Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 2, 273–286

Persistent URL: http://dml.cz/dmlcz/128066

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Czechoslovak Mathematical Journal, 56 (131) (2006), 273–286

EXISTENCE AND ASYMPTOTIC STABILITY FOR VISCOELASTIC PROBLEMS WITH NONLOCAL BOUNDARY DISSIPATION

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(Received June 6, 2003)

Abstract. We consider the damped semilinear viscoelastic wave equation

$$u'' - \Delta u + \int_0^t h(t-\tau)\operatorname{div}\{a\nabla u(\tau)\}\,\mathrm{d}\tau + g(u') = 0 \quad \text{in } \Omega\times(0,\infty)$$

with nonlocal boundary dissipation. The existence of global solutions is proved by means of the Faedo-Galerkin method and the uniform decay rate of the energy is obtained by following the perturbed energy method provided that the kernel of the memory decays exponentially.

Keywords: asymptotic stability, viscoelastic problems, boundary dissipation, wave equation

MSC 2000: 35L70, 35L15, 65M60

1. Introduction

In this paper we are concerned with the existence and uniform decay rates of the solutions for the damped semilinear viscoelastic wave equation of the following form:

$$u'' - \Delta u + \int_0^t h(t-\tau)\operatorname{div}\{a\nabla u(\tau)\}\,\mathrm{d}\tau + g(u') = 0 \quad \text{in } Q=\Omega\times(0,\infty), \tag{1.1}$$

$$u(x,0)=u_0(x), \quad u'(x,0)=u_1(x) \ \text{ on } \Omega, \tag{1.2}$$

$$u=0 \quad \text{on } \Sigma_1=\Gamma_1\times(0,\infty), \tag{1.3}$$

This work was supported by grant No. (R01-2002-000-00491-0) from the Korea Science and Engineering Foundation.

$$
(1.4) \qquad -\frac{\partial u}{\partial \nu} + \int_0^t h(t-\tau)(a\nabla u(\tau)) \cdot \nu \,\mathrm{d}\tau = |u|^\gamma u + M(\|u\|^2_{\Gamma_0})u' \quad \text{on } \Sigma_0 = \Gamma_0 \times (0,\infty),
$$

where $u'' = \partial^2 u/\partial t^2$, $u' = \partial u/\partial t$, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqslant 1)$ with sufficiently smooth boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset$ and Γ_0, Γ_1 have positive measures, M and g are functions of C^1-class, a is a positive funtion satisfying

$$
(1.5) \qquad a \in L^\infty(\Omega) \quad \text{and} \quad 0 < \|a\|_{L^\infty(\Omega)} \leqslant 1,
$$

$\|u\|^2_{\Gamma_0} = \int_{\Gamma_0} |u(x)|^2 d\Gamma$, $\Delta u = \sum\limits_{i=1}^{n} \partial^2 u/\partial x_i^2$ and ν denotes the unit outward normal vector to Γ. We assume that

$$
(1.6) \qquad 0 < \gamma \leqslant \frac{1}{n-2} \quad \text{if } n \geqslant 3 \quad \text{and} \quad \gamma > 0 \quad \text{if } n = 1, 2.
$$

This problem has its origin in the mathematical description of viscoelastic materials. From the physical point of view, the problems (1.1)–(1.4) describe the position $u(x,t)$ of the material particle x at time t, which is clamped in the internal portion Γ_1 of its boundary and its external portion Γ_0 is supported by elastic bearings with nonlinear boundary responses, represented by the function $|u|^\gamma u$ and is also subject to nonlinear and nonlocal dissipation represented by the function $M(\|u(t)\|^2_{\Gamma_0})u'$, which takes into consideration the distributional average on the whole portion Γ_0 of the boundary Γ. Recently, many authors have investigated viscoelastic problems with memory terms in the domain [3], [4], [5], [10]. M. L. Santos [10] and M. M. Cavalcanti et al. [4] proved the existence and uniform decay of solutions of problems (1.1)–(1.2) with $a(x) = 1$ in Ω, $g \equiv 0$ and Dirichlet boundary conditions. M. M. Cavalcanti et al. [5] studied the existence and uniform decay rates of solutions of problems (1.1)–(1.3) with $a(x) = 1$ in Ω, $g \equiv 0$ and nonlinear boundary damping. On the other hand, boundary stabilization has received considerable attention in the literature and among the numerous papers in this direction, we can cite the results of [6], [9], [11]. Related to viscoelastic problems with memory terms acting on the boundary, we can cite the papers of J. J. Bae et al. [1] and J. Y. Park and J. J. Bae [8]. In this paper we prove the existence of solutions $u = u(x,t)$ of the problems (1.1)–(1.4) when the positive function a satifies (1.5). Moreover the uniform decay of the energy

$$
E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}
$$

is proved.

