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ON A HOMOGENEITY CONDITION FOR MV-ALGEBRAS

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Abstract. In this paper we deal with a homogeneity condition for an MV-algebra concerning a generalized cardinal property. As an application, we consider the homogeneity with respect to α-completeness, where α runs over the class of all infinite cardinals.

Keywords: MV-algebra, generalized cardinal property, projectability, orthogonal completeness, direct product

MSC 2000: 06D35

1. Introduction

We denote by $\mathscr{M}$ the class of all MV-algebras. Further, let K be the class consisting of all cardinals and of the symbol ∞. For each cardinal α we put $\alpha < \infty$. A generalized cardinal property on the class $\mathscr{M}$ is a rule that assigns to each element $\mathscr{A} \in \mathscr{M}$ an element $f(\mathscr{A})$ of K such that, whenever $\mathscr{A}$ and $\mathscr{B}$ are isomorphic MV-algebras, then $f(\mathscr{A}) = f(\mathscr{B})$.

The underlying set of an MV-algebra $\mathscr{A}$ will be denoted by A. Let $a \in A$. We can define in a natural way an MV-algebra $\mathscr{A}_a$ whose underlying set is the interval $[0, a]$ of $\mathscr{A}$. (For definition, cf. Section 2 below.) $\mathscr{A}_a$ is a *substructure* of $\mathscr{A}$.

A generalized cardinal property on $\mathscr{M}$ is called *decreasing* if for each $\mathscr{A} \in \mathscr{M}$ and each substructure $\mathscr{A}_a$ of $\mathscr{A}$ the relation $f(\mathscr{A}_a) \geqslant f(\mathscr{A})$ is valid.

An MV-algebra $\mathscr{A}$ is *homogeneous* with respect to a generalized cardinal property f if, whenever $a(1)$ and $a(2)$ are nonzero elements of A, then $f(\mathscr{A}_{a(1)}) = f(\mathscr{A}_{a(2)})$.

Let $\mathscr{C}$ be the class of all MV-algebras $\mathscr{A} \neq \{0\}$ such that $\mathscr{A}$ is semisimple, projectable and orthogonally complete. Each complete MV-algebra belongs to $\mathscr{C}$.

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In the present paper sufficient conditions are found for a decreasing generalized cardinal property f (cf. the conditions (γ_1) and (γ_2) in Section 4) under which every MV-algebra belonging to $\mathscr{C}$ can be represented as a direct product of MV-algebras which are homogeneous with respect to f.

We apply this result to dealing with the generalized cardinal property f_1 which is defined by means of the notion of α-completeness (where α runs over the class of all infinite cardinals). It turns out that an MV-algebra $\mathscr{A}$ is homogeneous with respect to f_1 iff it satisfies the following condition: whenever α is an infinite cardinal and $a_1, a_2, b_1, b_2 \in A$, $a_1 < b_2$, $b_1 < b_2$ are such that the interval $[a_1, a_2]$ is α-complete, then $[b_1, b_2]$ is α-complete as well.

Cardinal properties of complete MV-algebras were studied by Pierce [14]. Cardinal properties and generalized cardinal properties of lattice ordered groups were investigated in [5] and [13]. The notion of α-completeness of pseudo MV-algebras was dealt with in [12].

2. Preliminaries

For the definition of MV-algebra, several equivalent systems of axioms were applied (cf., e.g., Cignoli, D'Ottaviano and Mundici [2], Dvurečenskij and Pulmanová [4]).

In the present paper the system from [2] will be used. Thus an *MV-algebra* $\mathscr{A} = (A; \oplus, \neg, 0)$ is an algebraic structure of type (2,1,0) such that the axioms (MV1)–(MV6) from [2] are satisfied. We put $\neg 0 = 1$.

Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For $x, y \in A$ we put

$$x \oplus y = (x + y) \wedge u, \quad \neg x = u - x, \quad 1 = u.$$

Then $(A; \oplus, \neg, 0)$ is an MV-algebra; it is denoted by $\Gamma(G, u)$.

For each MV-algebra $\mathscr{A}$ there exists an abelian lattice ordered group G with a strong unit u such that

$$\mathscr{A} = \Gamma(G, u) \tag{1}$$

(cf. [2]). In the sequel, when speaking about $\mathscr{A}$, we always suppose that the relation (1) is satisfied.

The partial order $\leqslant$ from G induces a partial order on the set A. Then $(A; \leqslant)$ is a distributive lattice with the least element 0 and the greatest element u; we denote this lattice by $\ell(\mathscr{A})$.

We say that $\mathscr{A}$ is complete if the lattice $\ell(\mathscr{A})$ is complete (and analogously for other lattice properties).

