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CONNECTED DOMINATION CRITICAL GRAPHS WITH RESPECT TO RELATIVE COMPLEMENTS

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Abstract. A dominating set in a graph G is a *connected dominating set* of G if it induces a connected subgraph of G. The minimum number of vertices in a connected dominating set of G is called the *connected domination number* of G, and is denoted by $\gamma_c(G)$. Let G be a spanning subgraph of $K_{s,s}$ and let H be the complement of G relative to $K_{s,s}$; that is, $K_{s,s} = G \oplus H$ is a factorization of $K_{s,s}$. The graph G is *k-γ_c-critical relative to $K_{s,s}$* if $\gamma_c(G) = k$ and $\gamma_c(G+e) < k$ for each edge $e \in E(H)$. First, we discuss some classes of graphs whether they are γ_c-critical relative to $K_{s,s}$. Then we study k-γ_c-critical graphs relative to $K_{s,s}$ for small values of k. In particular, we characterize the 3-γ_c-critical and 4-γ_c-critical graphs.

Keywords: connected domination number, connected domination critical graph relative to $K_{s,s}$, tree.

MSC 2000: 05C69, 05C35

1. Introduction

Let $G = (V, E)$ be a simple connected graph. The *degree, neighborhood and closed neighborhood of a vertex v* in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The *minimum degree and maximum degree* of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. Let P_n, C_n, $K_{1,n-1}$ and K_n denote the *path, cycle, star* and *complete graph* with n vertices, respectively. Let $K_{n,m}$ denote the *complete bipartite graph.*

A *dominating set S* is a set of vertices where every vertex of G is in $N[v]$ for some $v \in S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating

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set. A dominating set in a graph G is a *connected dominating set* of G if it induces a connected subgraph of G. The *connected domination number* $\gamma_c(G)$ is the minimum cardinality of a connected dominating set. If S is a minimum connected dominating set, we call S a γ_c-set of G.

If G is a spanning subgraph of F, then the graph $F - E(G)$ is the complement of G relative to F with respect to a fixed embedding of G into F. The idea of a relative complement of a graph was suggested by Cockayne [1] and is studied in [2]. We shall assume that the complete bipartite graph $K_{s,s}$ has partite sets A and B, and that $G \oplus H = K_{s,s}$ is a factorization of $K_{s,s}$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.) Notice that if G is uniquely embeddable in $K_{s,s}$, then H is unique. We henceforth consider only spanning subgraphs G of $K_{s,s}$ such that G is uniquely embeddable in $K_{s,s}$. We denote the relative complement H of G by $\overline{G}$.

Haynes and Henning [3] studied *domination critical graphs with respect to the relative complement*, that is, the graphs G such that $\gamma(G+e) = \gamma(G) - 1$ for all $e \in E(\overline{G})$. Hayness, Henning and Van der Merwe [4]–[5] studied *total domination edge critical graphs with respect to the relative complement*, or just k_t-critical graphs, that is, the graphs G such that $\gamma_t(G+e) < \gamma_t(G) = k$ for any edge $e \in E(\overline{G})$.

In this paper we study the same concept for connected domination. We say that a graph G is *connected domination critical relative to* $K_{s,s}$, or just k-γ_c-critical, if $\gamma_c(G+e) < \gamma_c(G) = k$ for any edge $e \in E(\overline{G})$.

We use the following notation. An *endvertex* is a vertex of degree one and its neighbor is called a *support vertex*. An endvertex of a tree is also called a *leaf*. For a set S, $X \subseteq V$, if S dominates X, then we write $S \succ X$, while if $\langle S \rangle$ is connected and S dominates X, we write $S \succ_c X$. If v, u are adjacent vertices, then we write $v \perp u$. Otherwise, we write $v \pm u$.

First, we discuss some classes of graphs whether they are γ_c-critical relative to $K_{s,s}$. Then we study k-γ_c-critical graphs relative to $K_{s,s}$ for small values of k. In particular, we characterize the 3-γ_c-critical and 4-γ_c-critical graphs.

2. Main results

Whereas the addition of an edge from the complement $\overline{G}$ can change the domination number of G by at most one, it can change the connected domination number by as much as two.