It is important to observe that as far as we know the system (1.1)–(1.2) with boundary dissipation has never been considered in the literature. To obtain the existence of solutions we make use of the Faedo-Galerkin approximation and also to show the uniform decay we use the perturbed energy method by assuming that the kernel h in the memory term decays exponentially. Our paper is organized as follows: In Section 2, we give some notations, assumptions and state the main results. In Section 3, we prove the existence of solution of the problems (1.1)–(1.4), and the uniform decay of energy is given in Section 4.

2. Assumptions and main result

Throughout this paper we define

$$V = \{u \in H^1(\Omega)\colon\ u = 0 \ \ on \ \ \Gamma_1\}, \quad (u, v) = \int_\Omega u(x)v(x)\,\mathrm{d}x,$$
$$(u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\,\mathrm{d}\Gamma, \quad \|u\|^p_{p,\Gamma_0} = \int_{\Gamma_0} |u(x)|^p\,\mathrm{d}\Gamma,$$

and

$$\|u\|_\infty = \|u\|_{L^\infty(\Omega)}.$$

To simplify the notations we denote $\|u\|_{L^2(\Omega)}$ and $\|u\|_{2,\Gamma_0}$ by $\|u\|$ and $\|u\|_{\Gamma_0}$, respectively. Moreover, we denote $\|h\|_{L^1(0,\infty)}$ and $\|h\|_{L^\infty(0,\infty)}$ by $\|h\|_{L^1}$ and $\|h\|_{L^\infty}$, respectively, for a real-valued function $h\colon\ \mathbb{R}_+ \to \mathbb{R}_+$.

In the sequel we state the general hypotheses.

(A1) $h\colon\ \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^2 function satisfying

$$1 - \int_0^\infty h(s)\,\mathrm{d}s = l > 0$$

and there exist positive constants ξ_1, ξ_2 and ξ_3 satisfying

$$-\xi_1 h(t) \leqslant h'(t) \leqslant -\xi_2 h(t), \ \ \forall\, t \geqslant 0,$$
$$0 \leqslant h''(t) \leqslant \xi_3 h(t), \ \ \forall\, t \geqslant 0.$$

(A2) The function g is a nondecreasing continuous function of C^1-class and $g(0) = 0$. Furthermore, there exist positive constants β and δ such that

$$g(s)s \geqslant \beta|s|^2, \ \ \forall\, s \in \mathbb{R},$$
$$|g(s)| \leqslant \delta|s|, \ \ \forall\, s \in \mathbb{R}.$$

(A3) The function $M\colon [0,\infty) \to [m_0,\infty)$ is a continuously differentiable function satisfying $M(s) \geqslant m_0 > 0$.

(A4) Let us consider the following assumptions on the initial data:

$$u_0 \in V \cap H^2(\Omega), \quad u_1 \in V,$$
$$\frac{\partial u_0}{\partial \nu} + M(\|u_0\|^2_{\Gamma_0})u_1 + |u_0|^\gamma u_0 = 0 \quad \text{on } \Gamma_0.$$

The variational formulation associated with the problems (1.1)–(1.4) is given by

$$(2.1) \qquad (u'',w) + (\nabla u, \nabla w) - \int_0^t h(t-\tau)(a\nabla u(\tau), \nabla w)\,\mathrm{d}\tau + (|u|^\gamma u, w)_{\Gamma_0} + M(\|u\|^2_{\Gamma_0})(u',w)_{\Gamma_0} + (g(u'),w) = 0, \quad \forall\, w \in V.$$

Now we are in a position to state our main results.

Theorem 2.1. *Under the assumptions* (A1)–(A4) *the problems* (1.1)–(1.4) *have a unique solution u such that*

$$u \in L^\infty(0,\infty;V), \quad u' \in L^\infty(0,\infty;V), \quad u'' \in L^\infty(0,\infty;L^2(\Omega)).$$

Moreover, if $\|h\|_{L^1}$ is sufficiently small, then the energy $E(t)$ has the following decay rates

$$E(t) \leqslant C_3 E(0) \exp\Bigl(-\frac{2}{3}C_2 \varepsilon t\Bigr), \quad \forall\, t \geqslant 0 \text{ and } \forall\, \varepsilon \in (0,\varepsilon_0],$$

where C_2, C_3 and ε_0 are positive constants.