Let α be an infinite cardinal. Recall that a lattice L is said to be α-complete if each nonempty subset X of L with $\operatorname{card} X \leqslant \alpha$ possesses a supremum and an infimum in L.

A nonempty subset Y of A is called *orthogonal* if $y_1 \wedge y_2 = 0$ whenever y_1 and y_2 are distinct elements of Y. We say that $\mathscr{A}$ is *orthogonally complete* if each orthogonal subset of $\mathscr{A}$ has a supremum in $\ell(\mathscr{A})$.

Let $X \subseteq A$. We put

$$X^{\delta(\mathscr{A})} = \{x_1 \in A \colon\ x_1 \wedge x = 0 \text{ for each } x \in X\}.$$

The set $X^{\delta(\mathscr{A})}$ is called a *polar* in $\mathscr{A}$. If X is a one-element set, $X = \{x\}$, then $(X^{\delta(A)})^{\delta(A)}$ is said to be a *principal polar* (generated by the element x).

Analogously, for $Y \subseteq G$ we denote

$$Y^{\delta(G)} = \{y_1 \in G \colon\ |y_1| \wedge |y| = 0 \text{ for each } y \in Y\}.$$

Then $Y^{\delta(G)}$ is a *polar* in G. The *principal polar* in G is defined similarly as in the case of $\mathscr{A}$.

Let $a_1 \in A$. Put $[0, a_1] = A_1$. For each $x, y \in A_1$ we set

$$x \oplus_{a_1} y = (x + y) \wedge a_1, \quad \neg_{a_1} x = a_1 - x.$$

Then $\mathscr{A}_1 = (A_1; \oplus_{a_1}, \neg_{a_1}, 0)$ is an MV-algebra. We say that $\mathscr{A}_1$ is a *substructure* (or an interval subalgebra) of $\mathscr{A}$. We denote $\mathscr{A}_1 = \mathscr{A}_{a_1}$.

Let $(\mathscr{B}_j)_{j \in J}$ be an indexed system of MV-algebras. The direct product of this system is defined in the usual way; we denote it by $\prod\limits_{j \in J} \mathscr{B}_j$. Direct product decompositions of MV-algebras have been investigated in [7]; cf. also [11].

Let I be a nonempty system of indices and for each $i \in I$ let $\mathscr{A}_i$ be a substructure of $\mathscr{A}$ with a greatest element a_i and an underlying set A_i. Denote $A' = \prod\limits_{i \in I} A_i$. Consider a mapping φ of A into A' such that

$$\varphi(x) = (x \wedge a_i)_{i \in I}$$

for each $x \in A$.

If the mapping φ is an MV-isomorphism of $\mathscr{A}$ onto the direct product $\prod\limits_{i \in I} \mathscr{A}_i$, then we say that the indexed system $(\mathscr{A}_i)_{i \in I}$ determines an *internal direct product* decomposition of $\mathscr{A}$. In such a case, the MV-algebras $\mathscr{A}_i$ are called *internal direct factors* of $\mathscr{A}$.

Let $\mathscr{A}_1$ be an internal direct factor of $\mathscr{A}$ with a greatest element a_1. For $x \in A$ we denote by $x(\mathscr{A}_1)$ the component of x in $\mathscr{A}_1$; i.e., $x(\mathscr{A}_1) = x \wedge a_1$.

In view of [7], if $x, y, z \in A$ and $x + y = z$, where $x + y$ means the group addition taken in G restricted to $\mathscr{A}$, then

$$z(\mathscr{A}_1) = x(\mathscr{A}_1) + y(\mathscr{A}_1).$$

Assume that I is a nonempty set of indices and that for each $i \in I$, $\mathscr{A}_i$ is an internal direct factor of $\mathscr{A}$ with a greatest element a_i. Suppose that the system $S = (\mathscr{A}_i)_{i\in I}$ has the following properties:

(i) if $i(1)$ and $i(2)$ are distinct elements of I, then $a_{i(1)} \wedge a_{i(2)} = 0$;

(ii) if $x \in A$ and $x \wedge a_i = 0$ for each $i \in I$, then $x = 0$.

Lemma 2.1. *Let $\mathscr{A}$ be an MV-algebra and let S be as above. Then* $\bigvee\limits_{i\in I} a_i = u$.

