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A SPACE OF GENERALIZED DISTRIBUTIONS

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To the memory of Laurent Schwartz

Abstract. In this paper we use a duality method to introduce a new space of generalized distributions. This method is exactly the same introduced by Schwartz for the distribution theory. Our space of generalized distributions contains all the Schwartz distributions and all the multipole series of physicists and is, in a certain sense, the smallest space containing all these series.

Keywords: distributions, generalized distributions, multipole series

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1. Introduction

Distribution theory was introduced by Laurent Schwartz [16] in order to allow the derivation of any order in the space of locally integrable functions in an open (non empty) subset Ω of $\mathbb{R}^N$. For this purpose Schwartz extended the concept of function and completely justified the use of the so called Dirac function (and its derivatives) introduced by Heaviside [8], [9] and largely used by Dirac in quantum mechanics [3]. The distribution space $\mathscr{D}'(\Omega)$ is defined by Schwartz as the strong dual of the space $\mathscr{D}(\Omega)$ of C^∞ functions with compact support in Ω, equipped with an appropriate topology. This topology, which is not an easy one, uses some properties of inductive limits established previously by Dieudonné and Schwartz [2]. The space $\mathscr{D}'(\Omega)$ is a Montel space and, in particular, is reflexive, its dual being $\mathscr{D}(\Omega)$, and a sequence $(T_n)_{n\in\mathbb{N}}$ tends to zero in $\mathscr{D}'(\Omega)$ iff the scalar sequence $T_n(\varphi)$ tends to zero for all $\varphi \in \mathscr{D}(\Omega)$.

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In order to define the Fourier transform, Schwartz introduced also the space $\mathscr{S}'$ of tempered distributions: it is the strong dual of the Schwartz space $\mathscr{S}$ of C^∞ rapidly decreasing functions in $\mathbb{R}^N$, equipped with an appropriate topology. The classical Fourier transform is a vectorial and topological isomorphism of $\mathscr{S}$ onto itself and Schwartz extended it, by duality, to the space $\mathscr{S}'$ of tempered distributions.

The motivation for the generalization of distributions is the series of multipoles

$$\sum_{\alpha \in \mathbb{N}^N} a_\alpha \partial^\alpha \delta_a$$

used by physicists, where the a_α are scalars and δ_a is the Dirac distribution at the point $a \in \Omega$. This series is not convergent in $\mathscr{D}'(\Omega)$ (except in the trivial case where all but a finite number of the a_α are null) and therefore we may be tempted to use other spaces than $\mathscr{D}'(\Omega)$. One way to do this is to modify the test function space $\mathscr{D}(\Omega)$, the main difficulty being the construction of subspaces still dense in $\mathscr{D}(\Omega)$. One attempt done by Lions and Magenes [11], using functions introduced by Gevrey [5], led to the so called Gevrey classes of ultradistributions. Komatsu [10] did the same with the classes of ultradifferentiable functions. Guelfand and Chilov [7] replaced $\mathscr{D}(\Omega)$ by the space of analytic functions; the disadvantage is that this space is not included in $\mathscr{D}(\Omega)$. Another way established by Sebastião e Silva [17, 18] and developed by Oliveira [13] is to extend the Fourier transform to a space bigger than $\mathscr{S}'$, maintaining, if possible, the isomorphism property in that bigger space; unfortunately this space does not contain $\mathscr{D}'(\Omega)$. We remark that an extension of the Fourier transform to $\mathscr{D}'(\Omega)$ was done by Ehrenpreis [4], but the automorphism property is lost. A space close to that of Sebastião e Silva and Oliveira was axiomatically introduced by Menezes [12] and revisited by Luísa Ribeiro [14]. For other generalizations of the distributions see the book of Lions and Magenes [11] or the thesis of Luísa Ribeiro [15] and the references presented in these works.

Our approach is the construction of a very simple subspace of $\mathscr{D}(\Omega)$, much simpler than those introduced by Gevrey or Komatsu. The advantages of our method are: (i) our space includes $\mathscr{D}'(\Omega)$ and we very easily characterize the distributions (Theorem 1); (ii) our theory is closely related to the well-known Schwartz theory; (iii) the notion of ultra-support localizes the singular behavior of our generalized distributions; (iv) the definition of support is quite natural and has nice properties (which is not the case with the spaces of Sebastião e Silva, Oliveira and Luísa Ribeiro); (v) the extension of the theory to differentiable manifolds seems straightforward; (vi) we think that the extension of the Fourier transform to a subspace of our space can be done by linearity and continuity in a natural way (we hope to publish a paper on that matter in the future).

We are grateful to the late prof. Laurent Schwartz who read the manuscript and made very useful suggestions in a letter dated 24 June 1993. In particular the theorems in Section 6 were established by prof. Schwartz.

2. The space $\mathscr{U}_W(\Omega)$

Let Ω be an open non empty subset of $\mathbb{R}^N$; we write $\mathscr{K}(\Omega)$ for the set of all compact subsets of $\mathbb{R}^N$ included in Ω and $\mathscr{W}(\Omega)$ for the set of all subsets W of Ω such that $K \cap W$ is finite for each $K \in \mathscr{K}(\Omega)$. Note that $\mathscr{W}(\Omega)$ is closed under intersections and finite unions. If $W \in \mathscr{W}(\Omega)$, then W is countable, closed in Ω, all its points are isolated, all its limit points belong to the boundary of Ω and $W_0 \in \mathscr{W}(\Omega)$ for all W_0 included in W. If $K \in \mathscr{K}(\Omega)$ and $W \in \mathscr{W}(\Omega)$, there exists an open neighbourhood $\mathscr{O}$ of K such that $\overline{\mathscr{O}} \in \mathscr{K}(\Omega)$ and $\overline{\mathscr{O}} \setminus K$ is disjoint from W. For $W \in \mathscr{W}(\Omega)$ and $m \in \mathbb{N}$, we define $\mathscr{U}_{W,m}(\Omega)$ as the subset of $\mathscr{D}(\Omega)$ of all φ such that $\partial^\alpha \varphi(x) = 0$ for all $\alpha \in \mathbb{N}^N$, $|\alpha| > m$, and all $x \in W$. It is a closed vector subspace of $\mathscr{D}(\Omega)$ and we have $\mathscr{U}_{W,m}(\Omega) = \mathscr{D}(\Omega)$ iff $W = \emptyset$. We denote by $\mathscr{U}_W(\Omega)$ the union of all $\mathscr{U}_{W,m}(\Omega)$, when $m \in \mathbb{N}$; as $\mathscr{U}_{W,m}(\Omega)$ is included in $\mathscr{U}_{W,m'}(\Omega)$ whenever $m \leqslant m'$, we see that $\mathscr{U}_W(\Omega)$ is a vector subspace of $\mathscr{D}(\Omega)$, equal to $\mathscr{D}(\Omega)$ iff $W = \emptyset$.

Proposition 1. *The space $\mathscr{U}_W(\Omega)$ is dense in $\mathscr{D}(\Omega)$.*

Proof. Let $\varphi \in \mathscr{D}(\Omega)$; we know, by the Weierstrass theorem, that there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that, for all $K \in \mathscr{K}(\Omega)$ and all $\alpha \in \mathbb{N}^N$, the sequence $(\partial^\alpha p_n)_{n\in\mathbb{N}}$ converges uniformly in K to $\partial^\alpha \varphi$. Fix a neighbourhood $\mathscr{O}$ of $\operatorname{supp}\varphi$, with $\overline{O} \in \mathscr{K}(\Omega)$, such that $\overline{O} \setminus \operatorname{supp}\varphi$ is disjoint from W and choose $\psi \in \mathscr{D}(\Omega)$ such that $\psi = 1$ in $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi \subset \mathscr{O}$; we have $\psi p_n \to \varphi$ in $\mathscr{D}(\Omega)$ and $\psi p_n \in \mathscr{U}_W(\Omega)$, and this shows that $\mathscr{U}_W(\Omega)$ is dense in $\mathscr{D}(\Omega)$. $\square$

We introduce in $\mathscr{U}_{W,m}(\Omega)$ the topology induced by $\mathscr{D}(\Omega)$. Recall the following result on inductive limits (for a proof see Grothendieck [6]; for a brief study of inductive limits see Viegas [19]): Let E be the inductive limit of a sequence of locally convex spaces $(E_n)_{n\in\mathbb{N}}$ such that $E_n \subset E_{n+1}$, the topology of E_n is the one induced by E_{n+1} and E_n is closed in E_{n+1}; then, if all the E_n are Montel spaces, E is also a Montel space.