Theorem 1. *Let G be a connected graph. Then for any edge $e \in E(\overline{G})$, $\gamma_c(G) - 2 \leqslant \gamma_c(G+e) \leqslant \gamma_c(G)$.*

P r o o f. It is clear that $\gamma_c(G+e) \leqslant \gamma_c(G)$. Now we only prove $\gamma_c(G) - 2 \leqslant \gamma_c(G+e)$ for any edge $e \in E(\overline{G})$. Let $e = vu$. Let S' be a connected dominating set of $G + e$ with minimum cardinality.

Case 1. $v, u \notin S'$. Then S' is a connected dominating set of G. Hence, $\gamma_c(G) \leqslant \gamma_c(G+e)$.

Case 2. $v \in S'$ and $u \notin S'$. If u is adjacent to at least one vertex in $S' - \{v\}$, then S' is a connected dominating set of G. Hence, $\gamma_c(G) \leqslant \gamma_c(G+e)$. So we assume that u is not adjacent to any vertex in $S' - \{v\}$. Since G is a connected graph, u is not an isolated vertex in G. Let $t \in N(u)$. Then $t \in V(G) - S'$ and t is dominated by at least one vertex in S'. Then $S' \cup \{t\}$ is a connected dominating set of G. Hence, $\gamma_c(G) \leqslant \gamma_c(G+e) + 1$.

Case 3. $u \in S'$ and $v \notin S'$. In a similar way as Case 2, it is easy to prove.

Case 4. $v \in S'$ and $u \in S'$. If vu is not a cut edge of $\langle S' \rangle$, then S' is a connected dominating set of G. Hence, $\gamma_c(G) \leqslant \gamma_c(G+e)$. If vu is a cut edge of $\langle S' \rangle$, then let $S_1^{'}$ and $S_2^{'}$ be the two components of $\langle S' \rangle - vu$. If there exists a vertex w in $V(G) - S'$ such that $w \in (N(S_1^{'}) \cap N(S_2^{'}))$, then $S' \cup \{w\}$ is a connected dominating set of G. Hence, $\gamma_c(G) \leqslant \gamma_c(G+e) + 1$. So we assume that there is no vertex w in $V(G) - S'$ such that $w \in (N(S_1^{'}) \cap N(S_2^{'}))$. Since G is a connected graph, there exist two vertices w_1 and w_2 such that $w_1 \in N(S_1^{'})$, $w_2 \in N(S_2^{'})$ and w_1 and w_2 are adjacent. Hence, $S' \cup \{w_1, w_2\}$ is a connected dominating set of G. Hence, $\gamma_c(G) \leqslant \gamma_c(G+e) + 2$. □

Observation 1. *If $\gamma_c(G + vu) < \gamma_c(G)$ for a connected graph and an edge $vu \in E(\overline{G})$, then every $\gamma_c(G+vu)$-set S contains at least one of u and v. Moreover, if without loss of generality, $v \in S$ and $u \notin S$, then v is the only neighbor of u in S.*

Observation 2. *If $\gamma_c(G + vu) = \gamma_c(G) - 2$ for a connected graph and an edge $vu \in E(\overline{G})$, then every $\gamma_c(G+vu)$-set S contains both v and u.*

For any edge $vu \in E(\overline{G})$, when we write $[v, S] \longmapsto_c u$ it is understood that $S \cup \{v\}$ is a connected dominating set of $G - \{u\}$ and u is not dominated by S.

Since adding the edge between the two end leaves of a path P_n yields a cycle C_n and $\gamma_c(P_n) = \gamma_c(C_n)$, we have the following lemma.

Lemma 1. *Let G be a path or a cycle. Then*
(1) *P_{2s} is not γ_c-critical relative to $K_{s,s}$ for $s \geqslant 2$.*
(2) *C_{2s} is γ_c-critical relative to $K_{s,s}$.*

Now, we prove that a tree is not γ_c-critical relative to $K_{s,s}$.

Theorem 2. *Let T be a tree with $n \geqslant 4$ vertices. Then T is not γ_c-critical relative to $K_{s,s}$.*

Proof. Suppose T is a γ_c-critical tree relative to $K_{s,s}$. Let $L = \{v \in V(T)\colon d(v) = 1\}$ and $I = V(T) - L$.