P r o o f. By way of contradiction, assume that the assertion of the lemma fails to hold. Hence there exists $b \in A$ such that $b < u$ and $a_i \leqslant b$ for each $i \in I$. Put $b - u = d$. Thus $d > 0$ and hence there exists $i \in I$ with $a_i \wedge d > 0$. We have

$$u(\mathscr{A}_i) = u \wedge a_i = a_i, \qquad b(\mathscr{A}_i) = b \wedge a_i = a_i,$$
$$u(\mathscr{A}_i) = (b + d)(\mathscr{A}_i) = b(\mathscr{A}_i) + d(\mathscr{A}_i),$$

whence $a_i = a_i + d_i > a_i$, which is a contradiction. □

Lemma 2.2. *Let $\mathscr{A}$ and S be as in* 2.1. *Then the system S determines an internal direct product decomposition of $\mathscr{A}$.*

P r o o f. For each $x \in A$ we put $\varphi(x) = (x \wedge a_i)_{i\in I}$. Then φ is a mapping of $\mathscr{A}$ into $\prod\limits_{i\in I} \mathscr{A}_i$ such that $x \leqslant y$ implies $\varphi(x) \leqslant \varphi(y)$. In view of 2.1 we have

$$x = x \wedge u = x \wedge \left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I} (x \wedge a_i),$$

thus

$$\varphi(x) \leqslant \varphi(y) \Rightarrow x \leqslant y.$$

Let $z_i \in A_i$ for $i \in I$. Then $z_{i(1)} \wedge z_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I, whence there exists $z \in A$ with $z = \bigvee\limits_{i\in I} z_i$. It is easy to verify that

$z(\mathscr{A}_i) = z_i$ for each $i \in I$. Therefore φ is an isomorphism of the lattice $\ell(\mathscr{A})$ onto the lattice $\ell\Big(\prod\limits_{i\in I} \mathscr{A}_i\Big)$. From this and from [7] we obtain that φ is an isomorphism of $\mathscr{A}$ onto $\prod\limits_{i\in I} \mathscr{A}_i$, completing the proof. $\square$

Let $c \in A$. In accordance with the terminology applied in the lattice theory we say that c is a *central element* of $\mathscr{A}$ if there exists an internal direct factor $\mathscr{A}_1$ of $\mathscr{A}$ such that c is the greatest element of $\mathscr{A}_1$.

All direct product decompositions and direct factors of $\mathscr{A}$ considered below will be assumed to be internal; therefore, the word 'internal' will be often omitted.

In the above definition of the class $\mathscr{C}$ (Section 1), the notions of semisimplicity, projectability and orthogonal completeness were used.

Let us remark that the notions of orthogonal completeness and of projectability have been investigated by several authors dealing with lattice ordered groups and with vector lattices (cf., e.g., Luxemburg and Zaanen [16], Bernau [1], Conrad [3] and the author [6]).

An MV-algebra $\mathscr{A}$ is *projectable* if each principal polar of $\mathscr{A}$ is the underlying set of some internal direct factor of $\mathscr{A}$.

Projectable MV-algebras have been dealt with by the author [10]; it was proved that for $\mathscr{A} = \Gamma(G, u)$, $\mathscr{A}$ is projectable if and only if G is projectable.

An MV-algebra $\mathscr{A}$ is *semisimple* (or *archimedean*) if, whenever $x \in A$ and $nx < u$ for each positive integer n, then $x = 0$. (Other formally different but equivalent definitions were used in literature.)

We conclude this section by giving two examples of decreasing generalized cardinal properties on the class $\mathscr{M}$ of all MV-algebras. Let $\mathscr{A} \in \mathscr{M}$.

Example 1. If the underlying lattice $\ell(\mathscr{A})$ of $\mathscr{A}$ is complete, then we put $f_1(\mathscr{A}) = \infty$. Otherwise, there exists a least cardinal α such that $\ell(\mathscr{A})$ fails to be α-complete; we put $f_1(\mathscr{A}) = \alpha$.

Example 2. For the notions of complete distributivity and of α-distributivity of a lattice (where α is an infinite cardinal) cf., e.g., [15]. We put $f_2(\mathscr{A}) = \infty$ if $\ell(\mathscr{A})$ is completely distributive; otherwise we set $f_2(\mathscr{A}) = \alpha$, where α is the least ordinal such that $\ell(\mathscr{A})$ is not α-distributive.

It is easy to verify that both f_1 and f_2 are decreasing generalized cardinal properties on the set $\mathscr{M}$.

3. Auxiliary results

In this section we assume that $\mathscr{A}$ is an MV-algebra belonging to the class $\mathscr{C}$.