3. Proof of Theorem 2.1

In this section we are going to show the existence of solution of the problems (1.1)–(1.4) using the Faedo-Galerkin approximation. For this we choose a basis $\{w_j\}_{j\geqslant 1}$ in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and let V_m the subspace of $V \cap H^2(\Omega)$ generated by the first m vectors. Next we define

$$u_m(t) = \sum_{j=1}^m f_{jm}(t) w_j,$$

where $u_m(t)$ is the solution of the following Cauchy problem:

$$(3.1) \qquad (u''_m, w) + (\nabla u_m, \nabla w) - \int_0^t h(t-\tau)(a\nabla u_m(\tau), \nabla w)\,\mathrm{d}\tau + (|u_m(t)|^\gamma u_m(t), w)_{\Gamma_0} + M(\|u_m(t)\|^2_{\Gamma_0})(u'_m(t), w)_{\Gamma_0} + (g(u'_m), w) = 0,$$

for all $w \in V$ with the initial conditions

$$
\text{(3.2)} \qquad \begin{cases} u_m(0) = u_{0m} = \sum\limits_{j=1}^{m} (u_0, w_j) w_j \to u_0 & \text{in } V \cap H^2(\Omega), \\ u_m'(0) = u_{1m} = \sum\limits_{j=1}^{m} (u_1, w_j) w_j \to u_1 & \text{in } V. \end{cases}
$$

Note that we can solve the system (3.1)–(3.2). In fact the problems (3.1)–(3.2) have a unique solution on some interval $[0, T_m)$. The extension of the solution to the whole interval $[0, \infty)$ is a consequence of the first estimate which we are going to prove below.

A priori estimate I

Replacing w by $u_m'(t)$ in (3.1) and using the assumption (A3), we have

$$
\begin{aligned}
\text{(3.3)} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2}\Big\} + m_0\|u_m'(t)\|_{\Gamma_0}^2 \\
+ (g(u_m'(t)), u_m'(t)) \\
\leqslant \int_0^t h(t-\tau)(a\nabla u_m(\tau), \nabla u_m'(t))\,\mathrm{d}\tau.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\int_0^t h(t-\tau)(a\nabla u_m(\tau), \nabla u_m'(t))\,\mathrm{d}\tau \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t h(t-\tau)(a\nabla u_m(\tau), \nabla u_m(t))\,\mathrm{d}\tau - h(0)(a\nabla u_m(t), \nabla u_m(t)) \\
&\quad\quad - \int_0^t h'(t-\tau)(a\nabla u_m(\tau), \nabla u_m(t))\,\mathrm{d}\tau,
\end{aligned}
$$

integrating (3.3) over $(0,t)$ and taking the assumption (A2) and (3.2) into account, we get

$$
\begin{aligned}
\text{(3.4)} \quad \frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + m_0\int_0^t \|u_m'(s)\|_{\Gamma_0}^2\,\mathrm{d}s \\
\leqslant k_1 + \int_0^t h(t-\tau)(a\nabla u_m(\tau), \nabla u_m(t))\,\mathrm{d}\tau \\
- h(0)\int_0^t (a\nabla u_m(s), \nabla u_m(s))\,\mathrm{d}s \\
- \int_0^t\int_0^s h'(s-\tau)(a\nabla u_m(\tau), \nabla u_m(s))\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}
$$

Using the assumption (A1) and the inequality $ab \leqslant (1/4\eta)a^2 + \eta b^2$, where η is an arbitrary positive number, we deduce that from (3.4)

$$\begin{aligned}\frac{1}{2}\|u_m'(t)\|^2 &+ \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + m_0\int_0^t \|u_m'(s)\|_{\Gamma_0}^2\,\mathrm{d}s\\ &\leqslant k_1 + \eta\|\nabla u_m(t)\|^2 + \frac{1}{4\eta}\|a\|_\infty^2\|h\|_{L^1}\|h\|_{L^\infty}\int_0^t \|\nabla u_m(\tau)\|^2\,\mathrm{d}\tau\\ &\quad + \frac{1}{2}\int_0^t \|\nabla u_m(s)\|^2\,\mathrm{d}s + \frac{1}{2}\xi_1^2\|a\|_\infty^2\|h\|_{L^1}^2\int_0^t \|\nabla u_m(\tau)\|^2\,\mathrm{d}\tau.\end{aligned}$$

Choosing $\eta > 0$ sufficiently small and employing Gronwall's lemma, we obtain the first estimate

$$\frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\|\nabla u_m(t)\|^2 + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + m_0\int_0^t \|u_m'(s)\|_{\Gamma_0}^2\,\mathrm{d}s \leqslant L_1,$$

where $L_1 > 0$ is a constant independent of $m \in \mathbb{N}$ and $t \in [0,T]$.

