mathfrak{P}M) = (\bigcap \mathfrak{P})M = \sqrt{(N : M)}M$. This completes the proof. □

Let N be a proper submodule of a module M. Now we give the following definition to prove our main aim in this paper: We say that N *satisfy the prime property* (*s.t.p.p.*) in M provided $(N : M) \subseteq \mathfrak{P}$ for a prime ideal $\mathfrak{P}$ of R, $N \subseteq \mathfrak{P}M$.

Example 2.6.

(i) Let M be an R-module and let I be an ideal of a ring R. Then it is easy to check that the submodule IM s.t.p.p. in M.

(ii) Let M be the $\mathbb{Z}$-module $\mathbb{Z} \bigoplus 36\mathbb{Z}$ and $N = 6\mathbb{Z} \bigoplus 36\mathbb{Z}$. Then $(N : M) = 12\mathbb{Z}$. It can be seen that N s.t.p.p. in M.

By using the prime property and projective modules, we obtain a characterization for a prime submodule. Now we recall the fact from [3] that if R is a domain and M is a torsion-free R-module then M is flat if and only if $(I \cap J)M = IM \cap JM$ for all ideals I and J of R. It is also known that a finitely generated flat module over a domain is projective. Hence we have

Proposition 2.7. *Let M be a finitely generated projective R-module and let N be a submodule which s.t.p.p. in M. Then N is a prime submodule if and only if $P = (N : M)$ is a prime ideal of R and M/N is a projective R/P-module.*

Proof. Sufficiency is evident. Let N be a prime submodule of M which s.t.p.p. in M. Then $\mathfrak{P} = (N : M)$ is a prime ideal of R and $(\mathcal{I} \cap \mathcal{J})(M/N) = \mathcal{I}(M/N) \cap \mathcal{J}(M/N)$ for all ideals $\mathcal{I}$ and $\mathcal{J}$ in $R/\mathfrak{P}$. Then the result follows from [3, Theorem 1 and Corollary 1]. □

Corollary 2.8. *Let M be a finitely generated projective R-module. Then $M/\mathfrak{P}M$ is a projective $R/\mathfrak{P}$-module for every prime ideal $\mathfrak{P}$ of R.*

The prime property gives also another characterization for radical submodules.

Theorem 2.9. *Let N be a submodule of a projective R-module M. Then N s.t.p.p. in M if and only if $M\text{-rad}_R N = \sqrt{(N : M)}M$. In particular, if N s.t.p.p. in M then N s.t.r.f. in M.*

Proof. Sufficiency is evident. Assume that N s.t.p.p. in M. Let P be a prime submodule of M containing $[(N : M)M]$. Thus we have $(N : M) \subseteq \mathfrak{P} = (P : M)$. Then by the prime property, we get $N \subseteq \mathfrak{P}M \subseteq P$. Therefore, $M\text{-rad}_R N \subseteq M\text{-rad}_R[(N : M)M]$. Now the result follows from Lemma 2.5. □

Corollary 2.10. *Let N be a submodule of an R-module M such that M/N is a projective R-module. Then $M\text{-rad}_R N = \sqrt{(N : M)}M + N = RE_M(N)$.*

Proof. Clearly the zero submodule of M/N s.t.p.p. in M/N. Since M/N is a projective R-module we have $M/N\text{-rad}_R 0 = \sqrt{0 : (M/N)}(M/N)$. On the other hand, $M/N\text{-rad}_R 0 = M\text{-rad}_R N/N$ and $\sqrt{0 : (M/N)}(M/N) = (\sqrt{N : M}M + N)/N$. Therefore, $M\text{-rad}_R N = \sqrt{N : M}M + N = RE_M(N)$. □

Using Corollary 2.10 we can improve the result [2, Corrollary 8].

Corollary 2.11. *If M is a projective R-module then $M\text{-rad}_R(0) = RE_M(0) = \sqrt{0 : M}M$.*

Compare the next corollary with [10, Corollary 1.5].