Let $0 < a \in A$. For $n \in \mathbb{N}$ we consider the element $na \in G$. Since $\mathscr{A}$ is semisimple, the lattice ordered group G is archimedean (cf., e.g., [9]). Thus there exists $n(1) \in \mathbb{N}$ such that $n(1) > 1$ and

$$n(1)a \nleqslant u, \quad (n(1)-1)a \leqslant u.$$

Hence $n(1)a - u \nleqslant 0$ and thus

$$(n(1)a-u)^+ > 0. \tag{1}$$

Further, we have

$$n(1)a = (n(1)-1)a + a, \quad 0 \leqslant (n(1)-1)a \leqslant u,$$

whence $0 < n(1)a \leqslant 2u$ and $n(1)a - u \leqslant u$. Thus $0 \leqslant (n(1)a-u) \vee 0 \leqslant u \vee 0 = u$. We obtain

$$(n(1)a-u)^+ = (n(1)a-u) \vee 0 \in A. \tag{2}$$

Put

$$X_1 = ((n(1)a-u)^+)^{\delta(\mathscr{A})\delta(\mathscr{A})}, \quad X_1' = ((n(1)a-u)^+)^{\delta(\mathscr{A})}.$$

Since $\mathscr{A}$ is projectable, it can be expressed as an internal direct product

$$\mathscr{A} = X_1 \times X_1'. \tag{3}$$

We denote $a_1 = a(X_1)$.

If $a(X_1') = 0$, then we stop our construction. (In this case we have $a = a_1$.)

Assume that $a(X_1') \neq 0$. In this case we perform an analogous step where instead of

$$\mathscr{A},\ a,\ u,\ n(1)$$

we take

$$X_1',\ a(X_1'),\ u(X_1'),\ n(2).$$

First we want to verify that $n(2) > n(1)$. By way of contradiction, suppose that $n(2) \leqslant n(1)$.

Denote $a(X_1') = a_2$, $u(X_1) = u_1$, $u(X_1') = u_2$.

a) Let $m \in \mathbb{N}$. If $ma \leqslant u$, then $ma_2 \leqslant u_2$. Therefore $n(2) \geqslant n(1)$.

b) It remains to verify that the relation $n(2) = n(1)$ cannot hold. In view of (3) and according to [7] we have

$$G = G_1 \times G_1', \tag{3'}$$

where G_1 is the convex ℓ-subgroup of G generated by the element u_1, and G_2 is the convex ℓ-subgroup of G generated by the element u_2. From (3′) we get

$$(ma - u)^+ = (ma_1 - u_1)^+ + (ma_2 - u_2)^+ \tag{4}$$

for each positive integer m. Take $m = n(1)$ and suppose that $n(1) = n(2)$. Then in view of the definition of X_1 we obtain $(ma - u)^+ \in X_1$, whence

$$(ma - u)^+ = (ma - u)^+(X_1) = (ma_1 - u_1)^+.$$

Further, the relation $m = n(2)$ yields

$$(ma_2 - u_2)^+ > 0.$$

In view of (4), we have arrived at a contradiction.

Now let us write $X_1 = \mathscr{A}_1$, $X_1' = \mathscr{A}_1'$. By applying the obvious induction we conclude that one of the following possibilities must occur:

α) There exists a positive integer k such that $\mathscr{A}$ can be expressed as an internal direct product

$$\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2 \times \ldots \times \mathscr{A}_k \tag{5a}$$

and $n(1) < n(2) < \ldots < n(k)$, where for $i \in \{1, 2, \ldots, k\}$ we denote by $n(i)$ the first positive integer with

$$(n(i)a_i - u)^+ > 0$$

(taking $a_i = a(\mathscr{A}_i)$, $u_i = u(\mathscr{A}_i)$);

β) for each positive integer k, the MV-algebra $\mathscr{A}$ can be expressed as a direct product

$$\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2 \times \ldots \times \mathscr{A}_k \times \mathscr{A}_k' \tag{5b}$$

and $n(i) < n(i+1)$ for each $i \in \mathbb{N}$, where $n(i)$ has the same meaning as in α).

In order to unify the notation, in the case α) we put $\mathscr{A}_m = \{0\}$ for $m \in \mathbb{N}$, $m > k$ and $\mathscr{A}_m' = \{0\}$ for $m \geqslant k$.

Let us apply the notation as in (5b). For a positive integer i with $i \leqslant k$ let A_i be the underlying set of the MV-algebra $\mathscr{A}_i$. Similarly, let A'_k be the underlying set of $\mathscr{A}'_k$. From the above construction we obtain:

(+) Let $0 < x \in A_i$ and let n_x be the first positive integer with $(n_x x - u)^+ > 0$. Then $n_x \geqslant n(i)$.

($+_1$) Let $0 < x \in A'_k$ and let n_x be as in (+). Then $n_x > n(k)$.

Further, we have

($+_2$) The indexed system $(a(\mathscr{A}_n))_{n \in \mathbb{N}}$ is orthogonal.