A priori estimate II

First of all, we are going to estimate $u_m''(0)$ in the L^2-norm. Replacing w by $u_m''(0)$ in (3.1) and considering $t = 0$, we arrive at

$$\begin{aligned}\|u_m''(0)\|^2 &- (\Delta u_m(0), u_m''(0)) + \Big(\frac{\partial u_m(0)}{\partial \nu}, u_m''(0)\Big)_{\Gamma_0} + (|u_m(0)|^\gamma u_m(0), u_m''(0))_{\Gamma_0}\\ &+ M(\|u_m(0)\|_{\Gamma_0}^2)(u_m'(0), u_m''(0))_{\Gamma_0} + (g(u_m'(0)), u_m''(0)) = 0.\end{aligned}$$

This equation and the assumptions (A2) and (A4) yield that

$$\|u_m''(0)\|^2 \leqslant L_2, \tag{3.6}$$

where $L_2 > 0$ is a constant independent of $m \in \mathbb{N}$ and $t \in [0,T]$.

Now, differentiating (3.1) with respect to t, substituting w by $u_m''(t)$ and using the fact that $M(s) \geqslant m_0 > 0$, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|u_m''(t)\|^2 &+ \|\nabla u_m'(t)\|^2\} + m_0\|u_m''(t)\|_{\Gamma_0}^2 + (g'(u_m'(t))u_m''(t), u_m''(t))\\ \leqslant &- (\gamma+1)(|u_m(t)|^\gamma u_m'(t), u_m''(t))_{\Gamma_0}\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t h'(t-\tau)(a\nabla u_m(\tau), \nabla u_m'(t))\,\mathrm{d}\tau\\ &- 2M'(\|u_m(t)\|_{\Gamma_0}^2)(u_m(t), u_m'(t))_{\Gamma_0}(u_m'(t), u_m''(t))_{\Gamma_0} - h(0)\|\sqrt{a}\nabla u_m'(t)\|^2\\ &+ h(0)\frac{\mathrm{d}}{\mathrm{d}t}(a\nabla u_m(t), \nabla u_m'(t)) - h'(0)(a\nabla u_m(t), \nabla u_m'(t))\\ &- \int_0^t h''(t-\tau)(a\nabla u_m(\tau), \nabla u_m'(t))\,\mathrm{d}\tau.\end{aligned} \tag{3.7}$$

Since $\frac{1}{2}\gamma(\gamma+1)^{-1}+\frac{1}{2}(\gamma+1)^{-1}+\frac{1}{2}=1$ and $H^1(\Omega)\hookrightarrow L^{2(\gamma+1)}(\Omega)$, Hölder's inequality and Young's inequality [2] give us

$$\tag{3.8} \begin{aligned} &|(\gamma+1)(|u_m(t)|^\gamma u_m'(t),u_m''(t))_{\Gamma_0}| \\ &\quad\leqslant (\gamma+1)\|u_m(t)\|^\gamma_{2(\gamma+1),\Gamma_0}\|u_m'(t)\|_{2(\gamma+1),\Gamma_0}\|u_m''(t)\|_{\Gamma_0} \\ &\quad\leqslant \theta_1(\eta)\|\nabla u_m'(t)\|^2+\eta\|u''(t)\|^2_{\Gamma_0}, \end{aligned}$$

where we have used the result of the first estimate (3.5) in the last inequality. We note that M is a C^1-class on $[0,\infty)$ and $H^1(\Omega)\hookrightarrow L^2(\Gamma)$. Then, making use of the result of (3.5) and Young's inequality, we get

$$\tag{3.9} \begin{aligned} &-2M'(\|u_m(t)\|^2_{\Gamma_0})(u_m(t),u_m'(t))_{\Gamma_0}(u_m'(t),u_m''(t))_{\Gamma_0} \\ &\qquad\leqslant C\|u_m(t)\|_{\Gamma_0}\|u_m'(t)\|_{\Gamma_0}\|u_m'(t)\|_{\Gamma_0}\|u_m''(t)\|_{\Gamma_0} \\ &\qquad\leqslant C\|\nabla u_m(t)\|\,\|\nabla u_m'(t)\|\,\|u_m'(t)\|_{\Gamma_0}\|u_m''(t)\|_{\Gamma_0} \\ &\qquad\leqslant \theta_2(\eta)\|\nabla u_m'(t)\|^2\|u_m'(t)\|^2_{\Gamma_0}+\eta\|u_m''(t)\|^2_{\Gamma_0}, \end{aligned}$$