Proposition 2. *The space $\mathscr{U}_{W,m}(\Omega)$ is a Montel space.*

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of non void relatively compact open subsets of Ω such that $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ and, for all $n \in \mathbb{N}$, $\overline{\Omega_n} \subset \Omega_{n+1}$ and $\partial\Omega_n \cap W = \emptyset$. Write $K_n = \overline{\Omega_n}$.

We first prove that the topology of $\mathscr{U}_{W,m}(\Omega)$ is the inductive limit topology of the spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ equipped with the topology induced by $\mathscr{D}_{K_n}(\Omega)$. From a general result on inductive limits (see [1]) we know that the inductive limit topology of the spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ is finer than the $\mathscr{U}_{W,m}(\Omega)$ topology. We have to show that the $\mathscr{U}_{W,m}(\Omega)$ topology is also finer than the inductive limit topology of the spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$. Let V be a closed absolutely convex neighbourhood in the inductive limit topology of the spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$; then, for each $n \in \mathbb{N}$, $V \cap \mathscr{D}_{K_n}(\Omega)$ is a neighbourhood in $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ and, as the topology of these spaces is the one induced by $\mathscr{D}_{K_n}(\Omega)$, we know that (see [16]), for each $n \in \mathbb{N}$, there exist $\varepsilon_n > 0$ and $k_n \in \mathbb{N}$ such that, if $\varphi \in \mathscr{U}_{W,m}(\Omega)$ verifies $|\partial^\alpha \varphi(x)| \leqslant \varepsilon_n$ for all $\alpha \in \mathbb{N}^N, |\alpha| \leqslant k_n$, and $\operatorname{supp} \varphi \subset K_n$, then $\varphi \in V$. Obviously we may choose the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ strictly decreasing converging to 0 and the sequence $(k_n)_{n\in\mathbb{N}}$ strictly increasing.

Now we fix once for all a sequence $(\psi_n)_{n\in\mathbb{N}}$ of functions such that:

$$\psi_n \in \mathscr{U}_{W,m}(\Omega);\ \operatorname{supp} \psi_n \subset (K_{n+2} \setminus \Omega_n);\ \psi_n \geqslant 0;\ \sum_{n=0}^{\infty} \psi_n = 1.$$

The existence of such a sequence is a consequence of the fact that the intersection of W with each K_n is finite. For $\varphi \in \mathscr{U}_{W,m}(\Omega)$ we have

$$\varphi = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} 2^{n+1} \psi_n \varphi$$

and, since V is closed and convex, this equality shows that $\varphi \in V$ whenever each function $2^{n+1}\psi_n\varphi$ belongs to V.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive real numbers, converging to 0, and let $(r_n)_{n\in\mathbb{N}}$ be an increasing and unbounded sequence of positive real numbers. As the topology of $\mathscr{U}_{W,m}(\Omega)$ is the one induced by $\mathscr{D}(\Omega)$, we know (see [16]) that the set V_0 of functions of $\mathscr{U}_{W,m}(\Omega)$ satisfying

$$\forall x \notin K_n\ \forall \alpha \in \mathbb{N}^N,\ |\alpha| \leqslant r_n \quad |\partial^\alpha \varphi(x)| \leqslant \lambda_n$$

is a neighbourhood in $\mathscr{U}_{W,m}(\Omega)$. But, if φ verifies the above condition, then

$$\forall x \notin K_n \quad \forall \alpha \in \mathbb{N}^N,\ |\alpha| \leqslant r_n \quad |2^{n+1}\partial^\alpha(\psi_n\varphi)(x)| \leqslant c_n\lambda_n$$

where, for each $n \in \mathbb{N}$, c_n is a constant $\geqslant 0$.

Choose $(\lambda_n)_{n\in\mathbb{N}}$ such that $c_n\lambda_n \leqslant \varepsilon_n$ for all $n \in \mathbb{N}$ and put $r_n = k_n$; if $\varphi \in V_0$, then $2^{n+1}\psi_n\varphi \in V$, and consequently $\varphi \in V$. We see that $V_0 \subset V$ and so V is a neighbourhood in $\mathscr{U}_{W,m}(\Omega)$.

We have just proved that the topology of $\mathscr{U}_{W,m}(\Omega)$ is the inductive limit topology of the spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ equipped with the topology induced by $\mathscr{D}_{K_n}(\Omega)$. But the spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ are Fréchet, thus they are barrelled; the $\mathscr{D}_{K_n}(\Omega)$ are Montel spaces, thus the $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ are also Montel spaces. For each $n \in \mathbb{N}$ the topology of $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ is the topology induced by $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_{n+1}}(\Omega)$, and the first space is closed in the second one. This shows that the inductive limit $\mathscr{U}_{W,m}(\Omega)$ of the Montel spaces $\mathscr{U}_{W,m}(\Omega) \cap \mathscr{D}_{K_n}(\Omega)$ is a Montel space.

$\square$

Note that, for all $m, m' \in \mathbb{N}$, $m \leqslant m'$, we have $\mathscr{U}_{W,m}(\Omega) \hookrightarrow \mathscr{U}_{W,m'}(\Omega) \hookrightarrow \mathscr{D}(\Omega)$, where $\hookrightarrow$ means continuous injection (we use $\hookrightarrow$ for continuous injections, $\overset{c}{\hookrightarrow}$ for compact injections and $\overset{d}{\hookrightarrow}$ for continuous and dense injections). We equip $\mathscr{U}_W(\Omega)$ with the inductive limit topology of the sequence $(\mathscr{U}_{W,m}(\Omega))_{m\in\mathbb{N}}$.

Proposition 3. *The space $\mathscr{U}_W(\Omega)$ is a Montel space.*

P r o o f. For each $m \in \mathbb{N}$, $\mathscr{U}_{W,m}(\Omega)$ is a Montel space (Proposition 2), $\mathscr{U}_{W,m+1}(\Omega)$ induces in $\mathscr{U}_{W,m}(\Omega)$ its topology and $\mathscr{U}_{W,m}(\Omega)$ is closed in $\mathscr{U}_{W,m+1}(\Omega)$.