Proof. By way of contradiction, assume that our assertion is not valid. Hence there exist $n(1), n(2) \in \mathbb{N}$ and $0 < x \in A$ such that

$$n(1) < n(2), \quad x \leqslant a(\mathscr{A}_{n(1)}) \wedge a(\mathscr{A}_{n(2)}).$$

Then $x \in A_{n(1)} \cap A_{n(2)}$. In view of (5b) we have

$$\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2 \times \ldots \times \mathscr{A}_{n(2)} \times \mathscr{A}'_{n(2)}.$$

This relation yields that $A_{n(1)} \cap A_{n(2)} = \{0\}$; thus we have arrived at a contradiction. □

Lemma 3.1. *Under the notation as above, we have* $a = \bigvee_{n=1}^{\infty} a(\mathscr{A}_k)$.

Proof. If α) is valid, then

$$a = a(\mathscr{A}_1) + a(\mathscr{A}_2) + \ldots + a(\mathscr{A}_k) = a(\mathscr{A}_1) \vee a(\mathscr{A}_2) \vee \ldots \vee a(\mathscr{A}_k).$$

If $m > k$, then $a(\mathscr{A}_m) = 0$. Thus the assertion of the lemma is valid.

Let β) be valid. By way of contradiction, assume that the relation $a = \bigvee_{n=1}^{\infty} a(\mathscr{A}_n)$ does not hold. We obviously have $a(\mathscr{A}_n) \leqslant a$. Hence there exists $b \in A$ such that $a(\mathscr{A}_n) \leqslant b < a$ for each $n \in \mathbb{N}$. Put $c = a - b$. Hence $c > 0$ and thus there exists the first positive integer m with $mc \nleqslant u$.

In view of ($+_2$), the indexed system $(a(\mathscr{A}_n))_{n \in \mathbb{N}}$ is orthogonal. From this we conclude that c is orthogonal to each $a(\mathscr{A}_n)$ ($n \in \mathbb{N}$). Hence $c \in \mathscr{A}'_j$ for each $j \in \mathbb{N}$. Thus we have $m \geqslant n(j)$ for each $j \in \mathbb{N}$, which is impossible. □

Now let $\mathscr{A} \in \mathscr{C}$ and let us apply the notation as above. From the construction of $\mathscr{A}_n$ ($n \in \mathbb{N}$) we infer that the system $(u_n)_{n \in \mathbb{N}}$ is orthogonal. Hence the element $\bigvee_{n=1}^{\infty} u_n$ exists; we will denote it by u_0.

Lemma 3.2. $u_0 \oplus u_0 = u_0$.

P r o o f. From α) and β) above we conclude that whenever n and m are distinct positive integers, then $u_n \wedge u_m = 0$. Thus

$$u_0 \oplus u_0 = (u_0 + u_0) \wedge u = \left(\left(\bigvee_{n\in\mathbb{N}} u_n\right) + \bigvee_{m\in\mathbb{N}} u_m\right) \wedge u$$
$$= \left(\bigvee_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} (u_n + u_m)\right) \wedge u.$$

We have

$$u_n + u_m = \begin{cases} u_n \vee u_m & \text{if } n \neq m, \\ 2u_n & \text{if } n = m. \end{cases}$$

Therefore, since $2u_n \vee 2u_m \geqslant u_n \vee u_m$, we obtain

$$u_0 \oplus u_0 = \left(\bigvee_{n\in\mathbb{N}} 2u_n\right) \wedge u = \bigvee_{n\in\mathbb{N}} (2u_n \wedge u).$$

We have already verified that the interval $[0, u_n]$ of $\mathscr{A}$ is an internal direct factor of $\mathscr{A}$. Hence there exists a complement u_n' of u_n in the lattice $\ell(\mathscr{A}) = [0, u]$. We get

$$2u_n \wedge u = 2u_n \wedge (u_n \vee u_n') = (2u_n \wedge u_n) \vee (2u_n \wedge u_n').$$

From $u_n \wedge u_n' = 0$ we get $2u_n \wedge u_n' = 0$, whence

$$2u_n \wedge u = 2u_n \wedge u_n = u_n.$$

Thus

$$u_0 \oplus u_0 = \bigvee_{n\in\mathbb{N}} u_n = u_0.$$

□

Lemma 3.3. *The interval $[0, u_0]$ is the underlying lattice of an internal direct factor of $\mathscr{A}$.*

P r o o f. This is a consequence of 3.2 and of the results of [11]. □

Corollary 3.4. *The interval $[0, u_0]$ is a principal polar of $\mathscr{A}$ generated by the element u_0.*

Corollary 3.5. *The element u_0 has a complement u_0' in' $\ell(\mathscr{A})$ and the interval $[0, u_0']$ is the underlying lattice of an internal direct factor of $\mathscr{A}$.*

4. A generalized cardinal property f

Let K and $\mathscr{M}$ be as in Section 1.

