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Donatella Bongiorno; Luisa Di Piazza; Valentin A. Skvortsov

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VARIATIONAL MEASURES RELATED TO LOCAL SYSTEMS AND THE WARD PROPERTY OF $\mathcal{P}$-ADIC PATH BASES

D. Bongiorno[1] and L. Di Piazza[2], Palermo, and V. A. Skvortsov[3], Moscow

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Abstract. Some properties of absolutely continuous variational measures associated with local systems of sets are established. The classes of functions generating such measures are described. It is shown by constructing an example that there exists a $\mathcal{P}$-adic path system that defines a differentiation basis which does not possess Ward property.

Keywords: local system, $\mathcal{P}$-adic system, differentiation basis, variational measure, Ward property

MSC 2000: 26A39, 26A42, 26A45, 28A12

1. Introduction

In this paper we proceed with the investigation of some properties of variational measures generated by functions and associated with local systems of sets. The notion of local system $\mathcal{S}$ was introduced by Thomson in [31]. Some properties of the Kurzweil-Henstock type integrals related to local systems ($\mathcal{S}$-integrals) were studied in [6], [11], [13] and, in the case of path systems in [6], [15].

In Section 3 we investigate classes of functions, generating absolutely continuous variational measures with respect to local systems, and we compare them with the correspondent classes of generalized $\mathcal{S}$-absolutely continuous functions. Both those classes play an important role in the descriptive characterization of the Kurzweil-Henstock type integrals (see [3]–[14], [17]–[24], [30], [33]). We show that, under

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some assumptions related to the σ-finiteness of the variational measure, those classes coincide.

In the next sections our primary concern is to study some properties of the differentiation bases defined by $\mathcal{P}$-adic path systems. Those bases, and especially the dyadic path basis, have important applications in harmonic analysis on $\mathcal{P}$-adic Cantor groups (see [1], [2], [5], [6], [16], [26], [27], [28]).

In Section 4 we prove that in the case of any $\mathcal{P}$-adic path system each absolutely continuous variational measure is σ-finite. So, in this case, the two classes considered in Section 3 for a general local system coincide without any additional assumption.

Section 5 is related to the Ward property which is shared by many differentiation bases (see for example [5], [7], [25]). In particular in [34] it is proved that each basis related to a $\mathcal{P}$-adic path system defined by a bounded sequence $\mathcal{P}$ possesses the Ward property. However, here we show by constructing an example that for the case of an unbounded sequence $\mathcal{P}$ this property can fail to be true (see Theorem 5.1).

In the last section, applying the above example to the theory of the $\mathcal{S}$-integral, we prove that, in the case of the $\mathcal{P}$-adic path systems defined by some unbounded sequences, the class of the indefinite $\mathcal{P}$-integrals is strictly included in the class of the $\mathcal{P}ACG$ functions.

2. Preliminaries

Throughout the paper all sets are subsets of the real line $\mathbb{R}$ and all the functions are real functions defined on $\mathbb{R}$, unless specified differently. If $E \subset \mathbb{R}$ then $|E|$ and ∂E denote, respectively, the outer Lebesgue measure and the boundary of E. By $\mathcal{L}$ we denote the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$. The terms "almost everywhere" (br. a.e.) is always used in the sense of the Lebesgue measure. If $|E| = 0$ then the set E is called *negligible*. If $A, B \subset \mathbb{R}$, then $d(A, B)$ denotes the euclidean distance from A to B. An *interval* is always a compact nondegenerate subinterval of $\mathbb{R}$. A collection of intervals is called *nonoverlapping* whenever their interiors are disjoint.

We shall use some basic measure theoretic notions. Given an outer measure Λ on the family of all subsets of $X \subset \mathbb{R}$, a set $E \subset X$ is said to be Λ-*measurable* (in the sense of Carathéodory) if for any set $A \subset X$:

$$\Lambda(A) = \Lambda(A \cap E) + \Lambda(A \setminus E). \tag{1}$$

An outer measure Λ is said to be *absolutely continuous with respect to* a measure μ on a set X if

$$\mu(N) = 0 \Rightarrow \Lambda(N) = 0$$

for any set $N \subset X$. We remind the following well known property (see [25]).

Proposition 2.1. *A finite measure Λ is absolutely continuous on a set X, with respect to a measure μ, if and only if for each $\varepsilon > 0$ there corresponds an $\eta > 0$ such that:*

$$\mu(E) < \eta \Rightarrow \Lambda(E) < \varepsilon$$

for every μ-measurable set $E \subset X$.

Throughout this paper each time we say that an outer measure Λ is *absolutely continuous* we have in mind that Λ is *absolutely continuous with respect to the Lebesgue measure.*

A *metric outer measure* Λ on a set X is an outer measure Λ on X such that if $A, B \subset X$, $A \neq \emptyset$, $B \neq \emptyset$ and $d(A, B) > 0$, then $\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)$.

It is well known that each Borel set is Λ-measurable for any metric outer measure Λ.

We recall that an outer measure Λ is *σ-finite* on a set X if

$$X = \bigcup_{i=1}^{\infty} X_i \quad \text{and} \quad \Lambda(X_i) < +\infty \quad \text{for each } i.$$

If in the previous definition the sets X and X_i are Lebesgue measurable, then we say that Λ is *strongly σ-finite.*

A Borel measure Λ on a set X is called *semi-moderated* if X is the union of closed sets C_1, $C_2, \ldots$ with $\Lambda(C_n) < +\infty$ for $n = 1, 2, \ldots$; it is called *semi-locally finite* if each nonempty closed set $C \subset X$ contains a non-empty relatively open subset U with $\Lambda(U) < +\infty$.

Proposition 2.2 (see [24])**.** *A Borel measure Λ on $X \subset \mathbb{R}$ is semi-moderated if and only if it is semi-locally finite.*

Definition 2.3. A family $\mathcal{S} = \{S(x)\colon x \in \mathbb{R}\}$ is said to be a *local system* if each $S(x)$ is a collection of sets with the following properties:

(i) $\{x\} \notin S(x)$, for all x;
(ii) if $s_1 \in S(x)$ and $s_1 \subseteq s_2$, then $s_2 \in S(x)$;
(iii) if $s \in S(x)$, then $x \in s$;
(iv) if $s \in S(x)$ and $\delta > 0$, then $s \cap (x - \delta, x + \delta) \in S(x)$.