where C is a generic positive constant independent of m and t. Integrating (3.7) over $(0,t)$, taking (3.2), (3.8) and (3.9) into account and using the fact that $(g'(u_m'(t))u_m''(t),u_m''(t))\geqslant 0$, we get

$$\begin{aligned} &\frac{1}{2}\|u_m''(t)\|^2+\frac{1}{2}\|\nabla u_m'(t)\|^2+m_0\int_0^t\|u_m''(s)\|^2_{\Gamma_0}\,\mathrm{d}s \\ &\quad\leqslant k_2+2\eta\int_0^t\|u_m''(s)\|^2_{\Gamma_0}\,\mathrm{d}s+\theta_1(\eta)\int_0^t\|\nabla u_m'(s)\|^2\,\mathrm{d}s \\ &\qquad+\theta_2(\eta)\int_0^t\|\nabla u_m'(s)\|^2\|u_m'(s)\|^2_{\Gamma_0}\,\mathrm{d}s-h(0)\int_0^t\|\sqrt{a}\nabla u_m'(s)\|^2\,\mathrm{d}s \\ &\qquad+h(0)(a\nabla u_m(t),\nabla u_m'(t))-h'(0)\int_0^t(a\nabla u_m(s),\nabla u_m'(s))\,\mathrm{d}s \\ &\qquad-\int_0^t\int_0^s h''(s-\tau)(a\nabla u_m(\tau),\nabla u_m'(s))\,\mathrm{d}\tau\,\mathrm{d}s \\ &\qquad+\int_0^t h'(t-\tau)(a\nabla u_m(\tau),\nabla u_m'(t))\,\mathrm{d}\tau \\ &\quad\leqslant k_2+2\eta\int_0^t\|u_m''(s)\|^2_{\Gamma_0}\,\mathrm{d}s+\theta_1(\eta)\int_0^t\|\nabla u_m'(s)\|^2\,\mathrm{d}s \\ &\qquad+k_3\int_0^t\|\nabla u_m'(\tau)\|^2\,\mathrm{d}\tau+\theta_2(\eta)\int_0^t\|\nabla u_m'(s)\|^2\|u_m'(s)\|^2_{\Gamma_0}\,\mathrm{d}s \\ &\qquad-\frac{h(0)}{2}\int_0^t\|\sqrt{a}\nabla u_m'(s)\|^2\,\mathrm{d}s+2\eta\|\nabla u_m'(t)\|^2+\frac{h(0)^2}{4\eta}\|a\|^2_\infty\|\nabla u_m(t)\|^2 \\ &\qquad-\frac{h'(0)}{2}\|\sqrt{a}\nabla u_m'(t)\|^2+h'(0)\|\sqrt{a}\nabla u_m'(0)\|^2, \end{aligned}$$

where $k_3 = \frac{1}{2}\xi_3^2 h(0)^{-1}\|h\|_{L^1}^2\|a\|_\infty + \frac{1}{4}\eta^{-1}\|h\|_{L^\infty}\|h\|_{L^1}\|a\|_\infty^2$. Therefore, choosing $\eta > 0$ small enough, considering the first estimate (3.5) and employing Gronwall's lemma, we obtain the second estimate

$$\|u_m''(t)\|^2 + \|\nabla u_m'(t)\|^2 + \int_0^t \|u_m''(s)\|_{\Gamma_0}^2 \, \mathrm{d}s \leqslant L_3, \tag{3.10}$$

whrere L_3 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0, T]$.

The above estimates (3.5) and (3.10) imply that

$$(u_m) \text{ is bounded in } L^\infty(0, T; V), \tag{3.11}$$

$$(u_m') \text{ is bounded in } L^\infty(0, T; V), \tag{3.12}$$

$$(u_m'') \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^2(\Gamma_0)). \tag{3.13}$$

These results are sufficient to pass to the limit in the linear terms of problem (3.1). Next we are going to consider the nonlinear ones.

Analysis of the nonlinear terms

From the above estimates (3.5) and (3.10), we have that

$$\begin{cases} (u_m) \text{ is bounded in } L^2(0, T; H^{1/2}(\Gamma_0)), \\ (u_m') \text{ is bounded in } L^2(0, T; H^{1/2}(\Gamma_0)), \\ (u_m'') \text{ is bounded in } L^2(0, T; L^2(\Gamma_0)). \end{cases} \tag{3.14}$$