Corollary 2.12. *Let M be a primary projective R-module. Then the radical of M is a prime submodule of M.*

P r o o f. Since M is a projective R-module M contains a prime submodule and so $M\text{-rad}_R 0$ is not equal to M. Therefore we can prove that $M\text{-rad}_R 0 = \sqrt{0 : M}M$ is a prime submodule of M by using the same argument as in Theorem 2.2. $\square$

If N is a primary submodule of a projective R-module M which s.t.p.p. in M then by Theorem 2.2 and Theorem 2.9, the radical of N in M is a prime submodule of M or $M\text{-rad}_R N = M$. On the other hand, the following example shows that a partial converse of Theorem 2.9 is not true in general.

Example 2.13. Let R be a principal ideal domain and $M = R \oplus R$. Let N be a non-zero cyclic submodule of M. It can be easily seen that $M\text{-rad}_R N \neq \sqrt{(N : M)}M$. Hence N s.t.r.f. but not s.t.p.p. in M.

Let $N = \bigoplus N_i$ be a submodule of an R-module M. Then provided N_i s.t.p.p. in M for all $i = 1, \ldots, n$, N s.t.p.p. in M. But the converse is not true in general (see Example 2.6 (ii)). However, for the converse we can state the following: Let $N = \bigoplus N_i$. Also assume that N s.t.p.p. in M. If $\sqrt{(N_i : M)} = \sqrt{(N : M)}$ for some i then N_i s.t.p.p. in M.

Let $M = \bigoplus M_i$ be an R-module. Consider the submodule $N = \bigoplus N_i$ of M such that N_i is a submodule of M_i for all $i \in I$. It can be proved that if N s.t.p.p. in M then N_i s.t.p.p. in M_i for all $i \in I$. For the converse, if N_i s.t.p.p. in M_i for all $i \in I$ with $\sqrt{N_i : M_i} = \sqrt{N : M}$, then N s.t.p.p. in M.

Now we turn our attention to primary submodules. First, we give a characterization for primary submodules of M such that $M = \bigoplus M_i$ is a direct sum of modules M_i $(i \in I)$. For each i, let N_i be a submodule of M_i and $N = \bigoplus N_i$.

Theorem 2.14. *Let M and N be as above. Assume that $\mathfrak{P}$ is a prime ideal of R. Then N is a $\mathfrak{P}$-primary submodule of M if and only if N_i is a $\mathfrak{P}$-primary submodule of M_i whenever $N_i \neq M_i$ for all i.*

P r o o f. Let N be a $\mathfrak{P}$-primary submodule of M. Since $N \neq M$, there is a non-empty subset J of I such that $N_j \neq M_j$ for all $j \in J$ and so $N = \bigoplus N_j \oplus (\bigoplus M_i)$. First we prove that $\sqrt{(N_t : M_t)} = \mathfrak{P}$ for all $t \in J$.

Let $r \in \mathfrak{P}$. Choose an element $m_t \in M_t$ but not in N_t. Let $m = (0, \ldots, m_t, \ldots 0) \in M$. Then for a positive integer l, we have $r^l(0, \ldots, m_t, \ldots 0) \in N$ and so $r^l m_t \in N_t$. Hence $r \in \sqrt{(N_t : M_t)}$ and so $\mathfrak{P} \subseteq \sqrt{(N_t : M_t)}$. Now take elements $r \in \sqrt{(N_t : M_t)} \setminus \mathfrak{P}$ and $m = (m_i) \in M$ such that

$$m = (m_i) = \begin{cases} m_i \in N_i & \text{if } i \neq t, \\ m_t \in M_t \setminus N_t & \text{if } i = t. \end{cases}$$

Then for a positive integers l_i we have $r^{l_i} m_i \in M_i$. Let $k = \max\{l_i\}$ and so $r^k m \subseteq N$. Since $m \notin N$, we get that $r \in \mathfrak{P}$. Therefore $\mathfrak{P} = \sqrt{(N_j : M_j)}$ for all $j \in J$.