As we have already remarked above, the element ∞ is considered to be the greatest element of K; for $k_1, k_2 \in K$ with $k_1 \neq \infty \neq k_2$, the relation $k_1 \leqslant k_2$ has the usual meaning. Assume that f is a generalized cardinal property on $\mathscr{M}$.

Further, let us consider the following conditions (γ_1) and (γ_2) for f.

(γ_1) Let $\mathscr{A} \in \mathscr{M}$, $\mathscr{A} = \prod_{i \in I} \mathscr{A}_i$, $\alpha \in K$. If $f(\mathscr{A}_i) \geqslant \alpha$ for each $i \in I$, then $f(\mathscr{A}) \geqslant \alpha$.

(γ_2) Let $\mathscr{A} \in \mathscr{M}$, $\mathscr{A} = \Gamma(G, u)$. Whenever n is a positive integer and $\mathscr{A}_1 = \Gamma(G, nu)$, then $f(\mathscr{A}_1) = f(\mathscr{A})$.

In what follows we assume that f is a decreasing generalized cardinal property satisfying the conditions (γ_1) and (γ_2).

Let $\mathscr{A}$ be a fixed element of $\mathscr{M}$, $\mathscr{A} = \Gamma(G, u)$. Further, let a be as in Section 3.

Consider the relations (3) and (3′) from Section 3, i.e.,

$$\mathscr{A} = X_1 \times X_1', \quad G = G_1 \times G_1'$$

(meaning the internal direct product decompositions of $\mathscr{A}$ or of G (respectively)). From the results of [7] we conclude that if $t \in A$, then we have

$$t(X_1) = t(G_1), \quad t(X_1') = t(G_1').$$

Let $n(1)$ be as above; put $b = (n(1)a - u)^+$. In view of (2) of Section 3, $0 < b \in A$. Denote

$$u(G_i) = u_1, \quad u(G_1') = u_1', \quad a(G_1) = a_1, \quad a(G_1') = a_1'.$$

According to the definition of G_1 and G_1' we obtain

$$G_1 = \{b\}^{\delta(G)\delta(G)}, \quad G_1' = \{b\}^{\delta(G)}.$$

Since $(n(1)a - u)^- \wedge b = 0$, we get

$$(n(1)a - u)^- \in \{b\}^{\delta(G)} = G_1',$$

whence $(n(1)a - u)^-(G_1) = 0$. In view of

$$n(1)a - u = (n(1)a - u)^+ - (n(1)a - u)^-$$

we obtain

$$(n(1)a-u)(G_1)=(n(1)a-u)^+(G_1).$$

Next, we have

$$n(1)a_1-u_1=((n(1)a-u)\vee 0)(G_1)=(n(1)a_1-u_1)\vee 0,$$

whence $n(1)a_1-u_1\geqslant 0$.

If $n(1)a_1-u_1=0$, then

$$\begin{aligned}n(1)a-u&=(n(1)a-u)(G_1)+(n(1)a-u)(G_1')=(n(1)a-u)(G_1')\\&=((n(1)a-u)\vee 0)(G_1')+((n(1)a-u)\wedge 0)(G_1').\end{aligned}$$

Since $(n(1)a-u)\vee 0\in G_1$, we get

$$((n(1)a-u)\vee 0)(G_1')=0,$$

thus

$$n(1)a-u=((n(1)a-u)\wedge 0)(G_1')\leqslant 0,$$

which is a contradiction. Therefore

$$n(1)a_1-u_1>0. \tag{1}$$

From the fact that u is a strong unit of G we obtain that u_1 is a strong unit of G_1. Since $n(1)a_1-u_1\in G_1$, (1) yields that $u_1\neq 0$. Then $a_1>0$. Clearly $a_1=a(G_1)\leqslant a$ and hence $[0,a_1]\subseteq[0,a]$.

For any $g\in G$, $g>0$ let G_g be the convex ℓ-subgroup of G generated by the element g. Further, we put

$$\mathscr{A}_g=\Gamma(G_g,g).$$

Assume that α is an element of K such that $f(\mathscr{A}_a)\geqslant\alpha$. We have $a\geqslant a_1>0$, whence $\mathscr{A}_{a_1}\subseteq\mathscr{A}_a$. Since f is decreasing, we get $f(\mathscr{A}_{a_1})\geqslant\alpha$. Further, in view of (γ_2) we have $f(\mathscr{A}_{n(1)a_1})\geqslant\alpha$. In view of (1) we have $\mathscr{A}_{u_1}\subseteq\mathscr{A}_{n(1)a_1}$. Thus we get

Lemma 4.1. $f(\mathscr{A}_{u_1})\geqslant\alpha$.