Claim 1. No two support vertices are adjacent.

Suppose that u and v are support vertices of u' and v', respectively, and that u and v are adjacent. Consider $T' = T + u'v'$ and let S' be a connected dominating set of T'. If both u' and v' are in S', then $(S' - \{u', v'\}) \cup \{u, v\}$ is a connected dominating set of T, a contradiction since $|S'| < \gamma_c(T)$. Hence we may assume that $u' \in S'$ and $v' \notin S'$, implying that $u \in S'$ and u' is the only neighbor of v' in T' that belongs to S'. But then $(S' - \{u'\}) \cup \{v\}$ is a connected dominating set of T, again a contradiction.

Claim 2. No vertex is adjacent to two or more leaves.

Let a support vertex $v \in A$ be adjacent to two leaves v_1 and v_2. Since a tree is a connected graph and $|A| = |B|$, v has at least one neighbor u in B that is not a leaf. Let $u_1 \in N(u) - \{v\}$. By Claim 1, u_1 is not a leaf. Consider $T' = T + u_1v_1$ and let S' be a γ_c-set of T'. Since v and u_1 are cutvertices of T', it is obvious that $v, u_1 \in S'$. If $v_1 \in S'$, then $(S' - \{v_1\}) \cup \{u\}$ is a connected dominating set of T, contradicting the fact that T is γ_c-critical. If $v_1 \notin S'$, then $u \in S'$ and S' is a connected dominating set of T, contradicting the fact that T is γ_c-critical. Hence, each support vertex is adjacent to only one leaf.

Let L_A and L_B denote the set of leaves in T that belong to A and B, respectively.

Claim 3. $L_A \neq \emptyset$ and $L_B \neq \emptyset$.

If there is no leaf in A, then each vertex in A has degree at least 2 in T, and so T has at least $2s$ edges, which contradicts the fact that T is a tree of order $2s$. Hence, $L_A \neq \emptyset$. Similarly, $L_B \neq \emptyset$.

Let $u \in L(A)$ and $v \in L(B)$, and let $P\colon\ u = v_1, v_2, \ldots, v_t = v$ denote the longest path in T between u and v. By Claim 1, $t \geqslant 6$. Since T is a γ_c-critical tree relative to $K_{s,s}$, T is not isomorphic to P_{2s} by Lemma 1. Hence, there exists at least one vertex $v_i \in V(P)$ such that $d(v_i) \geqslant 3$. Since $d(v_2) = d(v_{t-1}) = 2$ by Claim 2, we have $3 \leqslant i \leqslant t-2$. Consider $T' = T + v_1v_t$ and let S' be a γ_c-set of T'. It follows that $v_i \in S'$. If $v_1, v_t \in S'$, then either $\{v_1, v_2, \ldots, v_i\} \subseteq S'$ or $\{v_i, v_{i+1}, \ldots, v_t\} \subseteq S'$. Without loss of generality, assume $\{v_1, v_2, \ldots, v_i\} \subseteq S'$. Then $\{v_{i+1}, v_{i+2}, \ldots, v_t\}$ has at most two adjacent vertices, say v_j, v_{j+1}, such that $v_j \notin S'$ and $v_{j+1} \notin S'$. Hence, $(S' - \{v_1, v_t\}) \cup \{v_j, v_{j+1}\}$ is a connected dominating set of T, contradicting the fact that T is γ_c-critical. If there exists exactly one vertex in $\{v_1, v_t\}$, say $v_1 \in S'$, then

$\{v_1, v_2, \ldots, v_i\} \subseteq S'$. It follows that $\{v_{i+1}, v_{i+2}, \ldots, v_{t-1}\}$ has at most one vertex v_{t-1} such that $v_{t-1} \notin S'$. Then $(S' - \{v_1\}) \cup \{v_{t-1}\}$ is a connected dominating set of T, contradicting the fact that T is γ_c-critical. $\square$

It is obvious that 1-γ_c-critical graph relative to $K_{s,s}$ is $K_{1,1}$. For 2-γ_c-critical graphs relative to $K_{s,s}$ it is $K_{s,s}$ for $s \geqslant 2$. For 3-γ_c-critical graphs relative to $K_{s,s}$, we have the following theorem.