$\square$

We remark that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ tends to zero in $\mathscr{U}_W(\Omega)$ iff the following three conditions are satisfied:

$$
\begin{aligned}
&\text{(i) } \exists p \in \mathbb{N}\ \forall n \in \mathbb{N}\ \forall x \in W\ \forall \alpha \in \mathbb{N}^N,\ |\alpha| > p\ \partial^\alpha \varphi_n(x) = 0\\
&\text{(ii) } \exists K \in \mathscr{K}(\Omega)\ \forall n \in \mathbb{N}\ \operatorname{supp} \varphi_n \subset K\\
&\text{(iii) } \forall \alpha \in \mathbb{N}^N\ \partial^\alpha \varphi_n \to 0 \text{ uniformly in } \Omega.
\end{aligned}
\tag{1}
$$

The proof of this statement is very easy. First, if (i), (ii) and (iii) are satisfied, then φ_n tends to zero in $\mathscr{U}_{W,p}(\Omega)$ and, a fortiori, in $\mathscr{U}_W(\Omega)$. Conversely, if φ_n tends to zero in $\mathscr{U}_W(\Omega)$, then φ_n is bounded and, as $\mathscr{U}_W(\Omega)$ is the strict inductive limit of the spaces $\mathscr{U}_{W,m}(\Omega)$ and each $\mathscr{U}_{W,m}(\Omega)$ is closed in $\mathscr{U}_{W,m+1}(\Omega)$, there exists $p \in \mathbb{N}$ such that φ_n belongs to $\mathscr{U}_{W,p}(\Omega)$; for this kind of inductive limits the topology of each $\mathscr{U}_{W,m}(\Omega)$ is the one induced by $\mathscr{U}_W(\Omega)$, and so φ_n tends to zero in $\mathscr{U}_{W,p}(\Omega)$, that is (i), (ii) and (iii) are satisfied.

Let $W_1, W_2 \in \mathscr{W}(\Omega)$, with $W_1 \subset W_2$; then we have $\mathscr{U}_{W_2}(\Omega) \hookrightarrow \mathscr{U}_{W_1}(\Omega)$, and we are able to prove that this injection is dense.

Proposition 4. *Let $W_1, W_2 \in \mathscr{W}(\Omega)$, $W_1 \subset W_2$; then $\mathscr{U}_{W_2}(\Omega)$ is dense in $\mathscr{U}_{W_1}(\Omega)$.*

P r o o f. If W_1 is empty we are reduced to Proposition 1. Suppose that W_1 is not empty; given $\varphi \in \mathscr{U}_{W_1}(\Omega)$ choose a neighbourhood $\mathscr{O}$ of $\operatorname{supp}\varphi$, with $\overline{\mathscr{O}} \in \mathscr{K}(\Omega)$, such that $\mathscr{O} \setminus \operatorname{supp}\varphi$ does not intersect W_2, fix $\psi \in \mathscr{D}(\Omega)$ with $\psi = 1$ in an open neighbourhood of $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi \subset \mathscr{O}$, and choose a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials such that, for each $\alpha \in \mathbb{N}^N$, $\partial^\alpha p_n \to \partial^\alpha \varphi$ uniformly in the compact subsets of Ω. Choose disjoint open neighbourhoods Ω_1 and Ω_2 of $W_1 \cap \operatorname{supp}\varphi$ and $(W_2 \setminus W_1) \cap \operatorname{supp}\varphi$, respectively, with $\overline{\Omega_1}, \overline{\Omega_2} \in \mathscr{K}(\Omega)$, write $\Omega_0 = \Omega \setminus (W_2 \cap \operatorname{supp}\varphi)$, and let (ψ_0, ψ_1, ψ_2) be a partition of unity subordinate to the open cover $(\Omega_0, \Omega_1, \Omega_2)$ of Ω. The sequence $\psi_1\psi\varphi + \psi_2\psi p_n + \psi_0\psi p_n$ is in $\mathscr{U}_{W_2}(\Omega)$ and converges to φ in $\mathscr{U}_{W_1}(\Omega)$. $\square$

3. The Space $\mathscr{U}'_W(\Omega)$

Let $\mathscr{U}'_W(\Omega)$ be the strong dual of $\mathscr{U}_W(\Omega)$; we shall denote the duality product between $T \in \mathscr{U}'_W(\Omega)$ and $\varphi \in \mathscr{U}_W(\Omega)$ by $\langle T, \varphi\rangle_{W,\Omega}$ or simply by $\langle T, \varphi\rangle$. The space $\mathscr{U}'_W(\Omega)$ is a Montel space, because it is the dual of the Montel space $\mathscr{U}_W(\Omega)$. A sequence $(T_n)_{n\in\mathbb{N}}$ tends to zero in the space $\mathscr{U}'_W(\Omega)$ iff $\langle T_n, \varphi\rangle \to 0$ for all $\varphi \in \mathscr{U}_W(\Omega)$. From $\mathscr{U}_W(\Omega) \overset{d}{\hookrightarrow} \mathscr{D}(\Omega)$ we see that $\mathscr{D}'(\Omega) \hookrightarrow \mathscr{U}'_W(\Omega)$ and $\langle T, \varphi\rangle_{W,\Omega} = \langle T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}$ for all $T \in \mathscr{D}'(\Omega)$ and all $\varphi \in \mathscr{U}_W(\Omega)$. From $\mathscr{U}_W(\Omega) \subset \mathscr{U}'_W(\Omega)$ and from the reflexivity of $\mathscr{U}_W(\Omega)$ we conclude that $\mathscr{U}_W(\Omega) \overset{d}{\hookrightarrow} \mathscr{U}'_W(\Omega)$, which implies $\mathscr{D}'(\Omega) \overset{d}{\hookrightarrow} \mathscr{U}'_W(\Omega)$. Thus, for $W_1, W_2 \in \mathscr{W}(\Omega)$, with $W_1 \subset W_2$, we may write the inclusions:

$$\mathscr{U}_{W_2}(\Omega) \overset{d}{\hookrightarrow} \mathscr{U}_{W_1}(\Omega) \overset{d}{\hookrightarrow} \mathscr{D}(\Omega) \overset{d}{\hookrightarrow} \mathscr{D}'(\Omega) \overset{d}{\hookrightarrow} \mathscr{U}'_{W_1}(\Omega) \overset{d}{\hookrightarrow} \mathscr{U}'_{W_2}(\Omega).$$

An element $T \in \mathscr{U}'_{W_2}(\Omega)$ belongs to $\mathscr{U}'_{W_1}(\Omega)$ iff T can be extended to $\mathscr{U}_{W_1}(\Omega)$ as a linear continuous (for the $\mathscr{U}_{W_1}(\Omega)$ topology) functional; T belongs to $\mathscr{D}'(\Omega)$ iff T can be extended to $\mathscr{D}(\Omega)$ as a linear continuous (for the $\mathscr{D}(\Omega)$ topology) functional.

We denote by $\mathscr{U}'(\Omega)$ the union of all $\mathscr{U}'_W(\Omega)$, when $W \in \mathscr{W}(\Omega)$. It is a vector space that we call the space of generalized distributions in Ω.

Lemma. *For all $W_1, W_2 \in \mathscr{W}(\Omega)$ we have $\mathscr{U}'_{W_1}(\Omega) \cap \mathscr{U}'_{W_2}(\Omega) = \mathscr{U}'_{W_1\cap W_2}(\Omega)$.*

P r o o f. As $W_1 \cap W_2$ is included in W_1 and in W_2, we have $\mathscr{U}'_{W_1\cap W_2}(\Omega)$ included in both spaces $\mathscr{U}'_{W_1}(\Omega)$ and $\mathscr{U}'_{W_2}(\Omega)$, that is $\mathscr{U}'_{W_1\cap W_2}(\Omega) \subset (\mathscr{U}'_{W_1}(\Omega) \cap \mathscr{U}'_{W_2}(\Omega))$.