A local system $\mathcal{S}$ is said to be *bilateral* if, for each $x \in \mathbb{R}$, every set $s \in S(x)$ contains points on either sides of x. A local system $\mathcal{S}$ is said to be *filtering* if at each point $x \in \mathbb{R}$ we have $s_1 \cap s_2 \in S(x)$ whenever s_1 and s_2 belong to $S(x)$.

A function $\gamma\colon E \to 2^{\mathbb{R}}$ with $\gamma(x) \in S(x)$, is called an *$\mathcal{S}$-choice* or simply a *choice* on the set E.

Let $\mathcal{S} = \{S(x)\colon\ x \in \mathbb{R}\}$ be a local system and let F be a real function on $\mathbb{R}$. The function F is said to have *$\mathcal{S}$-limit c at a point x*, provided that for every $\varepsilon > 0$ the set $\{t\colon\ t = x \text{ or } |F(t) - c| < \varepsilon\} \in S(x)$. When $c = F(x)$ the function F is said to be *$\mathcal{S}$-continuous* at the point x. Notice that if the local system is filtering, then the $\mathcal{S}$-limit is unique.

A special case of local systems are the systems generated by paths (see [31]). A *path at x* is a set E_x such that $x \in E_x$ and x is a point of accumulation for E_x. Let $\mathcal{E} = \{E_x\colon\ x \in \mathbb{R}\}$ be a system of paths. Note that each choice from a local system is in fact a system of paths.

We say that a local system $\mathcal{S} = \{S(x)\colon\ x \in \mathbb{R}\}$ is *generated by the system of paths $\mathcal{E}$*, or simply is a *path system* if for every $x \in \mathbb{R}$, the family $S(x)$ is the filter generated by a filter base of the form $\{E_x \cap (x - \eta, x + \eta)\colon\ \eta > 0\}$.

Given a local system $\mathcal{S}$ we associate with each choice γ a family

$$\beta_\gamma = \{([u,v],x)\colon\ x = u, v \in \gamma(x) \text{ or } x = v,\ u \in \gamma(x);\ x \in \mathbb{R}\}.$$

Note that if a system $\mathcal{S}$ is filtering then the set of all families β_γ forms a *differentiation basis* in terms of Kurzweil-Henstock integration theory (see [22], [32]). For a set E put

$$\beta_\gamma[E] = \{([u,v],x) \in \beta_\gamma\colon\ x \in E\}.$$

A finite subset π of $\beta_\gamma[E]$ is called a *β_γ-partition on E* if for distinct elements (I', x') and (I'', x'') in π, the intervals I' and I'' are nonoverlapping. If $\bigcup_{(I,x)\in\pi} I = E$, for $\pi \subset \beta_\gamma$, then we say that π is a *β_γ-partition of E*.

For a function F and an interval $I = [u,v]$ we shall use the standard notation $\Delta F(I) = F(v) - F(u)$.

3. $\mathcal{S}$-variational measure

Throughout this section $\mathcal{S}$ is a fixed filtering local system. Given a function F, a set E and an $\mathcal{S}$-choice γ on E, we define the *γ-variation of F on E* by

$$\operatorname{Var}(\beta_\gamma, F, E) = \sup \sum_{(I,x)\in\pi} |\Delta F(I)|\,, \tag{2}$$

where "sup" is taken over all $\pi \subset \beta_\gamma[E]$. Then for each E we define

$$V_F^{\mathcal{S}}(E) = \inf \operatorname{Var}(\beta_\gamma, F, E),$$

where "inf" is taken over all choices γ on E. We call the set function $V_F^{\mathcal{S}}$ the *variational measure generated by* F *with respect to the local system* $\mathcal{S}$. It is known (see [31]) that $V_F^{\mathcal{S}}$ is a metric outer measure.

To investigate properties of $V_F^{\mathcal{S}}$, we start with a simple, but very useful observation which for the usual full interval basis is made in [21].

Lemma 3.1. *If the variational measure* $V_F^{\mathcal{S}}$ *is absolutely continuous on* $\mathbb{R}$ *then it is a measure on the* σ*-algebra* $\mathcal{L}$.

P r o o f. As $V_F^{\mathcal{S}}$ is a metric outer measure then each Borel set B is $V_F^{\mathcal{S}}$-measurable. So for any set $A \subset \mathbb{R}$, according to (1), we have

$$V_F^{\mathcal{S}}(A) = V_F^{\mathcal{S}}(A \cap B) + V_F^{\mathcal{S}}(A \setminus B). \tag{3}$$

It is known (see [25]) that any Lebesgue measurable set E can be represented as a difference $E = B \setminus N$ where B is a Borel set (in fact a G_δ set) and N is a negligible set. Because of the absolute continuity of $V_F^{\mathcal{S}}$ we have $V_F^{\mathcal{S}}(N) = 0$. Therefore by (3) we get

$$\begin{aligned} V_F^{\mathcal{S}}(A) &\leqslant V_F^{\mathcal{S}}(A \cap E) + V_F^{\mathcal{S}}(A \setminus E) \\ &\leqslant V_F^{\mathcal{S}}(A \cap B) + V_F^{\mathcal{S}}(A \setminus B) + V_F^{\mathcal{S}}(A \cap N) = V_F^{\mathcal{S}}(A), \end{aligned}$$

for any set $A \subset \mathbb{R}$. This implies the equality (1), with $V_F^{\mathcal{S}}$ substituted for Λ. So E is $V_F^{\mathcal{S}}$-measurable. □

It is possible to formulate a simple sufficient condition for the variational measure generated by a function F to be absolutely continuous, in terms of the upper and lower $\mathcal{S}$-derivatives of F.

We recall that the *lower* $\mathcal{S}$*-derivative of* a function F at a point x is defined by

$$\underline{D}_{\mathcal{S}}F(x) = \sup_{s \in S(x)} \inf \left\{ \frac{F(y) - F(x)}{y - x} :\ y \in s, y \neq x \right\}.$$

The *upper* $\mathcal{S}$*-derivative* $\overline{D}_{\mathcal{S}}F(x)$ is defined in a similar way. As the system $\mathcal{S}$ is filtering, then $\underline{D}_{\mathcal{S}}F(x) \leqslant \overline{D}_{\mathcal{S}}F(x)$. If $\underline{D}_{\mathcal{S}}F(x) = \overline{D}_{\mathcal{S}}F(x)$ and this value is finite, we say that F is $\mathcal{S}$-differentiable at x. Then the $\mathcal{S}$*-derivative* $D_{\mathcal{S}}F(x)$ is defined as the common value of the extreme derivatives.