From (3.14), taking into consideration that M is continuous and the imbedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact, and using Aubin's compactness theorem [7], we can extract subsequences (in the sequel we denote subsequences by the same symbols as the original sequences) such that

$$u_m \to u \text{ a.e. on } \Sigma_0 \quad \text{and} \quad u_m' \to u' \text{ a.e. on } \Sigma_0 \tag{3.15}$$

and therefore

$$|u_m|^\gamma u_m \to |u|^\gamma u \quad \text{a.e. on } \Sigma_0, \tag{3.16}$$

$$M(\|u_m\|_{\Gamma_0}^2)u_m' \to M(\|u\|_{\Gamma_0}^2)u' \quad \text{a.e. on } \Sigma_0. \tag{3.17}$$

Similarly, from (3.12) and (3.13), we have

$$g(u_m') \to g(u') \quad \text{a.e. on } Q. \tag{3.18}$$

On the other hand, from the first and the second estimate we obtain

$$(|u_m|^\gamma u_m) \text{ and } (M(\|u_m\|^2_{\Gamma_0})u'_m) \text{ are bounded in } L^2(\Sigma_0), \tag{3.19}$$

$$(g(u'_m)) \text{ is bounded in } L^2(Q). \tag{3.20}$$

Combining (3.16)–(3.20), we deduce that

$$\begin{gathered}
|u_m|^\gamma u_m \to |u|^\gamma u \quad \text{weakly in } L^2(\Sigma_0),\\
M(\|u_m\|^2_{\Gamma_0})u'_m \to M(\|u\|^2_{\Gamma_0})u' \quad \text{weakly in } L^2(\Sigma_0),\\
g(u'_m) \to g(u') \quad \text{weakly in} L^2(Q).
\end{gathered}$$

The last convergences are sufficient to pass to the limit in the nonlinear terms of problem (3.1). The uniqueness is obtained in the usual way, so we omit the proof here.

4. Uniform decay

We define the energy $E(t)$ of the problems (1.1)–(1.4) by

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}. \tag{4.1}$$

Then the derivative of the energy is given by

$$\begin{aligned}
E'(t) = &-(g(u'(t)), u'(t)) + \int_0^t h(t-\tau)(a\nabla u(\tau), \nabla u'(t))\,\mathrm{d}\tau\\
&- M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0}.
\end{aligned} \tag{4.2}$$

A direct computation shows that

$$\begin{aligned}
&\int_0^t h(t-\tau)(a\nabla u(\tau), \nabla u'(t))\,\mathrm{d}\tau\\
&\quad = -\frac{1}{2}h(t)\int_\Omega a(x)|\nabla u(x,t)|^2\,\mathrm{d}x + \frac{1}{2}(h' \,\Box\, \nabla u)(t) - \frac{1}{2}(h \,\Box\, \nabla u)'(t)\\
&\quad\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega a(x)\Big(\int_0^t h(s)\,\mathrm{d}s\Big)|\nabla u(x,t)|^2\,\mathrm{d}x
\end{aligned} \tag{4.3}$$

where

$$(h \,\Box\, y)(t) = \int_0^t\int_\Omega h(t-\tau)a(x)|y(x,t) - y(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

Define the modified energy by

$$
(4.4)\qquad e(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \frac{1}{2}(h\,\Box\,\nabla u)(t) - \frac{1}{2}\int_\Omega a(x)\Big(\int_0^t h(s)\,\mathrm{d}s\Big)|\nabla u(x,t)|^2\,\mathrm{d}x.
$$

Then, from (4.2) and (4.3) we have

$$
\begin{aligned}
(4.5)\qquad e'(t) = {}& -(g(u'(t)),u'(t)) - \frac{1}{2}\int_\Omega a(x)|\nabla u(x,t)|^2\,\mathrm{d}x \\
&+ \frac{1}{2}(h'\,\Box\,\nabla u)(t) - M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0} \\
\leqslant {}& -(g(u'(t)),u'(t) + \frac{1}{2}(h'\,\Box\,\nabla u)(t) - M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0}.
\end{aligned}
$$

From the assumptions (A1)–(A3) and the equations (4.4)–(4.5), we deduce that $e(t) \geqslant 0$ and $e'(t) \leqslant 0$.