We have to prove that $(\mathscr{U}'_{W_1}(\Omega) \cap \mathscr{U}'_{W_2}(\Omega)) \subset \mathscr{U}'_{W_1\cap W_2}(\Omega)$. Let $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ be such that: Ω_1 is an open neighbourhood of W_1 disjoint from $W_2 \setminus W_1$; Ω_2 is an open neighbourhood of W_2 disjoint from $W_1 \setminus W_2$. Write $\Omega_0 = \Omega \setminus (W_1 \cup W_2)$ and let (ψ_0, ψ_1, ψ_2) be a partition of the unity subordinate to the open cover $(\Omega_0, \Omega_1, \Omega_2)$ of Ω. Let $T \in \mathscr{U}'_{W_1}(\Omega) \cap \mathscr{U}'_{W_2}(\Omega)$; for each $\varphi \in \mathscr{U}_{W_1\cap W_2}(\Omega)$, we define $\langle T, \varphi\rangle_{\mathscr{U}'_{W_1\cap W_2}(\Omega),\mathscr{U}_{W_1\cap W_2}(\Omega)}$ by:

$$\langle T, \varphi\rangle_{\mathscr{U}'_{W_1\cap W_2}(\Omega),\mathscr{U}_{W_1\cap W_2}(\Omega)} = \langle T, \psi_0\varphi\rangle_{W_1\cup W_2,\Omega} + \langle T, \psi_1\varphi\rangle_{W_1,\Omega} + \langle T, \psi_2\varphi\rangle_{W_2,\Omega}\,.$$

The previous equation shows that T can be extended as a linear continuous functional on $\mathscr{U}_{W_1\cap W_2}(\Omega)$, and thus $T \in \mathscr{U}'_{W_1\cap W_2}(\Omega)$. $\square$

An immediate consequence of the previous lemma is that, if $T \in \mathscr{U}'(\Omega)$ belongs to $\mathscr{U}'_{W_1}(\Omega)\cap\mathscr{U}'_{W_2}(\Omega)$, where W_1 and W_2 are two disjoint sets in $\mathscr{W}(\Omega)$, then $T \in \mathscr{D}'(\Omega)$.

Theorem 1. *Let I be a nonempty index set and, for each $i \in I$, let $W_i \in \mathscr{W}(\Omega)$; write $W = \bigcap_{i\in I} W_i$. Then we have $\bigcap_{i\in I} \mathscr{U}'_{W_i}(\Omega) = \mathscr{U}'_W(\Omega)$.*

Proof. For each $i \in I$ we have $W \subset W_i$; this implies that, for each $i \in I$, $\mathscr{U}'_W(\Omega) \subset \mathscr{U}'_{W_i}(\Omega)$, that is $\mathscr{U}'_W(\Omega) \subset \bigcap_{i\in I} \mathscr{U}'_{W_i}(\Omega)$.

It remains to prove that $\bigcap_{i\in I} \mathscr{U}'_{W_i}(\Omega) \subset \mathscr{U}'_W(\Omega)$. Let $T \in \bigcap_{i\in I} \mathscr{U}'_{W_i}(\Omega)$; we have to prove that we can extend T as a linear functional in $\mathscr{U}_W(\Omega)$, continuous for the $\mathscr{U}_W(\Omega)$ topology.

Let $\varphi \in \mathscr{U}_W(\Omega)$ and let $K \in \mathscr{K}(\Omega)$ be such that $\operatorname{supp}\varphi \subset K$. As the set $K \cap W$ is finite we may fix a finite subset J_K of I such that $K \cap W = K \cap \Big(\bigcap_{j\in J_K} W_j\Big)$. From $\varphi \in \mathscr{U}_W(\Omega)$ we see that $\varphi \in \mathscr{U}_{K\cap W}(\Omega)$ (because $(K \cap W) \subset W$), that is $\varphi \in \mathscr{U}_{K\cap(\bigcap_{j\in J_K} W_j)}(\Omega)$. As K contains the support of φ, this function is identically null outside of K. This implies that $\varphi \in \mathscr{U}_{\bigcap_{j\in J_K} W_j}(\Omega)$. Now, as T belongs to $\bigcap_{i\in I} \mathscr{U}'_{W_i}(\Omega)$ and this set is included in the finite intersection $\bigcap_{j\in J_K} \mathscr{U}'_{W_j}(\Omega)$, we have $T \in \bigcap_{j\in J_K} \mathscr{U}'_{W_j}(\Omega)$. By the lemma, taking into account that J_K is a finite set, we have $T \in \mathscr{U}'_{\bigcap_{j\in J_K} W_j}(\Omega)$. So far we have proved that T is in $\mathscr{U}'_{\bigcap_{j\in J_K} W_j}(\Omega)$ and φ is in $\mathscr{U}_{\bigcap_{j\in J_K} W_j}(\Omega)$; thus we may write, by definition,

$$\langle T, \varphi\rangle_{W,\Omega} = \langle T, \varphi\rangle_{\bigcap_{j\in J_K} W_j,\Omega}\,.$$

The previous equality shows that T can be extended as a linear continuous functional on $\mathscr{U}_W(\Omega)$, and thus $T \in \mathscr{U}'_W(\Omega)$. $\square$

Next we give an important example of a generalized distribution which is not a distribution. Let $a \in \Omega$, $W = \{a\}$ and $(\beta_\alpha)_{\alpha\in\mathbb{N}^N}$ a given multi-sequence of complex numbers. We define $T \in \mathscr{U}'_W(\Omega)$ by

$$\langle T, \varphi\rangle = \sum_{\alpha\in\mathbb{N}^N} (-1)^{|\alpha|}\beta_\alpha\partial^\alpha\varphi(a) \tag{2}$$

for all $\varphi \in \mathscr{U}_W(\Omega)$. Note that (2) defines indeed an element T of $\mathscr{U}'_W(\Omega)$ that cannot be extended as a linear continuous functional to $\mathscr{D}(\Omega)$ unless all but a finite number of the scalars β_α are null; note also that the previous series reduces, for each

$\varphi \in \mathscr{U}_W(\Omega)$, to a finite sum with a number of terms depending on φ. The generalized distribution T defined by (2) is called the multipole series at the point a associated with the multi-sequence $(\beta_\alpha)_{\alpha\in\mathbb{N}^N}$.

4. Ultra-support. Restriction. Support

Let $T \in \mathscr{U}'(\Omega)$; we call ultra-support of T, and we denote by $\operatorname{usupp} T$, the intersection of all $W \in \mathscr{W}(\Omega)$ such that $T \in \mathscr{U}'_W(\Omega)$. We have $\operatorname{usupp} T \in \mathscr{W}(\Omega)$ and $T \in \mathscr{D}'(\Omega)$ iff $\operatorname{usupp} T = \emptyset$; by Theorem 1 we see that $T \in \mathscr{U}'_{\operatorname{usupp} T}(\Omega)$. For notational purposes, we adopt the following convention: if $T \in \mathscr{U}'(\Omega)$ and $\varphi \in \mathscr{U}_{\operatorname{usupp} T}(\Omega)$, the symbol $\langle T, \varphi\rangle$ means always $\langle T, \varphi\rangle_{\operatorname{usupp} T, \Omega}$.

Let $T \in \mathscr{U}'(\Omega)$ and let Ω_0 be an open non empty subset of $\mathbb{R}^N$ included in Ω. We introduce the restriction of T to Ω_0 as the element $\varrho T \in \mathscr{U}'_W(\Omega_0)$, where $W \in \mathscr{W}(\Omega_0)$ is the set $W = \Omega_0 \cap \operatorname{usupp} T$, defined by $\langle \varrho T, \varphi\rangle_{W,\Omega_0} = \langle T, \varphi_0\rangle$ for all $\varphi \in \mathscr{U}_W(\Omega_0)$, where φ_0 is the trivial extension of φ to Ω. The mapping $\varrho\colon \mathscr{U}'_W(\Omega) \to \mathscr{U}'_W(\Omega_0)$ just defined is linear continuous and $\operatorname{usupp}(\varrho T) \subset (\operatorname{usupp} T \cap \Omega_0)$.