Theorem 3.2. *If both the upper and the lower $\mathcal{S}$-derivatives of a function F are finite on a set E, then the variational measure $V_F^{\mathcal{S}}$ is absolutely continuous on E.*

Proof. Take a negligible set $N \subset E$ and write $N = \bigcup_{n=1}^{\infty} N_n$, where N_n is the set of all $x \in N$ such that for a certain set $s_x \in \mathcal{S}(x)$ we have

$$\left|\frac{F(y)-F(x)}{y-x}\right| < n \quad \text{if } y \in s_x,\ y \neq x.$$

Take $\varepsilon > 0$ and choose an open set O_n such that $N_n \subset O_n$ and $|O_n| < \varepsilon/n$. Define a choice γ on N_n by putting $\gamma(x) = s_x \cap (x - d(x, \partial O_n), x + d(x, \partial O_n))$. By an easy computation we get $\mathrm{Var}(\beta_\gamma, F, N_n) \leqslant \varepsilon$. Then the thesis follows. $\square$

Definition 3.3 (see [13]). Given a local system $\mathcal{S}$, a function F is said to be $\mathcal{S}$AC on a set E if for any $\varepsilon > 0$ there exist $\delta > 0$ and an $\mathcal{S}$-choice γ on E such that $\sum_{(I,x)\in\pi} |\Delta F(I)| < \varepsilon$, for any partition $\pi \in \beta_\gamma[E]$ with $\sum_{(I,x)\in\pi} |I| < \delta$. Moreover F is said to be $\mathcal{S}ACG$ on E if $E = \bigcup_n E_n$, and F is $\mathcal{S}AC$ on E_n, for each n. In case the sets E_n can be taken Lebesgue measurable or closed, F is said to be $(\mathcal{S}ACG)$ or $[\mathcal{S}ACG]$, respectively.

It is known that for some particular local system $\mathcal{S}$, the class of $\mathcal{S}ACG$ functions coincides with the class of functions F generating absolutely continuous variational measure $V_F^{\mathcal{S}}$ (see [14]). We consider here some conditions under which those classes coincide for a general local system.

Lemma 3.4. *Let the variational measure $V_F^{\mathcal{S}}$ be finite on $X \in \mathcal{L}$. If $V_F^{\mathcal{S}}$ is absolutely continuous on X, then F is $\mathcal{S}AC$ on X.*

Proof. Our argument is similar to the ones used in [21] and in [29] for the full interval bases and for some other class of bases, respectively.

Fix $\varepsilon > 0$ and take an $\mathcal{S}$-choice γ such that

$$\mathrm{Var}(\beta_\gamma, F, X) < V_F^{\mathcal{S}}(X) + \frac{\varepsilon}{3}. \tag{4}$$

We can suppose that $|X| > 0$, because the case $|X| = 0$ is trivial. Applying Lemma 3.1 and Proposition 2.1 to $\mathrm{V}_F^{\mathcal{S}}$, we choose $0 < \eta < |X|/2$ such that $|E| < \eta$ implies

$$V_F^{\mathcal{S}}(E) < \frac{\varepsilon}{3}, \tag{5}$$

for every Lebesgue measurable set $E \subset X$. Let $\pi = \{(I_i, x_i)\}_{i=1}^{p}$ be any partition from $\beta_\gamma[X]$, with $\sum_{i=1}^{p} |I_i| < \eta$. Consider the set $Y = X \setminus \left(\bigcup_{i=1}^{p} I_i\right)$. Because of the choice of η, we have $|Y| > 0$. Now we define an $\mathcal{S}$-choice γ_0 on Y such that $\operatorname{Var}(\beta_{\gamma_0}, F, Y) < +\infty$, $\gamma_0(x) \subset \gamma(x)$, and $\gamma_0(x) \cap \left(\bigcup_{i=1}^{p} I_i\right) = \emptyset$ for each $x \in Y$ (see property (iv) in Definition 2.3). There exists a partition $\pi_0 = \{(J_j, y_j)\}_{j=1}^{q} \in \beta_{\gamma_0}[Y]$ such that

$$\sum_{j=1}^{q} |F(J_j)| > \operatorname{Var}(\beta_{\gamma_0}, F, Y) - \frac{\varepsilon}{3}. \tag{6}$$

By construction each interval J_j has no common point with $\bigcup_{i=1}^{p} I_i$. So $\pi_0 \cup \pi \in \beta_\gamma[X]$. Therefore by (4)

$$\sum_{i=1}^{p} |F(I_i)| + \sum_{j=1}^{q} |F(J_j)| \leqslant \operatorname{Var}(\beta_\gamma, F, X) < V_F^{\mathcal{S}}(X) + \frac{\varepsilon}{3}.$$

Then by (6)

$$\begin{aligned}\sum_{i=1}^{p} |F(I_i)| &< V_F^{\mathcal{S}}(X) + \frac{\varepsilon}{3} - \sum_{j=1}^{q} |F(J_j)| \\ &< V_F^{\mathcal{S}}(X) - \operatorname{Var}(\beta_{\gamma_0}, F, Y) + \frac{2\varepsilon}{3} \leqslant V_F^{\mathcal{S}}(X) - V_F^{\mathcal{S}}(Y) + \frac{2\varepsilon}{3}.\end{aligned} \tag{7}$$

As $X \in \mathcal{L}$ and $V_F^{\mathcal{S}}$ is a measure on the σ-algebra $\mathcal{L}$ (see Lemma 3.1), we have

$$V_F^{\mathcal{S}}(X) - V_F^{\mathcal{S}}(Y) = V_F^{\mathcal{S}}\left(X \cap \left(\bigcup_{i=1}^{p} I_i\right)\right).$$

So we apply (5) with $X \cap (\bigcup_{i=1}^{p} I_i)$ instead of E, to get from (7)

$$\sum_{i=1}^{p} |F(I_i)| \leqslant V_F^{\mathcal{S}}\left(X \cap \left(\bigcup_{i=1}^{p} I_i\right)\right) + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

It means that F is $\mathcal{S}AC$ on X. □

The following lemma was in fact proved in [13].

Lemma 3.5. *If a function F is $\mathcal{S}AC$ on a set X, then $V_F^{\mathcal{S}}$ is absolutely continuous on X.*

Theorem 3.6. *A function F is $(\mathcal{S}ACG)$ on a set X if and only if the variational measure $V_F^{\mathcal{S}}$ is absolutely continuous and strongly σ-finite on X.*

Proof. As $V_F^{\mathcal{S}}$ is strongly σ-finite on X, we have $X = \bigcup\limits_{i=1}^{\infty} X_i$, where $X_i \in \mathcal{L}$, and $V_F^{\mathcal{S}}$ is finite on X_i, for each $i = 1, 2, \ldots$.