Theorem 3. *Let $K_{s,s}$ have partite sets A and B. For $s \geqslant 3$, a graph G is 3-γ_c-critical relative to $K_{s,s}$ if and only if*

(1) *there exists a vertex v of A such that $d(v) = s$, and*

(2) *each vertex of B has degree $s - 1$.*

P r o o f. We first prove the necessity. Assume that G is 3-γ_c-critical relative to $K_{s,s}$ and let $S = \{x, y, z\}$ be a $\gamma_c(G)$-set. Since S induces a P_3, we may assume that $x \in A$ and $\{y, z\} \subset B$. So, $d(x) = s$.

Let v be a vertex of degree s in G. We may assume that $v \in A$, that is, $v \succ B$. Since $\gamma_c(G) = 3$, no vertex in B dominates A. Hence, $d(u) \leqslant s - 1$ for each $u \in B$. For each $u \in B$, let $\bar{u}$ denote a vertex in A that is not adjacent to u in G. Let S be a $\gamma_c(G + u\bar{u})$-set. Since G is 3-γ_c-critical relative to $K_{s,s}$, we have $|S| = 2$ and at least one of u and $\bar{u}$ is in S. If $u \notin S$, then $S = \{\bar{u}, x\}$ where $x \in B - \{u\}$. But then $d(x) = s$, a contradiction. If $u \in S$, then $S = \{u, x\}$ where $x \in A$. Hence, $d(u) = s - 1$ for all $u \in B$.

Conversely, let G be a graph with the two properties listed in the theorem. Clearly, no two adjacent vertices dominate G, and so $\gamma_c(G) \geqslant 3$. For each $u \in B$, let $\bar{u}$ denote a vertex in A that is not adjacent to u in G. Let $w \in N(\bar{u})$. Then $\{v, u, w\}$ is a connected dominating set of G. Hence, $\gamma_c(G) = 3$. For every edge $u\bar{u} \in E(\overline{G})$, $\{v, u\}$ is a connected dominating set of $G + u\bar{u}$. Hence, $\gamma_c(G + u\bar{u}) = 2$. Hence, the graph G is 3-γ_c-critical relative to $K_{s,s}$. $\square$

Let A and B be partite sets of $K_{s,s}$, and let η be the family of graphs G such that G is a connected spanning subgraph of $K_{s,s}$ for $s \geqslant 3$ and the following conditions hold:

(1) there exists a vertex in A with degree s,

(2) no pair of vertices in B dominates A, and

(3) for each nonadjacent pair $u \in A$ and $v \in B$, there exists a vertex $w \in B$ such that $\{v, w\} \succ A - \{u\}$.

Let τ be the family of spanning subgraphs G of $K_{s,s}$ such that the relative complement of G is the disjoint union of at least three nontrivial stars.

Theorem 4. *A connected graph G is 4-γ_c-critical relative to $K_{s,s}$ if and only if $G \in \eta \cup \tau$.*

Proof. Suppose $G \in \eta \cup \tau$. We first show that $\gamma_c(G) \geqslant 4$. Clearly, no two adjacent vertices dominate G, and so $\gamma_c(G) \geqslant 3$. Suppose that $S = \{x, y, z\}$ is a $\gamma_c(G)$-set. Since S induces a P_3, we may assume that $x \in A$ and $\{y, z\} \subset B$. Hence, $x \succ B$, and so $d(x) = s$, while $\{y, z\} \succ A$. But then $G \notin \eta \cup \tau$, a contradiction. Hence, $\gamma_c(G) \geqslant 4$.

Case 1. $G \in \eta$. Let $x \in A$ be a vertex of G such that $x \succ B$. Since $\gamma_c(G) \geqslant 4$, there exists a pair of nonadjacent vertices $u \in A$ and $v \in B$. Moreover, there is a vertex $w \in B$ such that $\{v, w\} \succ A - \{u\}$. Thus, $\{x, v, w, z\} \succ_c G$ where $z \in N(u)$ implying that $\gamma_c(G) = 4$. By condition (3), for each nonadjacent pair $u \in A$ and $v \in B$ there exists a vertex $w \in B$ such that $\{v, w\} \succ A - \{u\}$. Thus, $\{v, w, x\} \succ_c G + uv$, and so $\gamma_c(G + uv) \leqslant 3$. Then G is 4-γ_c-critical relative to $K_{s,s}$.