Let $T \in \mathscr{U}'(\Omega)$. We write Ω_T for $\Omega \setminus \operatorname{usupp} T$. We associate to T the distribution $\gamma T \in \mathscr{D}'(\Omega_T)$ defined by $\langle \gamma T, \varphi\rangle_{\mathscr{D}'(\Omega_T), \mathscr{D}(\Omega_T)} = \langle T, \varphi_0\rangle$, where φ_0 is the trivial extension of φ to Ω (φ_0 equals φ on Ω_T and φ_0 is zero outside Ω_T); in other words, γT is the restriction of T to the open set Ω_T. We define the support of T by $\operatorname{supp} T = \operatorname{usupp} T \cup \operatorname{supp}(\gamma T)$, where $\operatorname{supp}(\gamma T)$ is the usual support of the distribution γT. Obviously, if $T \in \mathscr{D}'(\Omega)$, the support of T as a generalized distribution is the same as the support as a distribution. For $T, S \in \mathscr{U}'(\Omega)$ and a scalar $\alpha \neq 0$, we have the relations $\operatorname{usupp}(T + S) \subset (\operatorname{usupp} T \cup \operatorname{usupp} S)$ and $\operatorname{usupp}(\alpha T) = \operatorname{usupp} T$; for the supports we have also $\operatorname{supp}(T + S) \subset (\operatorname{supp} T \cup \operatorname{supp} S)$ and $\operatorname{supp}(\alpha T) = \operatorname{supp} T$.

Theorem 2. *Let $T \in \mathscr{U}'(\Omega)$ and let $\varphi \in \mathscr{D}(\Omega)$ be such that $\operatorname{supp}\varphi \cap \operatorname{usupp} T = \emptyset$. Then $\langle T, \varphi\rangle = \langle \gamma T, \varphi\rangle_{\mathscr{D}'(\Omega), \mathscr{D}(\Omega)}$.*

Proof. This is a consequence of the definition of γT. □

The support of a generalized distribution $T \in \mathscr{U}'(\Omega)$ may be defined without the help of the distribution γT. We say that T is null on an open subset $\mathscr{O}$ of Ω if $\langle T, \varphi\rangle = 0$ for all $\varphi \in \mathscr{U}_{\operatorname{usupp} T(\Omega)}$ such that $\operatorname{supp}\varphi$ is included in $\mathscr{O}$. We say that $T, S \in \mathscr{U}'(\Omega)$ are equal on $\mathscr{O}$ iff $T - S$ is null on $\mathscr{O}$. Let $(\Omega_i)_{i\in I}$ be a family of open subsets of Ω, covering Ω, and let $(T_i)_{i\in I}$ be a family (indexed by the same index set I) of generalized distributions, with $T_i \in \mathscr{U}'(\Omega_i)$; then, if T_i and T_j are equal in $\Omega_i \cap \Omega_j$ whenever this intersection is non void and if $\bigcup\limits_{i\in I} \operatorname{usupp} T_i \in \mathscr{W}(\Omega)$, there is one and only one generalized distribution $T \in \mathscr{U}'(\Omega)$ such that T is equal to T_i in Ω_i.

Before the proof of this property, we mention that it is the analog of the "principe du recollement des morceaux" for Schwartz distributions, and that it enables us to given another (equivalent) definition of the support of T: it is the complement of the union of all open subsets of Ω where T is null.

Now we have to prove the property mentioned above. Let $(\psi_i)_{i\in I}$ be a partition of the unity subordinated to the open cover $(\Omega_i)_{i\in I}$ of Ω. Write $W = \bigcup_{i\in I} \operatorname{usupp} T_i$; recall that, by assumption, $W \in \mathscr{W}(\Omega)$. Let $\varphi \in \mathscr{U}_W(\Omega)$ and write $K = \operatorname{supp}\varphi$. Because K is compact we know that there exists a finite subset of I, say J_K, such that, if $i \notin J_K$, ψ_i is identically null on K. Thus we may write

$$\varphi = \sum_{j\in J_K} \psi_j\varphi.$$

This equation shows that, if there exists a generalized distribution $T \in \mathscr{U}'(\Omega)$ with the required properties, T is in $\mathscr{U}'_W(\Omega)$ and

$$\langle T,\varphi\rangle_{W,\Omega} = \sum_{j\in J_K} \langle T,\psi_j\varphi\rangle_{W,\Omega} = \sum_{j\in J_K} \langle T_j,\psi_j\varphi\rangle_{W_j,\Omega_j}\,. \tag{3}$$

Equation (3) shows the uniqueness of $T \in \mathscr{U}'(\Omega)$ such that T is equal to T_i in Ω_i. To prove the existence it is sufficient to see that equation (3) defines indeed a linear and continuous functional T on $\mathscr{U}_W(\Omega)$ such that T is equal to T_i in Ω_i. This is straightforward and we leave the proof to the reader.

It is easily seen that $\langle T,\varphi\rangle = 0$ whenever $\varphi = 0$ in a neighbourhood of $\operatorname{supp} T$. As φ has a compact support, this implies that $\langle T,\varphi\rangle = 0$ whenever $\operatorname{supp} T$ and $\operatorname{supp}\varphi$ are disjoint sets.

Let $\mathscr{E}(\Omega)$ be the vector space of C^∞ functions in Ω with the (usual) topology of uniform convergence of the function and its derivatives in the compact subsets of Ω. For $W \in \mathscr{W}(\Omega)$, we denote by $\mathscr{V}_W(\Omega)$ the vector subspace of $\mathscr{E}(\Omega)$ of the functions ψ such that there is $m \in \mathbb{N}$ (depending on ψ) with $\partial^\alpha\psi(x) = 0$ for all $\alpha \in \mathbb{N}^N$, $|\alpha| > m$ and all $x \in W$. Obviously we have $\mathscr{U}_W(\Omega) \subset \mathscr{V}_W(\Omega)$.

Let $T \in \mathscr{U}'_W(\Omega)$, $\varphi \in \mathscr{V}_W(\Omega)$ and $K = \operatorname{supp} T \cap \operatorname{supp}\varphi$. We are going to show that, if $K \in \mathscr{K}(\Omega)$, we still can define $\langle T,\varphi\rangle$. We fix an open neighbourhood $O \subset \Omega$ of K and a function $\mu \in \mathscr{D}(\Omega)$ such that $\mu = 1$ in O. We put, by definition,

$$\langle T,\varphi\rangle = \langle T,\mu\varphi\rangle\,. \tag{4}$$

It is easily seen that the value $\langle T,\varphi\rangle$ does not depend on the choices of O and μ. In fact, if $O' \subset \Omega$ is another open neighbourhood of K and $\nu \in \mathscr{D}(\Omega)$ another function such that $\nu = 1$ in O', we have

$$\langle T,\mu\varphi\rangle - \langle T,\nu\varphi\rangle = \langle T,(\mu-\nu)\varphi\rangle = 0$$

because $\operatorname{supp}(\mu-\nu)$ is included in $\Omega\backslash(O\cap O')$ which is disjoint form a neighbourhood of K; a fortiori, $\operatorname{supp}(\mu-\nu)\varphi$ is disjoint from a neighbourhood of the support of the generalized distribution T.