Now, as $V_F^{\mathcal{S}}$ is also supposed to be absolutely continuous, it is enough to apply Lemma 3.4 to the set X_i to get that F is $\mathcal{S}AC$ on X_i, for each i. Therefore F is $(\mathcal{S}ACG)$ on X.

In the opposite direction suppose that F is $(\mathcal{S}ACG)$. Then $X = \bigcup\limits_{i=1}^{\infty} X_i$ with $X_i \in \mathcal{L}$ and F being $\mathcal{S}AC$ on each X_i. Now using Lemma 3.5 we get that the measure $V_F^{\mathcal{S}}$ is absolutely continuous on each X_i, and consequently on X.

To obtain the strong σ-finiteness of $V_F^{\mathcal{S}}$, take for each i a choice γ_i on X_i and $\eta > 0$ such that, for any partition $\pi = \{(I_i, x_i)\}_{i=1}^{p} \in \beta_{\gamma_i}[X_i]$, the inequality $\sum\limits_{i=1}^{p} |I_i| < \eta$ implies $\sum\limits_{i=1}^{p} |F(I_i)| < 1$. Now choose $1/n < \eta$. Then for each integer j we have

$$V_F^{\mathcal{S}}\Big(X_i \cap \Big[\frac{j}{n}, \frac{j+1}{n}\Big]\Big) \leqslant \operatorname{Var}\Big(\beta_{\gamma_i}, F, X_i \cap \Big[\frac{j}{n}, \frac{j+1}{n}\Big]\Big) \leqslant 1.$$

As each set $X_i \cap [\frac{j}{n}, \frac{j+1}{n}]$ is Lebesgue measurable, this completes the proof. □

In the case the set X is $\mathcal{F}_\sigma$ we have the following version of the previous theorem.

Theorem 3.7. *A function F is $[\mathcal{S}ACG]$ on a set X if and only if the variational measure $V_F^{\mathcal{S}}$ is absolutely continuous and semi-moderated on X.*

Proof. Follow the lines of the previous proof, using closed sets X_i instead of Lebesgue measurable sets. □

In the case of $\mathcal{S}ACG$ functions we have only the following result that can be obtained by using an argument similar to the one of the necessity part of Theorem 3.6.

Theorem 3.8. *If a function F is $\mathcal{S}ACG$ on a set X then the variational measure $V_F^{\mathcal{S}}$ is absolutely continuous and σ-finite on X.*

It is not known whether it is possible to strengthen the above result (under the respective assumption related to X), stating that the measure $V_F^{\mathcal{S}}$ is also semi-moderated or at least strongly σ-finite for any local system. But in the particular case of the $\mathcal{P}$-adic path system (see Section 4 below), we are proving that the absolute continuity of the corresponding variational measure implies that it is also strongly

σ-finite or semi-moderated and so Theorem 3.8 is true for this local system, together with its converse.

4. $\mathcal{P}$-adic path variational measure

Now we consider the results of the previous section in the case of the $\mathcal{P}$-adic path system. We remind some notations.

Let $\mathcal{P} = \{p_j\}_{j=0}^{\infty}$ be a fixed sequence of integers, with $p_j > 1$ for $j = 0, 1, \ldots$. We set $m_0 = 1$, $m_k = p_0 p_1 \ldots . p_{k-1}$, for $k \geqslant 1$. We call the closed intervals

$$\left[\frac{r}{m_k}, \frac{r+1}{m_k}\right] = I_r^{(k)}, \quad r \in \mathbb{Z} \tag{8}$$

for fixed $k = 0, 1, \ldots$, the *$\mathcal{P}$-adic intervals* (or simply *$\mathcal{P}$-intervals*) *of rank k*.

The points r/m_k where $r \in \mathbb{Z}$ and $k = 0, 1, \ldots$, constitute the set of all *$\mathcal{P}$-adic rationals* on $\mathbb{R}$. Its complementary set on $\mathbb{R}$ is the set of all *$\mathcal{P}$-adic irrationals* on $\mathbb{R}$. For each $\mathcal{P}$-adic irrational point x, there exists only one $\mathcal{P}$-interval $I_x^{(k)} = [a_x^{(k)}, b_x^{(k)}]$ of rank k containing x so that $\{x\} = \bigcap\limits_{k=0}^{\infty} [a_x^{(k)}, b_x^{(k)}]$. We say that the sequence $\{[a_x^{(k)}, b_x^{(k)}]\}$ of nested $\mathcal{P}$-intervals is the *basic sequence of $\mathcal{P}$-intervals convergent to x*. If x is a $\mathcal{P}$-adic rational point, then there exist two decreasing sequences of $\mathcal{P}$-intervals for which x is a common end-point starting with some k; i.e. for such a point we have two basic sequences convergent to x: the left one and the right one.

Now we define the system of $\mathcal{P}$-adic paths. If x is a $\mathcal{P}$-adic irrational we denote by $P_-(x)$ and $P_+(x)$ the convergent to x sequences $\{a_x^{(k)}\}$ and $\{b_x^{(k)}\}$, respectively, which are given by the definition of the basic sequence of $\mathcal{P}$-intervals. Then the set $P_x = \{x\} \cup P_-(x) \cup P_+(x)$ is the *$\mathcal{P}$-adic path at x*. In the case of a $\mathcal{P}$-adic rational x, we denote by $P_-(x)$ (respectively by $P_+(x)$) the sequence of the left (respectively right) end-points of the intervals from the left (respectively right) basic sequence. The definition of the $\mathcal{P}$-adic path P_x at x is the same as in the case of the $\mathcal{P}$-adic irrationals. We denote by $\mathcal{P}$ the local path system generated by these $\mathcal{P}$-adic paths.

Theorem 4.1. *Let F be a $\mathcal{P}$-continuous function and let X be a closed set. If the variational measure $V_F^{\mathcal{P}}$ is σ-finite on all negligible Borel subsets of X then it is semi-moderated on X.*

P r o o f. Suppose that $V_F^{\mathcal{P}}$ is not *semi-moderated* on X. Let $Q \subset X$ be the set of all points $x \in X$ for which $V_F^{\mathcal{P}}$ is not *semi-moderated* on $X \cap (c, d)$, for every interval (c, d) containing x. If Q is empty, the thesis follows easily. Indeed it is enough to use a compactness argument and the fact that if a measure is *semi-moderated* on each set X_n, then it is *semi-moderated* on $\bigcup\limits_n X_n$.