Take a submodule N_j for any $j \in J$ and $rm_j \in N_j$ where $r \in R$ and $m_j \in M_j$. Choose an element

$$m = (m_i) = \begin{cases} m_i = m_j & \text{if } i = j, \\ m_i = 0 & \text{if } i \neq j \end{cases}$$

of M. Then $rm \in N$. Since N is primary, it follows that either $m \in N$ or $r \in \mathfrak{P}$. Hence either $m_j \in N_j$ or $r \in \mathfrak{P}$. Therefore N_j is a primary submodule of M_j for all $j \in J$.

Conversely, assume that $N = \bigoplus N_j \oplus (\bigoplus M_i)$ is such that for all $j \in J \subseteq I$, N_j is a $\mathfrak{P}$-primary submodule in M_j. Take elements $r \in R$ and $m \in M$ such that $rm \in N$. Then $m = (m_i) \in \bigoplus M_i$ and so $rm = (rm_i) \in \bigoplus N_j \oplus (\bigoplus M_i)$. Now assume that for some $j \in J$, we have $m_j \notin N_j$. Then $rm_j \in N_j$ and so $r \in \mathfrak{P}$. This means that N is a $\mathfrak{P}$-primary submodule of M. □

Corollary 2.15. *Let N and M be as in Theorem 2.14. Also suppose that N is a primary submodule of M. Then N s.t.p.p. in M if and only if N_i s.t.p.p. in M_i for all $i \in I$.*

For the rest of this section, we assume R to be a principal ideal domain and $M = R \oplus R$. We close this paper by giving equivalent conditions to the prime property. Let N be a submodule of M. If N is generated by (a, b) then clearly $M\text{-rad}_R N = \gcd\{a, b\}R$ and it does not satisfy the prime property. Now assume N is generated by $\{(a, b), (c, d)\}$ and let $\Delta = ad - bc$. Then it is routine to check that there is an element k of R such that $\Delta = k \gcd(a, b, c, d)$ and $(N : M) = kR$ where $\gcd(a, b, c, d)$ denotes the greatest common divisor of the elements a, b, c and d. Let $\Delta = p^t$ where p is a prime element of R and $t \in \mathbb{N}$. Then N is a prime submodule whenever $t = 1$. Otherwise N is a primary submodule of M.

Theorem 2.16. *Let N be a submodule of $M = R \oplus R$ generated by $\{(a, b), (c, d)\}$ and let $(N : M) = kR$ for some $k \in R$. Let $k = p_1^{t_1} \dots p_n^{t_n}$ and $s = p_1 \dots p_n$ where for each i, p_i is a prime element in R and $t_i \in \mathbb{N}$ for all $i = 1, 2, \dots, n$. Then the following statements are equivalent.*

(1) *s divides a, b, c and d.*

(2) *N s.t.p.p. in M.*

(3) *$M\text{-rad}_R N = RE_M(N) = \sqrt{kR}M$.*

Proof. It is sufficient to prove the equivalence of (1) and (2).

(1) $\Rightarrow$ (2): Let pR be a prime ideal containing $(N : M)$. Then $p = p_i$ for some $1 \leqslant i \leqslant n$ and so there exist $t_1, t_2, t_3, t_4 \in R$ such that $(a, b) = p(t_1, t_2)$, $(c, d) = p(t_3, t_4)$. Hence N s.t.p.p. in M.

(2) $\Rightarrow$ (1): Let p be a prime element of R such that p divides s. Then $kR \subseteq sR \subseteq pR$. Hence $(a,b),(c,d) \in pRM$. It follows that $(a,b) = p(t_1,t_2)$, $(c,d) = p(t_3,t_4)$ for $t_1,t_2,t_3,t_4 \in R$. Therefore, s divides a, b, c and d. □

Now we close this paper by the following observation.

Let M, N and Δ be as above. If $\Delta = p^t$ where p is a prime element of R and p divides a, b, c and d, then $M\text{-rad}_R N = pM$ is a prime submodule of M.

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