Case 2. $G \in \tau$. Each vertex of G is either the center of a star or an endvertex of a star in $\overline{G}$. If $\overline{G} = sK_2$, then it is clear that G is 4-γ_c-critical relative to $K_{s,s}$. Hence we may assume that there is a vertex $u \in A$ that is the center of a star, say S_1, in $\overline{G}$ of order at least 3. Since $|A| = |B|$, there is a vertex $v \in B$ that is the center of a star, S_2, in $\overline{G}$ of order at least 3. Let u_1 (v_1) be adjacent to u (v, respectively) in $\overline{G}$. Let S_3 be another star in $\overline{G}$ distinct from S_1 and S_2. Let $x, y \in V(S_3)$ and $x \in A$, $y \in B$. Then $\{x, y, u_1, v_1\}$ is a connected dominating set of G. Hence, $\gamma_c(G) = 4$. For an arbitrary edge $uv \in \overline{G}$, assume $u \in A$ and $v \in B$. Suppose u is the center and v is the endvertex of the same star in $\overline{G}$. Then $\{u, u', v\} \succ_c G + uv$ for any vertex $u' \in A - \{u\}$, and so $\gamma_c(G + uv) \leqslant 3$. Then G is 4-γ_c-critical relative to $K_{s,s}$.

Conversely, we consider two cases.

Claim 1. If G has a vertex of degree s, then $G \in \eta$.

Without loss of generality, we may assume that $z \in A$ has degree s. Since $\gamma_c(G) = 4$, it follows that no vertex in B has degree s, and no pair of vertices in B dominate A. Hence conditions (1) and (2) hold. Since G is connected, every vertex in A has a neighbor in B implying that no vertex in B can have degree $s-1$. Hence, $d(v) \leqslant s-2$ for each $v \in B$.

Let $u \in A$ be a vertex not adjacent to $v \in B$. Since $d(v) \leqslant s - 2$, it is impossible that $\{v, u\} \succ_c G + vu$. If there exists a vertex x such that $\{v, u, x\} \succ_c G + vu$, then $x \in B$. So $\{x, v\} \succ A$ and $\{x, v, z\} \succ_c G$, which is a contradiction. Hence there exist two vertices x and y such that $\{v, x, y\} \succ_c G + vu$ or $\{u, x, y\} \succ_c G + vu$.

If $\{u, x, y\} \succ_c G + vu$, then, since each vertex in B has degree at most $s - 2$, both x and y must belong to B. But then $\{x, y, z\} \succ_c G$, a contradiction. Hence,

$\{v,x,y\} \succ_c G+vu$. Then, we may assume that $x \in B$ and $y \in A$. Thus, $\{v,x\} \succ A-\{u\}$, and condition (3) holds. Hence, $G \in \eta$.

Claim 2. If G has no vertex of degree s, then $G \in \tau$.

Let $u \in A$ and $v \in B$ be two nonadjacent vertices in G. We first prove that at least one of u and v has degree $s-1$ in G. Suppose $d(u) \leqslant s-2$. Hence, $\{u,v\} \not\succ_c G+uv$. If there exists a vertex $w \in A$ such that $\{u,v,w\} \succ_c G+uv$, then $d(v)=s-1$. If there exist two vertices w, z distinct from u and v such that $\{u,w,z\} \succ_c G+uv$, then there exists exactly one vertex in $\{w,z\}$ that belongs to A. Without loss of generality, assume $w \in A$ and $z \in B$. Then $d(z)=s$, which is a contradiction. If there exist two vertices w, z distinct from u and v such that $\{v,w,z\} \succ_c G+uv$, then $w,z \in A$. Hence $d(v)=s-1$.

It follows from the above fact that at least one of u and v is a leaf in $\overline{G}$. This is true for every pair of nonadjacent vertices with one vertex in A and the other in B. Hence, since each vertex of $\overline{G}$ has degree at least 1, $\overline{G}$ is the disjoint union of nontrivial stars. Moreover, since G is a connected subgraph of $K_{s,s}$, $\overline{G}$ is the disjoint union of at least three nontrivial stars. Thus, $G \in \tau$.

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