5. Derivatives, multiplication and translation

Let $T \in \mathscr{U}'(\Omega)$ and let $\alpha \in \mathbb{N}^N$. As the operator ∂^α is linear continuous from $\mathscr{U}_{\operatorname{usupp} T}(\Omega)$ into $\mathscr{U}_{\operatorname{usupp} T}(\Omega)$, we may extend it, by duality, to the space $\mathscr{U}'_{\operatorname{usupp} T}(\Omega)$, defining $\partial^\alpha T$ as follows: $\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha\varphi\rangle$, for all $\varphi \in \mathscr{U}_{\operatorname{usupp} T}(\Omega)$. The operator ∂^α is linear continuous from $\mathscr{U}'_{\operatorname{usupp} T}(\Omega)$ into $\mathscr{U}'_{\operatorname{usupp} T}(\Omega)$ and, on $\mathscr{D}'(\Omega)$, equals the usual derivative operator. Note that the generalized distribution T defined by (2), that is the multipole series at the point a associated with the sequence $(\beta_\alpha)_{\alpha\in\mathbb{N}^N}$, verifies

$$T = \sum_{\alpha\in\mathbb{N}^N} \beta_\alpha \partial^\alpha \delta_a. \tag{5}$$

This series is convergent in $\mathscr{U}'_{\{a\}}(\Omega)$ but is not convergent in $\mathscr{D}'(\Omega)$, except in the trivial case where all but a finite number of the β_α are null. It is very easy to prove the following properties of the multipole series T defined by (5):

$$\begin{aligned}
\operatorname{supp} T &\subset \{a\};\\
\operatorname{supp} T &= \emptyset \text{ iff } \forall\alpha \in \mathbb{N}^N\, \beta_\alpha = 0;\\
\operatorname{usupp} T &= \{a\} \text{ iff there are infinitely many non null scalars } \beta_\alpha;\\
\operatorname{usupp} T &= \emptyset \text{ iff all but a finite number of the scalars } \beta_\alpha \text{ are null};\\
T &= \sum_{\alpha\in\mathbb{N}^N} \gamma_\alpha \partial^\alpha \delta_b \text{ iff } a = b \wedge \forall\alpha \in \mathbb{N}^N\, \beta_\alpha = \gamma_\alpha.
\end{aligned}$$

As $\psi\varphi$ belongs to $\mathscr{U}_W(\Omega)$ whenever $\varphi \in \mathscr{U}_W(\Omega)$ and $\psi \in \mathscr{V}_W(\Omega)$, we may define, by duality, the product of ψ by an element $T \in \mathscr{U}'_W(\Omega)$: $\langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle$, for all $\varphi \in \mathscr{U}_W(\Omega)$. The mapping $T \to \psi T$ is linear continuous from $\mathscr{U}'_W(\Omega)$ into itself and $\operatorname{usupp}(\psi T) \subset \operatorname{usupp} T$. The Leibniz formulas for the derivatives of the product remain valid.

Let $f \in C(\mathbb{R}^N)$ and let $h \in \mathbb{R}^N$; the translation operator $\tau_h\colon C(\mathbb{R}^N) \to C(\mathbb{R}^N)$ is defined by $\tau_h f(x) = f(x-h)$. Obviously, if $f \in C^\infty(\mathbb{R}^N)$, we have $\partial^\alpha(\tau_h f) = \tau_h(\partial^\alpha f)$ for all $\alpha \in \mathbb{N}^N$. This implies that, if $W \in \mathscr{W}(\mathbb{R}^N)$ and $\varphi \in \mathscr{U}_W(\mathbb{R}^N)$, then $\tau_h\varphi \in \mathscr{U}_{h+W}(\mathbb{R}^N)$, where $h+W$ is the subset of $\mathbb{R}^N$ of points x of the form $x = y+h$ with $y \in W$. We define the operator $\tau_{h,W}\colon \mathscr{U}'_W(\mathbb{R}^N) \to \mathscr{U}'_{h+W}(\mathbb{R}^N)$ by $\langle \tau_h T, \varphi\rangle_{h+W,\mathbb{R}^N} = \langle T, \tau_{-h}\varphi\rangle_{W,\mathbb{R}^N}$ for all $\varphi \in \mathscr{U}_{h+W}(\mathbb{R}^N)$. Obviously $\tau_{h,W}$ is a vectorial and topological isomorphism from $\mathscr{U}'_W(\mathbb{R}^N)$ onto $\mathscr{U}'_{h+W}(\mathbb{R}^N)$, the inverse being the operator $\tau_{-h,h+W}$. We now define the operator $\tau_h\colon \mathscr{U}'(\mathbb{R}^N) \to \mathscr{U}'(\mathbb{R}^N)$ as follows: for $T \in \mathscr{U}'(\mathbb{R}^N)$, $\tau_h T = \tau_{h,\operatorname{usupp} T} T$. The operator τ_h is a vectorial and topological isomorphism from $\mathscr{U}'_{\operatorname{usupp} T}(\mathbb{R}^N)$ onto $\mathscr{U}_{h+\operatorname{usupp} T}(\mathbb{R}^N)$, the inverse being τ_{-h}. The translation operator in $\mathscr{U}'(\mathbb{R}^N)$ has the following properties:

$$\tau_h \tau_k T = \tau_{h+k} T \text{ for all } T \in \mathscr{U}'(\mathbb{R}^N) \text{ and all } h, k \in \mathbb{R}^N;$$
$$\partial^\alpha(\tau_h T) = \tau_h(\partial^\alpha T) \text{ for all } T \in \mathscr{U}'(\mathbb{R}^N) \text{ and all } \alpha \in \mathbb{N}^N;$$
$$\operatorname{usupp}(\tau_h T) = h + \operatorname{usupp} T \text{ for all } T \in \mathscr{U}'(\mathbb{R}^N).$$

Theorem 3. *Let $T \in \mathscr{U}'(\mathbb{R}^N)$ be such that $\tau_{ch} T = T$ for some $h \in \mathbb{R}^N \setminus \{0\}$ and all $c \in \mathbb{R}$; then $T \in \mathscr{D}'(\mathbb{R}^N)$.*

P r o o f. From $\tau_{ch} T = T$ we have $\operatorname{usupp}(\tau_{ch} T) = \operatorname{usupp} T$; then $\operatorname{usupp} T = ch + \operatorname{usupp} T$ for all $c \in \mathbb{R}$ and, since the intersections of $\operatorname{usupp} T$ with the compacts are finite, this implies $\operatorname{usupp} T = \emptyset$, that is (Theorem 1) $T \in \mathscr{D}'(\mathbb{R}^N)$. □

Lemma. *Let $T \in \mathscr{U}'(\mathbb{R}^N)$ be such that $\partial_j T = 0$ (∂_j is the derivative with respect to x_j) for some $j \in \{1, \ldots, N\}$. Then $\tau_{ce_j} T = T$ for all $c \in \mathbb{R}$, where e_j is the j-vector of the canonical basis of $\mathbb{R}^N$.*

P r o o f. Suppose, for simplicity, that $j = 1$. Fix $c \in \mathbb{R}$ and write $W = \operatorname{usupp} T \cup \operatorname{usupp}(\tau_{ce_1} T)$. Suppose that $\tau_{ce_1} T \neq T$; then there is a function $\varphi \in \mathscr{U}_W(\mathbb{R}^N)$ such that

(6) $$\langle \tau_{ce_1} T - T, \varphi \rangle_{W,\mathbb{R}^N} = \langle T, \tau_{-ce_1} \varphi - \varphi \rangle_{W,\mathbb{R}^N} \neq 0.$$

Define $\psi\colon \mathbb{R}^N \to \mathbb{R}$ by $\psi(x) = \int_{-\infty}^{x_1} \varphi(y, x_2, \ldots, x_N) dy$; we have

$$(\tau_{-ce_1} \varphi - \varphi)(x) = \varphi(x + ce_1) - \varphi(x) = \partial_1(\psi(x + ce_1) - \psi(x)) = \partial_1 \lambda(x)$$

with $\lambda(x) = \psi(x + ce_1) - \psi(x)$. But $\lambda \in \mathscr{U}_W(\mathbb{R}^N)$, then, since $\partial_1 T = 0$,

$$\langle T, \tau_{-ce_1} \varphi - \varphi \rangle_{W,\mathbb{R}^N} = \langle T, \partial_1 \lambda \rangle_{W,\mathbb{R}^N} = -\langle \partial_1 T, \lambda \rangle_{W,\mathbb{R}^N} = 0$$

which contradicts (6). □

Theorem 4. *Let $T \in \mathscr{U}'(\Omega)$ and let $\alpha \in \mathbb{N}^N$. If $\partial^\alpha T = 0$, then $T \in \mathscr{D}'(\Omega)$.*

P r o o f. The theorem is a consequence of the lemma and Theorem 3. □

Note that, if $T \in \mathscr{U}'(\Omega)$ verifies $\partial^\alpha T = 0$ for all α such that $|\alpha|$ is grater then a given $m \in \mathbb{N}$, and if Ω is connected, then T is a polynomial of degree less then $|\alpha|$. If $T \in \mathscr{U}'(\Omega)$ verifies $\partial^\alpha T = 0$ for some $\alpha \in \mathbb{N}^N$, then T is a pseudopolynomial of degree less then $|\alpha|$.