So let us show that Q is empty by obtaining a contradiction from the assumption that $Q \neq \emptyset$. Clearly Q is closed. It is easy to check that Q has no isolated points and therefore it is a perfect set. Let T be the countable set of all one-side isolated points of Q and put $P = Q \setminus T$. As F is $\mathcal{P}$-continuous $V_F^{\mathcal{P}}(\{x\}) = 0$ for any singleton. Then since $V_F^{\mathcal{P}}$ is a Borel measure, we have $V_F^{\mathcal{P}}(T) = 0$. We note that under our assumption, for any interval I with $I \cap P \neq \emptyset$ we have $V_F^{\mathcal{P}}(I \cap P) = +\infty$. Indeed if $V_F^{\mathcal{P}}$ were finite on $I \cap P$ it would be finite also on $I \cap Q$ because

$$V_F^{\mathcal{P}}(I \cap Q) = V_F^{\mathcal{P}}(I \cap P) + V_F^{\mathcal{P}}(I \cap T) = V_F^{\mathcal{P}}(I \cap P).$$

Then $V_F^{\mathcal{P}}$ would be semi-moderated on $I \cap Q$. This implies easily that it would be semi-moderated on $I \cap X$, in contradiction with the definition of Q.

Now we can end the proof by repeating word by word the argument in the proof of Theorem 4.3 of [6], using the corresponding $\mathcal{P}$-adic path version of Lemma 4.2 of [6]. □

In a similar way we can prove the following theorem.

Theorem 4.2. *Let F be a $\mathcal{P}$-continuous function and let $X \in \mathcal{L}$. If the variational measure $V_F^{\mathcal{P}}$ is σ-finite on all negligible Borel subsets of X then it is strongly σ-finite on X.*

The following simple example of a variational measure which is σ-finite but not semi-moderated, shows that we cannot drop the assumption of $\mathcal{P}$-continuity in Theorem 4.1.

Example 4.3. Let T be a countable dense set of $\mathcal{P}$-adic irrationals, and

$$F(x) = \begin{cases} 1 & \text{if } x \in T, \\ 0 & \text{if } x \in \mathbb{R} \setminus T. \end{cases}$$

Obviously $V_F^{\mathcal{P}}(\{x\}) = 1$ if $x \in T$ and $V_F^{\mathcal{P}}(E) = 0$, for any set $E \subset \mathbb{R} \setminus T$. Then $V_F^{\mathcal{P}}$ is σ-finite on $\mathbb{R}$ and on any negligible set, but it is not *semi-moderated*. In fact $V_F^{\mathcal{P}}(I) = +\infty$ for any interval $I \subset \mathbb{R}$, hence $V_F^{\mathcal{P}}$ is not *semi-locally finite* on $\mathbb{R}$. So we can apply Proposition 2.2.

Corollary 4.4. *Let F be a $\mathcal{P}$-continuous function and let X be an $\mathcal{F}_\sigma$ set. If the variational measure $V_F^{\mathcal{P}}$ is σ-finite on all negligible Borel subsets of X then it is semi-moderated on X.*

Proof. Since X is an $\mathcal{F}_\sigma$ set, we have that $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is closed. Therefore to end the proof it is enough to apply Theorem 4.1 to each X_n. □

Corollary 4.5. *Let X be an $\mathcal{F}_\sigma$ set. If the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on X, then it is semi-moderated.*

Proof. It is enough to note that the absolute continuity of $V_F^{\mathcal{P}}$ implies the $\mathcal{P}$-continuity of F and to apply Corollary 4.4. □

In a similar way we get from Theorem 4.2

Corollary 4.6. *If the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on $X \in \mathcal{L}$, then it is strongly σ-finite.*

Corollary 4.7. *A function F is $(\mathcal{P}ACG)$ on $X \in \mathcal{L}$ (or $[\mathcal{P}ACG]$ on an $\mathcal{F}_\sigma$ set X) if and only if the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on X.*

Proof. Apply Corollary 4.6 and Theorem 3.6 (or Corollary 4.5 and Theorem 3.7) □

5. The ward property for a $\mathcal{P}$-adic path system

As we have mentioned already in Section 2, the family $\{\beta_\gamma\}$, defined by a local system, represents a special case of a differentiation basis on $\mathbb{R}$. In a more general setting (see [22], [32]), a *differentiation basis* or simply a *basis* on $\mathbb{R}$ is defined as a filtering down nonempty family $\mathcal{B}$ of nonempty subsets β of the product $\mathcal{I} \times \mathbb{R}$ where $\mathcal{I}$ is the set of all compact intervals of $\mathbb{R}$ and $x \in I$ for any $(I, x) \subset \beta$. The $\mathcal{B}$-derivative and the extreme $\mathcal{B}$-derivatives with respect to such a basis are defined similarly to the definitions of the $\mathcal{S}$-derivatives given in Section 3.

We say that a given differentiation basis $\mathcal{B}$ possesses the *Ward property* whenever *each function is $\mathcal{B}$-differentiable almost everywhere on the set of all points at which at least one of its extreme $\mathcal{B}$-derivatives is finite.*

We note that Corollary 4.7 and other results of the previous section were obtained without using the Ward property which is usually employed in this type of results (see [3]– [7], [10], [12]).

In this section we are showing that the Ward property can fail to be true for bases defined by some $\mathcal{P}$-adic path systems. But this can happen only in the case of unbounded sequences $\mathcal{P}$. In fact, as it is proved in [34], *each $\mathcal{P}$-adic path system defined by a bounded sequence $\mathcal{P}$, possesses the Ward property.*

Theorem 5.1. *For some unbounded sequence $\mathcal{P} = \{p_k\}_{k=0}^{\infty}$ there exist a closed set S of positive Lebesgue measure and*

(i) *a continuous function F, piecewise linear on each interval contiguous to S, such that*

$$-\frac{3}{2} \leqslant \underline{D}_{\mathcal{P}} F(x) < \overline{D}_{\mathcal{P}} F(x) \leqslant \frac{3}{2} \tag{9}$$

for any point $x \in S$;

(ii) *a continuous function G such that*

$$\overline{D}_{\mathcal{P}} G(x) = \underline{D}_{\mathcal{P}} G(x) = +\infty \tag{10}$$

for any point $x \in S$.