On the other hand, we note from the assumption (A1) that

$$
\begin{aligned}
(4.6)\qquad E(t) &= \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&\leqslant \frac{l^{-1}}{2}\|u'(t)\|^2 + \frac{l^{-1}}{2}\Big(1-\int_0^t h(s)\,\mathrm{d}s\Big)\|\nabla u(t)\|^2 \\
&\quad + \frac{l^{-1}}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \frac{l^{-1}}{2}(h\,\Box\,\nabla u)(t) \\
&\leqslant \frac{l^{-1}}{2}\|u'(t)\|^2 + \frac{l^{-1}}{2}\int_\Omega\Big(1-a(x)\int_0^t h(s)\,\mathrm{d}s\Big)|\nabla u(x,t)|^2\,\mathrm{d}x \\
&\quad + \frac{l^{-1}}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \frac{l^{-1}}{2}(h\,\Box\,\nabla u)(t) = l^{-1}e(t),
\end{aligned}
$$

where have we have used (1.5). Therefore it is enough to obtain the desired exponential decay for the modified energy $e(t)$ which will be done below. For every $\varepsilon > 0$ let us define the perturbed modified energy by

$$
e_\varepsilon(t) = e(t) + \varepsilon\psi(t),
$$

where $\psi(t) = (u'(t),u(t))$.

Proposition 4.1. *There exists $C_1 > 0$ such that for each $\varepsilon > 0$*

$$|e_\varepsilon(t) - e(t)| \leqslant \varepsilon C_1 e(t), \quad \forall\, t \geqslant 0.$$

Proof. Applying Cauchy-Schwarz's inequality and the inequality (4.6), we have

$$|\psi(t)| \leqslant \|u'(t)\|\,\|u(t)\| \leqslant \lambda\|u'(t)\|\,\|\nabla u(t)\| \leqslant \frac{\lambda}{2}(\|u'(t)\|^2 + \|\nabla u(t)\|^2) \leqslant \lambda l^{-1} e(t),$$

where λ is a positive constant satisfying $\|y\| \leqslant \lambda\|\nabla y\|$ for all $y \in V$. By taking $C_1 = \lambda l^{-1}$, we have

$$|e_\varepsilon(t) - e(t)| = \varepsilon|\psi(t)| \leqslant \varepsilon C_1 e(t).$$

□

Proposition 4.2. *There exist $C_2 > 0$ and ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$*

$$e'_\varepsilon(t) \leqslant -\varepsilon C_2 e(t).$$

Proof. Using the problems (1.1)–(1.4), we have

$$\begin{aligned}(4.7)\quad \psi'(t) &= \|u'(t)\|^2 - \|\nabla u(t)\|^2 + \int_0^t h(t-\tau)(a\nabla u(\tau), \nabla u(t))\,\mathrm{d}\tau \\ &\quad - \|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} - M(\|u(t)\|^2_{\Gamma_0})(u'(t), u(t))_{\Gamma_0} - (g(u'(t)), u(t)) \\ &= -e(t) + \frac{3}{2}\|u'(t)\|^2 - \frac{1}{2}\|\nabla u(t)\|^2 + \int_0^t h(t-\tau)(a\nabla u(\tau), \nabla u(t))\,\mathrm{d}\tau \\ &\quad - \frac{\gamma+1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} - M(\|u(t)\|^2_{\Gamma_0})(u'(t), u(t))_{\Gamma_0} + \frac{1}{2}(h \,\Box\, \nabla u)(t) \\ &\quad - \frac{1}{2}\int_\Omega a(x)\Big(\int_0^t h(s)\,\mathrm{d}s\Big)|\nabla u(x,t)|^2\,\mathrm{d}x - (g(u'(t)), u(t)).\end{aligned}$$

From (A2), we note that

$$(4.8)\qquad (g(u'(t)), u(t)) \leqslant \delta\lambda\|u'(t)\|\,\|\nabla u(t)\| \leqslant \eta\|\nabla u(t)\|^2 + \frac{\delta^2\lambda^2}{4\eta}\|u'(t)\|^2.$$

The condition (A1), (1.5), Young's inequality and the inequality (4.6) imply

$$\begin{aligned}(4.9)\quad &\Bigl|\int_0^t h(t-\tau)(a\nabla u(\tau),\nabla u(t))\,\mathrm{d}\tau\Bigr|\\ &\leqslant \eta\int_\Omega a(x)|\nabla u(x,t)|^2\,\mathrm{d}x+\Bigl(\int_0^t h(s)\,\mathrm{d}s\Bigr)\int_\Omega a(x)|\nabla u(x,t)|^2\,\mathrm{d}x\\ &\quad+\frac{1}{4\eta}(h\,\Box\,\nabla u)(t)\\ &\leqslant \eta\|\nabla u(t)\|^2+\Bigl(\int_0^t h(s)\,\mathrm{d}s\Bigr)\|\nabla u(t)\|^2+\frac{1}{4\eta}(h\,\Box\,\nabla u)(t)\\ &\leqslant 2(\eta+\|h\|_{L^1})l^{-1}e(t)+\frac{1}{4\eta}(h\,\Box\,\nabla u)(t).\end{aligned}$$