Theorem 5. *Let $T \in \mathscr{U}'(\mathbb{R})$ be such that $x^n T = 0$ for some $n \in \mathbb{N} \setminus \{0\}$; then $T = a_1\delta + \ldots + a_n\delta^{(n)}$.*

Proof. Consider first the case $n = 1$. We know that $0 = \langle xT, \varphi\rangle = \langle T, x\varphi\rangle$ for all $\varphi \in \mathscr{U}_{\operatorname{usupp} T}(\mathbb{R})$; writing $\mathscr{H} = \{\psi \in \mathscr{U}_{\operatorname{usupp} T}(\mathbb{R}); \psi(x) = x\varphi(x), \varphi \in \mathscr{U}_{\operatorname{usupp} T}(\mathbb{R})\}$ we have $\langle T, \psi\rangle = 0$ for all $\psi \in \mathscr{H}$. But $\psi \in \mathscr{H}$ iff $\psi \in \mathscr{U}_{\operatorname{usupp} T}(\mathbb{R})$ and $\psi(0) = 0$; thus $\mathscr{H}$ is a hyperplane of $\mathscr{U}_{\operatorname{usupp} T}(\mathbb{R})$ and T is entirely determined if we fix its value on a function $\varphi_0 \in \mathscr{U}_{\operatorname{usupp} T}(\mathbb{R}) \setminus \mathscr{H}$. Let us fix $\langle T, \varphi_0\rangle = \lambda$, with $\lambda \in \mathbb{C}$. Now, for $\varphi \in \mathscr{U}_{\operatorname{usupp} T}(\mathbb{R})$, we have the unique decomposition $\varphi = \psi + c\varphi_0$ ($c \in \mathbb{C}$) and

$$\langle T, \varphi\rangle = \langle T, \psi\rangle + \langle T, c\varphi_0\rangle = c\lambda = \lambda\frac{\varphi(0)}{\varphi_0(0)} = \langle c_1\delta, \varphi\rangle$$

with $c_1 = \lambda/\varphi_0(0)$. This shows that $T = c_1\delta$. As $x\delta^{(n)} = -n\delta^{(n-1)}$, the theorem can easily be proved by induction on n. □

6. Structure of $\mathscr{U}'_W(\Omega)$

In this section we prove a structure theorem for the space $\mathscr{U}'_W(\Omega)$. As we mentioned in the introduction this theorem was first proved by Laurent Schwartz; our proof is slightly different.

Proposition 5. *Let $T \in \mathscr{U}'(\Omega)$ and $a \in \Omega$. We have* $\operatorname{supp} T \subset \{a\}$ *iff* $T = \sum\limits_{\alpha\in\mathbb{N}^N} c_\alpha \partial^\alpha \delta_a$, *where the c_α are scalars uniquely determined by T.*

Proof. If $T = \sum\limits_{\alpha\in\mathbb{N}^N} c_\alpha \partial^\alpha \delta_a$ we know that $\operatorname{supp} T \subset \{a\}$. Suppose now that $\operatorname{supp} T \subset \{a\}$. Let $\varphi \in \mathscr{U}_{\{a\}}(\Omega)$; using the Taylor expansion of φ around the point a, we have

$$\varphi(x) = \sum_{\alpha\in\mathbb{N}^N} \frac{(x-a)^\alpha}{\alpha!}\partial^\alpha\varphi(a) + \psi(x),$$

where $\psi \in \mathscr{E}(\Omega)$ and $\partial^\alpha\psi(a) = 0$ for all $\alpha \in \mathbb{N}^N$. Note that, for each $\varphi \in \mathscr{U}_{\{a\}}(\Omega)$, the series reduces to a finite sum. As T has a compact support we may write

$$\begin{aligned}
\langle T, \varphi\rangle &= \left\langle T, \sum_{\alpha\in\mathbb{N}^N} \frac{(x-a)^\alpha}{\alpha!}\partial^\alpha\varphi(a)\right\rangle + \langle T, \psi\rangle \\
&= \sum_{\alpha\in\mathbb{N}^N} \langle T, (x-a)^\alpha\rangle \frac{1}{\alpha!}\partial^\alpha\varphi(a) + \langle T, \psi\rangle \\
&= \sum_{\alpha\in\mathbb{N}^N} \langle T, (x-a)^\alpha\rangle \frac{(-1)^{|\alpha|}}{\alpha!} \langle \partial^\alpha\delta_a, \varphi\rangle + \langle T, \psi\rangle = \sum_{\alpha\in\mathbb{N}^N} c_\alpha \partial^\alpha\delta_a + \langle T, \psi\rangle
\end{aligned}$$

where

$$c_\alpha = \frac{(-1)^{|\alpha|}}{\alpha!} \langle T, (x-a)^\alpha \rangle .$$

Now we have to prove that $\langle T, \psi \rangle = 0$.

For each $\varepsilon > 0$ we write $B_\varepsilon = \{x \in \mathbb{R}^N;\ |x-a| < \varepsilon\}$. We fix $r > 0$ such that $B_r \subset \Omega$. Let $\mu \in \mathscr{D}(\Omega)$ be such that $\operatorname{supp} \mu \subset B_r$ and $\mu = 1$ in $B_{r/2}$. For each $k \in \mathbb{N} \setminus \{0\}$ we define $\psi_k \in \mathscr{D}(\Omega)$ by

$$\psi_k(x) = \mu(x)\psi(x)[1 - \mu(k(x-a)+a)].$$

We have $\operatorname{supp} \psi_k \subset B_{r/k}$ and $\psi_k = 0$ in $B_{r/2k}$. This last condition implies that, for all $k \in \mathbb{N} \setminus \{0\}$, $\langle T, \psi_k \rangle = 0$.

Now it is sufficient to prove that $\psi_k \to \mu\psi$ in $\mathscr{U}_{\{a\}}(\Omega)$ because, in that case,

$$\langle T, \psi \rangle = \langle T, \mu\psi \rangle = \lim_{k \to +\infty} \langle T, \psi_k \rangle = 0.$$

Let us prove the convergence $\psi_k \to \mu\psi$ in $\mathscr{U}_{\{a\}}(\Omega)$. We have to show that $g_k \to 0$ in $\mathscr{U}_{\{a\}}(\Omega)$, where

$$g_k(x) = \mu(x)\psi(x) - \psi_k(x) = \mu(x)\psi(x)\mu(k(x-a)+a).$$

We have to verify conditions (1). Conditions (i) and (ii) are trivial, since $\partial^\alpha g_k(a) = 0$ for all $\alpha \in \mathbb{N}^N$ and $\operatorname{supp} g_k \subset B_{r/k} \subset B_r$. It remains to prove condition (iii); for $\alpha \in \mathbb{N}^N$ we have

$$\begin{aligned} \partial^\alpha g_k(x) &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta}(\mu\psi))(x) \partial^\beta (\mu(k(x-a)+a)) \\ &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta}(\mu\psi))(x) k^{|\beta|} (\partial^\beta \mu)(k(x-a)+a). \end{aligned}$$

Now, since $\mu\psi = \psi$ in $B_{r/2}$, we have, for $k \geqslant 2$:

$$\sup_{x \in \Omega} |\partial^\alpha g_k(x)| = \sup_{x \in B_{r/k}} |\partial^\alpha g_k(x)| \leqslant \sum_{\beta \leqslant \alpha} c_\beta k^{|\beta|} \sup_{x \in B_{r/k}} |\partial^{\alpha-\beta} \psi(x)|$$

where $c_\beta = \binom{\alpha}{\beta} \sup_{x \in \Omega} |\partial^\beta \mu(x)|$.