Proof. Consider the sequence $\mathcal{P} = \{p_k\}_{k=0}^{\infty}$ where $p_0 = p_1 = 2$ and $p_k = k^4$ for $k \geqslant 2$. Then

$$m_0 = 1,\ m_1 = 2,\ m_2 = 2^2,\ m_{k+1} = 2^2 (k!)^4 \quad \text{if } k \geqslant 2.$$

For the corresponding $\mathcal{P}$-adic intervals on $[0, 1]$ we have

$$I_i^{(k)} = \bigcup_{j=ik^4}^{(i+1)k^4-1} I_j^{(k+1)} \ \text{ for } i = 0, 1, \ldots, m_k - 1.$$

We introduce some auxiliary intervals (note that they are not $\mathcal{P}$-adic!)

$$J_r^{(k)} = \bigcup_{j=rk^2}^{(r+1)k^2-1} I_j^{(k+1)} \quad \text{for } r = 0, 1, \ldots, m_k k^2 - 1.$$

Then

$$I_i^{(k)} = \bigcup_{r=ik^2}^{(i+1)k^2-1} J_r^{(k)} \ \text{ for } i = 0, 1, \ldots, m_k - 1.$$

Put, for each $k \geqslant 3$,

$$P_k = \bigcup_{i=0}^{m_k-1} \bigcup_{r=ik^2+4}^{(i+1)k^2-5} (J_r^{(k)} \setminus I_{(r+1)k^2-1}^{(k+1)}),$$

and note that if $x \in P_k \cap I_i^{(k)}$ and $I_i^{(k)} = [a_i^{(k)}, b_i^{(k)}]$ then

$$x - a_i^{(k)} \geqslant 4\,|J_r^{(k)}| = \frac{4}{k^2}\,|I_i^{(k)}| \ \text{ and } \ b_i^{(k)} - x \geqslant \frac{4}{k^2} |I_i^{(k)}|. \tag{11}$$

Put also

$$S_l = \bigcap_{k=3}^{l} P_k, \quad \text{and} \quad S = \bigcap_{k=3}^{\infty} P_k.$$

An easy computation shows that

$$|P_k| = 1 - \frac{8}{k^2} - \frac{k^2 - 8}{k^4} \quad \text{for } k \geqslant 3$$

and

$$|S_l| = \prod_{k=3}^{l} \Big(1 - \frac{8}{k^2} - \frac{k^2 - 8}{k^4}\Big).$$

Then

$$|S| = \prod_{k=3}^{\infty} \Big(1 - \frac{8}{k^2} - \frac{k^2 - 8}{k^4}\Big).$$

The series $\sum_{k=3}^{\infty} (8/k^2 + (k^2 - 8)/k^4)$ is obviously convergent. Then S is of positive Lebesgue measure. Moreover S is closed (even perfect) and contains only $\mathcal{P}$-adic irrational points.

Now we fix an arbitrary $k \geqslant 3$ and we define on $[0, 1]$ the following continuous and piecewise linear function:

$$\Phi_k(x) = \begin{cases} x - \alpha_r & \text{if } x \in J_r^{(k)} \setminus I^{(k+1)}_{(r+1)k^2-1},\ r = 0, 1, \dots, m_k k^2 - 1, \\ (k^2 - 1)(\beta_r - x) & \text{if } x \in I^{(k+1)}_{(r+1)k^2-1},\ r = 0, 1, \dots, m_k k^2 - 1, \end{cases}$$

where α_r and β_r are the end-points of the interval $J_r^{(k)}$, for each r.

Note that if $I_i^{(k)} = [a_i^{(k)}, b_i^{(k)}]$, then $\Phi_j(a_i^{(k)}) = \Phi_j(b_i^{(k)}) = 0$ for $j \geqslant k$. If $x \in S \cap I_x^{(k)}$ and $I_x^{(k)} = [a_x^{(k)}, b_x^{(k)}]$, then

$$\text{(12)} \qquad \frac{\Delta\Phi_j(I_x^{(k)})}{|I_x^{(k)}|} = 1 \quad \text{for all } j < k.$$

For each $x \in [0, 1]$ and for $l \geqslant 4$ we set

$$\text{(13)} \qquad F_l(x) = \Phi_3(x) + \sum_{k=4}^{l} 2\,(-1)^{k+1}\, \Phi_k(x) \chi_{S_{k-1}}(x).$$

Then we define on $[0, 1]$

$$\text{(14)} \qquad F(x) = \Phi_3(x) + \sum_{k=4}^{\infty} 2\,(-1)^{k+1}\, \Phi_k(x) \chi_{S_{k-1}}(x).$$

Since, for $k = 3, 4, \ldots$, we have, for any $x \in [0, 1]$

$$\max_x |\Phi_k(x)| = |J_r^{(k)} \setminus I_{(r+1)k^2-1}^{(k+1)}| \leqslant \frac{1}{m_k k^2}, \tag{15}$$

then the function F, as a sum of a uniformly convergent series of continuous functions, is continuous. In particular, since the sum in (14) is finite at each point of any interval contiguous to S, then F is piecewise linear on these intervals and differentiable everywhere outside S, except on a countable set of "corner" points of the graph, where F has finite one-side ordinary derivatives (and so also $\mathcal{S}$-derivatives). By (11) and by (15) we get for $x \in S \subset P_k \cap I_i^{(k)}$ and $I_i^{(k)} = [a_i^{(k)}, b_i^{(k)}]$

$$0 \leqslant \frac{\Phi_k(a_i^{(k)}) - \Phi_k(x)}{a_i^{(k)} - x} = \frac{\Phi_k(x)}{x - a_i^{(k)}} \leqslant \frac{1}{m_k k^2} \frac{k^2}{4|I_i^{(k)}|} = \frac{1}{4}, \tag{16}$$

$$0 \geqslant \frac{\Phi_k(b_i^{(k)}) - \Phi_k(x)}{b_i^{(k)} - x} \geqslant -\frac{1}{4}. \tag{17}$$

As we have already mentioned, any point $x \in S$ is $\mathcal{P}$-adic irrational. In the rest of this proof we shall denote by $\{[a^{(k)}, b^{(k)}]\}$ the unique basic sequence $\{I_x^{(k)}\}$ of $\mathcal{P}$-intervals convergent to a fixed $x \in S$.