For $n\in\mathbb{N}$ let u_n be as in Section 3. By analogous reasoning as for u_1 we get

Lemma 4.2. *For each $n \in \mathbb{N}$, $f(\mathscr{A}_{u_n}) \geqslant \alpha$.*

Let u_0 be as in Section 3, i.e., $u_0 = \bigvee_{n \in \mathbb{N}} u_n$. We have already verified that the indexed system $(u_n)_{n \in \mathbb{N}}$ is orthogonal. From this we obtain by a simple calculation that the mapping $\varphi(x) = (x \wedge u_n)_{n \in \mathbb{N}}$ for $x \in [0, u_0]$ is an isomorphism of the lattice $[0, u_0]$ onto the lattice $\prod_{n \in \mathbb{N}} [0, u_n]$. Thus according to [7], the mapping φ is an internal product decomposition of the MV-algebra $\mathscr{A}_{u_0}$ onto $\prod_{n \in \mathbb{N}} \mathscr{A}_n$. Hence 4.2 and (γ_1) yield

Lemma 4.3. $f(\mathscr{A}_{u_0}) \geqslant \alpha$.

Lemma 4.4. *Let $0 < x \leqslant u_0$. Then $x \wedge a > 0$.*

Proof. By way of contradiction, assume that $x \wedge a = 0$. We have

$$x = x \wedge u_0 = x \wedge \left(\bigvee_{n \in \mathbb{N}} u_n \right) = \bigvee_{n \in \mathbb{N}} (x \wedge u_n).$$

Hence there is $n \in \mathbb{N}$ with $x \wedge u_n > 0$.

In view of (1) and of the analogous relation cocnerning u_n, a_n we conclude that there exists $m \in \mathbb{N}$ such that

$$ma_n - u_n > 0.$$

Clearly $a_n \leqslant a$, hence $ma > u_n \geqslant x \wedge u_n > 0$. Thus $ma \wedge x > 0$ yielding that $a \wedge x > 0$, which is a contradiction. $\square$

Lemma 4.5. $\{a\}^{\delta(\mathscr{A})\delta(\mathscr{A})} = [0, u_0]$.

Proof. From the relation $\mathscr{A} = [0, u_0] \times [0, u_0']$ we obtain

$$[0, u_0]^{\delta(\mathscr{A})} = [0, u_0'].$$

Since $a \in [0, u_0]$, we get

$$\{a\}^{\delta(\mathscr{A})} \supseteq [0, u_0]^{\delta(\mathscr{A})}.$$

Let $x_1 \in \{a\}^{\delta(\mathscr{A})}$. We have

$$x_1 = x_1([0, u_0]) \vee x_1([0, u_0']).$$

Then $x_1([0, u_0]) \leqslant x_1$, whence $x_1([0, u_0]) \in \{a\}^{\delta(\mathscr{A})}$. Put $x_1([0, u_0]) = x$. We have $x \leqslant u_0$. If $x > 0$, then in view of 4.4 we get $x \wedge a > 0$, yielding that $x \notin \{a\}^{\delta(\mathscr{A})}$, which is a contradiction. Thus $x_1([0,u_0])=0$. Hence $x_1=x_1([0,u_0'])\in[0,u_0']$. We obtain $\{a\}^{\delta(\mathscr{A})}\subseteq[0,u_0']$. Therefore

$$\{a\}^{\delta(\mathscr{A})}=[0,u_0'].$$

We get

$$\{a\}^{\delta(\mathscr{A})\delta(\mathscr{A})}=[0,u_0']^{\delta(\mathscr{A})}=[0,u_0].$$

□

Now, 4.3 and 4.5 yield

Theorem 4.6. *Let $\mathscr{A}\in\mathscr{C}$, $0<a\in A$. Assume that f is a decreasing generalized cardinal property on $\mathscr{C}$ satisfying the conditions (γ_1) and (γ_2). Let $\alpha\in K$, $f(\mathscr{A}_a)\geqslant\alpha$. Then $f(\{a\}^{\delta(\mathscr{A})\delta(\mathscr{A})})\geqslant\alpha$.*

5. The system $S(\alpha)$

Again, let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$. Let $\alpha\in K$ and let f be a decreasing generalized cardinal property on $\mathscr{C}$ satisfying the conditions (γ_1) and (γ_2). We denote by $S(\alpha)$ the system of all elements $a\in A$ such that $a>0$ and $f(\mathscr{A}_a)\geqslant\alpha$. We assume that $S(\alpha)\neq\emptyset$.

We modify the notation from Section 3 and Section 4 as follows. If $a\in S(\alpha)$, then instead of the symbols u_0 and u_0' used above we write

$$u_0(a,\mathscr{A}),\quad u_0'(a,\mathscr{A});$$

if no misunderstanding can occur, then we write briefly $u_0(a),u_0'(a)$.