All the derivatives of the function ψ are equal to zero at the point a; the Taylor expansion of $\partial^{\alpha-\beta}\psi$ around the point a shows the existence of a constant $C > 0$ such that

$$|\partial^{\alpha-\beta}\psi(x)| \leqslant C|x-a|^{|\alpha|+1}.$$

This implies that

$$0 \leqslant \sup_{x \in B_{r/k}} |\partial^\alpha g_k(x)| \leqslant \sum_{\beta \leqslant \alpha} c_\beta k^{|\beta|} C \sup_{x \in B_{r/k}} |x-a|^{|\alpha|+1} \leqslant \sum_{\beta \leqslant \alpha} c_\beta k^{|\alpha|} C \frac{r^{|\alpha|+1}}{k^{|\alpha|+1}} \leqslant \frac{D}{k}$$

where $D = Cr^{|\alpha|+1} \sum_{\beta \leqslant \alpha} c_\beta$. This shows that $\partial^\alpha g_k \to 0$ uniformly in Ω. □

Proposition 6. *Let $T \in \mathscr{U}'(\Omega)$ and $a \in \Omega$; then $T \in \mathscr{U}'_{\{a\}}(\Omega)$ iff*

$$T = S + \sum_{\alpha \in \mathbb{N}^N} c_\alpha \partial^\alpha \delta_a \tag{7}$$

where $S \in \mathscr{D}'(\Omega)$ and the c_α are scalars.

Proof. If T is of the form (7), then $T \in \mathscr{U}'_{\{a\}}(\Omega)$. Suppose now that $T \in \mathscr{U}'_{\{a\}}(\Omega)$; for $m \in \mathbb{N}$, let T_m be the restriction of the functional T to the space $\mathscr{U}_{\{a\},m}(\Omega)$. We have $T \in \mathscr{U}'_{\{a\},m}(\Omega)$ and, as the topology of $\mathscr{U}_{\{a\},m}(\Omega)$ is the one induced by $\mathscr{D}(\Omega)$, the Hahn-Banach theorem tells us that T_m can be extended to $\mathscr{D}(\Omega)$ as a linear continuous functional, which we denote by $\widetilde{T}_m$. We write

$$T = \widetilde{T}_m + (T - \widetilde{T}_m)$$

and, as $\widetilde{T}_m \in \mathscr{D}'(\Omega)$ and $T - \widetilde{T}_m \in \mathscr{U}'_{\{a\}}(\Omega)$, we just have to show, by Proposition 5, that $\operatorname{supp}(T - \widetilde{T}_m) \subset \{a\}$. But this is obvious because, for all $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp} \varphi \subset \Omega \setminus \{a\}$, $\langle \widetilde{T}_m, \varphi \rangle = \langle T_m, \varphi \rangle = \langle T, \varphi \rangle$, and this proves that $T - \widetilde{T}_m$ is null in the open set $\Omega \setminus \{a\}$, that is $\operatorname{supp}(T - \widetilde{T}_m) \subset \{a\}$. □

Proposition 7. *Let $T \in \mathscr{U}'_W(\Omega)$ and let $(\Omega_i)_{i \in I}$ be an open cover of W such that $\Omega_i \subset \Omega$ and $\Omega_i \cap \Omega_j \cap W = \emptyset$ whenever $i \neq j$. Then*

$$T = S + \sum_{i \in I} T_i \tag{8}$$

where $S \in \mathscr{D}(\Omega)$, $T_i \in \mathscr{U}'_{W \cap \Omega_i}(\Omega)$ with $\operatorname{supp} T_i \subset \Omega_i$, and the series $\sum_{i \in I} T_i$ is convergent in $\mathscr{U}'_W(\Omega)$.

Proof. We see that $\Omega \setminus W$ together with $(\Omega_i)_{i \in I}$ is an open cover of Ω. We consider a partition of unity relative to this open cover, let us say, $\psi_{\Omega \setminus W}$ and $(\psi_i)_{i \in I}$.

We define $S \in \mathscr{D}'(\Omega)$ by $\langle S, \varphi \rangle = \langle T, \psi_{\Omega \setminus W} \varphi \rangle$, for all $\varphi \in \mathscr{D}(\Omega)$. For each $i \in I$ we define $T_i \in \mathscr{U}'_{W \cap \Omega_i}(\Omega)$ by $\langle T_i, \varphi \rangle = \langle T_i, \psi_i \varphi \rangle$, for all $\varphi \in \mathscr{U}_{W \cap \Omega_i}(\Omega)$. We have $\operatorname{supp} T_i \subset \Omega_i$.

The series $\sum_{i \in I} T_i$ is convergent in $\mathscr{U}'_W(\Omega)$ because, for each $\varphi \in \mathscr{U}_W(\Omega)$, it reduces to a finite sum. Finally it is easily seen that the formula (8) is true. □

Theorem 6. *Let $T \in \mathscr{U}'(\Omega)$ and $W \in \mathscr{W}(\Omega)$; then $T \in \mathscr{U}'_W(\Omega)$ iff*

$$T = S + \sum_{w \in W} \sum_{\alpha \in \mathbb{N}^N} c_{w,\alpha} \partial^\alpha \delta_w \tag{9}$$

where $S \in \mathscr{D}'(\Omega)$ and the $c_{w,\alpha}$ are scalars.

P r o o f. For each $w \in W$ we fix an open subset Ω_w of Ω in such a way that $\Omega_w \cap W = \{w\}$ and, for all $K \in \mathscr{K}(\Omega)$, all but a finite number of the sets Ω_w are disjoint from K. By formula (8) we see that

$$T = S_0 + \sum_{w \in W} T_w$$

with $S_0 \in \mathscr{D}'(\Omega)$, $T_w \in \mathscr{U}'_{\{w\}}(\Omega)$ and $\operatorname{supp} T_w \subset \Omega_w$. By (7) we see that

$$T_w = S_w + \sum_{\alpha \in \mathbb{N}^N} c_{w,\alpha} \partial^\alpha \delta_w$$

where $S_w \in \mathscr{D}'(\Omega)$ and the $c_{w,\alpha}$ are scalars. Then we have

$$T = S_0 + \sum_{w \in W} S_w + \sum_{w \in W} \sum_{\alpha \in \mathbb{N}^N} c_{w,\alpha} \partial^\alpha \delta_w$$

and it remains to prove the convergence of the series $\sum\limits_{w \in W} S_w$. But this is obvious because, for each $\varphi \in \mathscr{D}(\Omega)$, the series reduces to a finite sum. This proves (9), with $S = S_0 + \sum\limits_{w \in W} S_w$. $\square$

From this theorem we can see that $\langle T, \varphi \rangle = 0$ for all $T \in \mathscr{U}'_W(\Omega)$ and all $\varphi \in \mathscr{U}_W(\Omega)$ such that, for all $\alpha \in \mathbb{N}^N$, $\partial^\alpha \varphi = 0$ in the support of T.

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