For $x \in S$ and for $k > 4$, we have by (12) and (13)

$$\frac{\Delta F_{k-1}(I_x^{(k)})}{|I_x^{(k)}|} = 1 + \sum_{j=4}^{k-1} 2(-1)^j.$$

Note that F_{k-1} is linear on $I_x^{(k)}$. So, if k is even, we have

$$\frac{F_{k-1}(a^{(k)}) - F_{k-1}(x)}{a^{(k)} - x} = \frac{F_{k-1}(b^{(k)}) - F_{k-1}(x)}{b^{(k)} - x} = \frac{\Delta F_{k-1}(I_x^{(k)})}{|I_x^{(k)}|} = 1, \tag{18}$$

and, if k is odd, we have

$$\frac{F_{k-1}(a^{(k)}) - F_{k-1}(x)}{a^{(k)} - x} = \frac{F_{k-1}(b^{(k)}) - F_{k-1}(x)}{b^{(k)} - x} = \frac{\Delta F_{k-1}(I_x^{(k)})}{|I_x^{(k)}|} = -1. \tag{19}$$

It is easy to compute that $\min\{\Phi_{k-1}(x)\colon\ x \in S_k\} = (4/m_k k^2)$.

This together with (15) gives, for $k > 4$

$$0 \leqslant \Phi_k(x) \leqslant \frac{1}{4}\Phi_{k-1}(x), \quad x \in S_k. \tag{20}$$

Then for any $k > 2$ we have

$$F_{2k}(x) < F(x) < F_{2k-1}(x), \quad F_{2k}(x) < F(x) < F_{2k+1}(x). \tag{21}$$

Now we estimate the upper and the lower derivative of F on S. Fix $x \in S$ and the basic sequence $\{I_i^k\} = \{[a^{(k)}, b^{(k)}]\}$. Note that $F(a^{(k)}) = F_{k-1}(a^{(k)})$ and $F(b^{(k)}) = F_{k-1}(b^{(k)})$. If k is even, then by (18), (21), (16) and (17), we have:

$$\begin{aligned} 1 &= \frac{F_{k-1}(a^{(k)}) - F_{k-1}(x)}{a^{(k)} - x} \geqslant \frac{F(a^{(k)}) - F(x)}{a^{(k)} - x} \geqslant \frac{F_{k-1}(a^{(k)}) - F_k(x)}{a^{(k)} - x} \\ &= \frac{F_{k-1}(a^{(k)}) - F_{k-1}(x)}{a^{(k)} - x} - \frac{2\Phi_k(a^{(k)}) - 2\Phi_k(x)}{a^{(k)} - x} \geqslant 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} 1 &= \frac{F_{k-1}(b^{(k)}) - F_{k-1}(x)}{b^{(k)} - x} \leqslant \frac{F(b^{(k)}) - F(x)}{b^{(k)} - x} \leqslant \frac{F_{k-1}(b^{(k)}) - F_k(x)}{b^{(k)} - x} \\ &= \frac{F_{k-1}(b^{(k)}) - F_{k-1}(x)}{b^{(k)} - x} - \frac{2\Phi_k(b^{(k)}) - 2\Phi_k(x)}{b^{(k)} - x} \leqslant 1 + \frac{1}{2} = \frac{3}{2}. \end{aligned}$$

In a similar way, if k is odd by (18), (21), (16) and (17), we have

$$\begin{aligned} -1 &= \frac{F_{k-1}(a^{(k)}) - F_{k-1}(x)}{a^{(k)} - x} \leqslant \frac{F(a^{(k)}) - F(x)}{a^{(k)} - x} \leqslant \frac{F_{k-1}(a^{(k)}) - F_k(x)}{a^{(k)} - x} \\ &= \frac{F_{k-1}(a^{(k)}) - F_{k-1}(x)}{a^{(k)} - x} + \frac{2\Phi_k(a^{(k)}) - 2\Phi_k(x)}{a^{(k)} - x} \leqslant -1 + \frac{1}{2} = -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} -1 &= \frac{F_{k-1}(b^{(k)}) - F_{k-1}(x)}{b^{(k)} - x} \geqslant \frac{F(b^{(k)}) - F(x)}{b^{(k)} - x} \geqslant \frac{F_{k-1}(b^{(k)}) - F_k(x)}{b^{(k)} - x} \\ &= \frac{F_{k-1}(b^{(k)}) - F_{k-1}(x)}{b^{(k)} - x} + \frac{2\Phi_k(b^{(k)}) - 2\Phi_k(x)}{b^{(k)} - x} \geqslant -1 - \frac{1}{2} = -\frac{3}{2}. \end{aligned}$$

Those estimations imply for each $x \in S$ the inequalities

$$-\frac{3}{2} \leqslant \underline{D}_{\mathcal{P}} F(x) \leqslant -\frac{1}{2} < \frac{1}{2} \leqslant \overline{D}_{\mathcal{P}} F(x) \leqslant \frac{3}{2},$$

which complete the proof of (i) part for the interval $[0, 1]$. Extending F periodically, with period equal to 1, we obtain the function with the desired property on the whole $\mathbb{R}$.

To prove the (ii)-part, we define

$$G(x) = \sum_{j=3}^{\infty} \Phi_j(x)\chi_{S_{j-1}}(x),$$

$$G_k(x) = \sum_{j=3}^{k} \Phi_j(x)\chi_{S_{j-1}}(x),$$

and

$$H_k(x) = \sum_{j=k}^{\infty} \Phi_j(x)\chi_{S_{j-1}}(x).$$

So that $G(x) = G_k(x) + H_k(x)$.

Because of (15), the series defining G and H_k are uniformly convergent. We observe that, if $I_j^k = [a^{(k)}, b^{(k)}]$, then $G(a^{(k)}) = G_j(a^{(k)})$ and $G(b^{(k)}) = G_j(b^{(k)})$ for all $j \geqslant k-1$. Moreover we note that $G_{k-1}(x)$ is linear on I_x^k. So if $x \in I_x^k \cap S_k$, by (13) for $k > 3$, we get:

$$\begin{aligned}
\text{(22)} \qquad \frac{G_{k-1}(b^{(k)}) - G_{k-1}(x)}{b^{(k)} - x} &= \frac{G_{k-1}(a^{(k)}) - G_{k-1}(x)}{a^{(k)} - x} \\
&= \frac{\Delta G_{k-1}(I_x^{(k)})}{|I_x^{(k)}|} = \frac{\sum\limits_{j=4}^{k-1} \Phi_j(I_x^k)}{|I_x^{(k)}|} = k-3.
\end{aligned}$$

By (20) we also have:

$$0 \leqslant H_k(x) \leqslant 2\Phi_k(x).$$

So, for $x \in S$ and $k > 3$, by this inequality and also by (22), (16) and (17) we obtain:

$$\begin{aligned}
\frac{G(b^{(k)}) - G(x)}{b^{(k)} - x} &= \frac{G_{k-1}(b^{(k)}) - G_{k-1}(x)}{b^{(k)} - x} - \frac{H_k(x)}{b^{(k)} - x} \\
&\geqslant k - 3 - \frac{2\Phi_k(x)}{b^{(k)} - x} > k - 3 - \frac{1}{2}; \\
\frac{G(a^{(k)}) - G(x)}{a^{(k)} - x} &= \frac{G_{k-1}(a^{(k)}) - G_{k-1}(x)}{a^{(k)} - x} - \frac{H_k(x)}{a^{(k)} - x} \geqslant k - 3.
\end{aligned}$$

Since k is arbitrary we get the desired result. $\square$

By Theorem 3.2 the variational measure $V_F^{\mathcal{P}}$ generated by the function F in the claim of Theorem 5.1 is absolutely continuous. Then, by Corollary 4.7, F is $[\mathcal{P}ACG]$. So we get the following theorem that gives a positive answer to the question stated in Problem 3.2 of [6].