Also, considering the first estimate (3.5), by using the assumption (A3) and Young's inequality, we get

$$(4.10)\qquad |M(\|u(t)\|^2_{\Gamma_0})(u'(t),u(t))_{\Gamma_0}|\leqslant \eta\|\nabla u(t)\|^2+\theta_3(\eta)M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0}.$$

Applying the inequalities (4.8)–(4.10) to (4.7), we have

$$\begin{aligned}(4.11)\quad \psi'(t)\leqslant\ &-(1-2(\eta+\|h\|_{L^1})l^{-1})e(t)+\theta_4(\eta)\|u'(t)\|^2\\ &+\Bigl(2\eta-\frac{1}{2}\Bigr)\|\nabla u(t)\|^2+\frac{1+2\eta}{4\eta}(h\,\Box\,\nabla u)(t)-\frac{\gamma+1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}\\ &+\theta_3(\eta)M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0}\\ &-\frac{1}{2}\int_\Omega a(x)\Bigl(\int_0^t h(s)\,\mathrm{d}s\Bigr)|\nabla u(x,t)|^2\,\mathrm{d}x\\ \leqslant\ &-(1-2(\eta+\|h\|_{L^1})l^{-1})e(t)+\theta_4(\eta)\|u'(t)\|^2\\ &+\Bigl(2\eta-\frac{1}{2}\Bigr)\|\nabla u(t)\|^2+\frac{1+2\eta}{4\eta}(h\,\Box\,\nabla u)(t)-\frac{\gamma+1}{\gamma+2}\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}\\ &+\theta_3(\eta)M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0},\end{aligned}$$

where $\theta_4(\eta)=\frac{3}{2}+\frac{1}{4}\delta^2\lambda^2\eta^{-1}$. By the assumptions (A1) and (A2), it follows from (4.5) and (4.11) that

$$\begin{aligned}(4.12)\ e'_\varepsilon(t)&=e'(t)+\varepsilon\psi'(t)\\ &\leqslant -\varepsilon C_2e(t)-(\beta-\varepsilon\theta_4(\eta))\|u'(t)\|^2+\varepsilon\Bigl(2\eta-\frac{1}{2}\Bigr)\|\nabla u(t)\|^2\\ &\quad-\Bigl(\frac{\xi_2}{2}-\varepsilon\frac{1+2\eta}{4\eta}\Bigr)(h\,\Box\,\nabla u)(t)-(1-\varepsilon\theta_3(\eta))M(\|u(t)\|^2_{\Gamma_0})\|u'(t)\|^2_{\Gamma_0},\end{aligned}$$

where $C_2=1-2(\eta+\|h\|_{L^1})l^{-1}$. Choose η and $\|h\|_{L^1}$ sufficiently small so that

$$C_2>0\quad\text{and}\quad 2\eta-\frac{1}{2}<0$$

and define

$$\varepsilon_1 = \min\Big\{\frac{\beta}{\theta_4(\eta)}, \frac{2\eta\xi_2}{1+2\eta}, \frac{1}{\theta_3(\eta)}\Big\}.$$

Then for each $\varepsilon \in (0, \varepsilon_1]$, we have

$$e_\varepsilon'(t) \leqslant -\varepsilon C_2 e(t).$$

□

Continuing the proof of Theorem 2.1

Let $\varepsilon_0 = \min\big\{\frac{1}{2C_1}, \varepsilon_1\big\}$ and let us consider $\varepsilon \in (0, \varepsilon_0]$. As we have $\varepsilon < (2C_1)^{-1}$, we conclude from Proposition 4.1 that

$$(1-\varepsilon C_1)e(t) < e_\varepsilon(t) < (1+\varepsilon C_1)e(t)$$

and so

$$\frac{1}{2}e(t) < e_\varepsilon(t) < \frac{3}{2}e(t). \tag{4.13}$$

Thus we have

$$e_\varepsilon'(t) \leqslant -\frac{2}{3}C_2\varepsilon e_\varepsilon(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[e_\varepsilon(t)\exp\Big(\frac{2}{3}C_2\varepsilon t\Big)\Big] \leqslant 0. \tag{4.14}$$

Integrating (4.14), the inequality (4.13) implies

$$e(t) \leqslant 3e(0)\exp\Big(-\frac{2}{3}C_2\varepsilon t\Big). \tag{4.15}$$

Hence, from (4.6), (4.15) and the fact that $e(0) = E(0)$, we have

$$E(t) \leqslant l^{-1}e(t) \leqslant 3E(0)l^{-1}\exp\Big(-\frac{2}{3}C_2\varepsilon t\Big).$$

This completes the proof of Theorem 2.1. □

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