Theorem 5.2. *There exists a $[\mathcal{P}ACG]$ function (associated with some unbounded sequence $\mathcal{P}$) which is not $\mathcal{P}$-differentiable on a set of positive measure.*

However the following problem is still open:

Problem 1. Do the statements of Theorems 5.1 and 5.2 hold for any unbounded sequence $\mathcal{P}$?

6. Application to the $\mathcal{S}$-integral

Let $\mathcal{S}$ be a local system which is bilateral, filtering and possessing the *partitioning property* in the following form: for any $\mathcal{S}$-choice γ there exists a β_γ-partition of any interval $[a,b]$. So for this basis we can define in the usual way the correspondent Kurzweil-Henstock type integral (see [11]) on an interval.

Definition 6.1. A function $f\colon [a,b] \to \mathbb{R}$ is said to be $\mathcal{S}$-*integrable* on $[a,b]$, with integral A, if for every $\varepsilon > 0$ there exists a choice γ on $[a,b]$ such that

$$\Bigl|\sum_{(I,x)\in\pi} f(x)|I| - A\Bigr| < \varepsilon,$$

for any partition $\pi \subset \beta_\gamma$ of $[a,b]$. We write $A = (\mathcal{S})\int_a^b f$.

For this integral many of the usual properties, known also for more general classes of bases, hold. In particular (see [6], [11], [13]):

P_1) If a function f is $\mathcal{S}$-integrable on $[a,b]$, then it is also $\mathcal{S}$-integrable on each subinterval of $[a,b]$. Therefore the indefinite $\mathcal{S}$-integral $F(x) = (\mathcal{S})\int_a^x f$ is defined for any $x \in [a,b]$.

P_2) The $\mathcal{S}$-indefinite integral F of f is $\mathcal{S}$-continuous at each point of $[a,b]$ and it is $\mathcal{S}$-differentiable a.e. with $D_{\mathcal{S}}F(x) = f(x)$ a.e. on $[a,b]$.

P_3) A function F is the indefinite $\mathcal{S}$-integral of a function f on $[a,b]$ if and only if F generates a variational measure absolutely continuous and F is $\mathcal{S}$-differentiable a.e. with $D_{\mathcal{S}}F(x) = f(x)$ a.e. on $[a,b]$.

P_4) A function $f\colon [a,b] \to \mathbb{R}$ is $\mathcal{S}$-integrable on $[a,b]$ if and only if there exists an $\mathcal{S}ACG$ function F on $[a,b]$ such that F is $\mathcal{S}$-differentiable a.e. and $D_{\mathcal{S}}F = f$ a.e. on $[a,b]$.

The class of all $\mathcal{S}ACG$ functions and the class of all functions generating absolutely continuous variational measure is widely used in the Kurzweil-Henstock theory of integration. In particular properties P_3) and P_4) represent examples of partial descriptive characterizations of the indefinite $\mathcal{S}$-integral.

In the case of many differentiation bases each of those classes coincides with the class of the indefinite integrals associated with the respective bases (see [3]– [7], [12], [21], [14], [29]). In particular it is true for the dyadic path integral (see [6]) and for the $\mathcal{P}$-adic path integral defined by a bounded sequence $\mathcal{P}$ (see [34]). But this result cannot be extended for an arbitrary local system. Indeed Theorem 5.2 and property P_2) implies the following statement.

Theorem 6.2. *For some unbounded sequence $\mathcal{P}$ the class of the indefinite $\mathcal{P}$-integrals is strictly included into the class of the $\mathcal{P}ACG$ functions (and even more so, into the class of the $[\mathcal{P}ACG]$ functions).*

That means that we cannot drop the assumption of $\mathcal{S}$-differentiability in the "if" part of P_3) and P_4), for a general local system $\mathcal{S}$.

If we know that a differentiation basis defined by a particular $\mathcal{P}$-adic path system has the Ward property, then it is possible to give the following full descriptive characterization of the respective $\mathcal{P}$-integral.

Theorem 6.3. *Let us assume that a basis defined by a $\mathcal{P}$-adic path system has the Ward property and let F be a real function on $[a,b]$. Then the following assertions are equivalent*

(i) *F is an indefinite $\mathcal{P}$-integral of some $\mathcal{P}$-integrable function on $[a,b]$;*
(ii) *F generates an absolute continuous variational measure $V_F^{\mathcal{P}}$ on $[a,b]$;*
(iii) *F is $\mathcal{P}ACG$ on $[a,b]$.*

Proof. (ii) $\Leftrightarrow$ (iii) by Corollary 4.7.

The implication (i) $\Rightarrow$ (ii) follows from property P_3).

To prove (ii) $\Rightarrow$ (i) note that, by Corollary 4.6 we get the σ-finiteness of the $V_F^{\mathcal{P}}$. Then by a standard argument, we infer that $\overline{\mathcal{D}}_F^{\mathcal{P}} < +\infty$ a.e. on $[a,b]$. To complete the proof we apply the Ward property together with property P_3). □

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Authors' addresses: D. Bongiorno, Dipartimento di Metodi e Mod. Mat., Viale delle Scienye, 90100, Palermo, Italy, e-mail: d.bongiorno@unipa.it; L. Di Piazza, Dipartimento di Matem. ed Applic., Via Archirafi 34, 90123 Palermo, Italy, e-mail: dipiazza@math.unipa.it; V. A. Skvortsov, Department of Mathematics, Moscow State University, Moscow 119899 (Russia), and Instytut Matematyki, Uniwersytet Kazimierza Wielkego, 85-072, Bydgoszcz, Poland e-mail: vaskvor2000@yahoo.com.