We apply Axiom of Choice; thus we can suppose that the system $S(\alpha)$ is written in the form

$$S(\alpha)=(a^i)_{i<m},$$

where m is an appropriate ordinal and i runs over the set of all ordinals less than m.

We put

$$u_0^1=u_0(a^1),\quad {u_0'}^1=u_0'(a^1),\quad v_0^1=u_0^1.$$

For each ordinal $i(1)<m$ we define by transfinite induction an element $v_0^{i(1)}$ of A such that

α_1) the system $(v_0^i)_{i\leqslant i(1)}$ is orthogonal;

α_2) for $b^{i(1)}=\bigvee\limits_{i\leqslant i(1)}v_0^i$ we have $a^{i(1)}\leqslant b^{i(1)}$;

α_3) the interval $[0, b^{i(1)}]$ is the underlying set of a direct factor of $\mathscr{A}$;

α_4) $f(\mathscr{A}_{v_0^{i(1)}}) \geqslant \alpha$.

From the results of Section 4 it follows that the conditions α_1)–α_4) are valid for $i(1) = 1$ (where v_0^1 is defined as above). Suppose that $i(1)$ is an ordinal with $1 < i(1) < m$ and that we have defined the elements v_0^i for $i < i(1)$ such that the conditions α_1)–α_4) are satisfied (in the sense that instead of $i(1)$ we take the element i under consideration, and instead of the symbol i from $\alpha 1$)–$\alpha 4$) we take an ordinal j with $j \leqslant i$). Then the system $(v_0^i)_{i<i(1)}$ is orthogonal, hence the element

$$b_o^{i(1)} = \bigvee_{i<i(1)} v_0^i$$

exists. Similarly as we did above for the element b_0 we can verify that the interval $[0, b_0^{i(1)}]$ is the underlying set of an internal direct factor of $\mathscr{A}$. Then $u - b_0^{i(1)} = b_0^{*i(1)}$ is the complement of $b_0^{i(1)}$ and $[0, b_0^{*i(1)}]$ is also an underlying set of a direct factor of $\mathscr{A}$; we have

$$\mathscr{A} = [0, b_0^{i(1)}] \times [0, b_0^{*i(1)}]. \tag{1}$$

Put

$$b^{i(1)} = a^{i(1)}[0, b_0^{*i(1)}].$$

If $b^{i(1)} = 0$, then we put $v_0^{i(1)} = 0$. Further, assume that $b^{i(1)} > 0$. In this case we proceed as in Section 3 and Section 4 with the distinction that instead of the element a we now have the element $b^{i(1)}$. Instead of u_0 we now obtain an element which will be denoted by $v_0^{i(1)}$.

From the definition of $v_0^{i(1)}$ we conclude that this element is orthogonal to all v_0^i for $i < i(1)$. Hence the system $(v_0^i)_{i \leqslant i(1)}$ is orthogonal. Thus α_1) is valid. Put

$$c^{i(1)} = a^{i(1)}[0, b_0^{i(1)}].$$

Then

$$a^{i(1)} = b^{i(1)} \vee c^{i(1)}.$$

The definition of $v_0^{i(1)}$ also yields that $b^{i(1)} \leqslant v_0^{i(1)}$ (cf. the analogous relation concerning a and u_0). Further, $c^{i(1)} \leqslant b_0^{i(1)}$. Therefore α_2) holds.

Similarly as $u_0, v_0^{i(1)}$ is also a central element of $\mathscr{A}$ (i.e., it belongs to the centre of the lattice $\ell(\mathscr{A})$). Then according to α_1), $b^{i(1)}$ is a central element of $\mathscr{A}$ as well. Hence α_3) holds.

In view of 4.3 (applied to $v_0^{i(1)}$), the relation $f(\mathscr{A}_{v_0^{i(1)}}) \geqslant \alpha$ is valid. From this and from α_1) we obtain $f(\mathscr{A}_b^{i(1)}) \geqslant \alpha$; hence α_4) holds.

Thus we have defined the system $(v_0^i)_{i<m}$. All elements of this system are central. Moreover, this system is orthogonal. Thus the element

$$v(\alpha) = \bigvee_{i<m} v_0^i$$

exists and it is central. By analogous argument as applied in connection with 4.3 we conclude that the relation $f(\mathscr{A}_{v(\alpha)}) \geqslant \alpha$ is valid. Let $v'(\alpha)$ be the complement of $v(\alpha)$. Hence we have

$$\mathscr{A} = [0, v(\alpha)] \times [0, v'(\alpha)]. \tag{2}$$

The previous consideration was performed under the assumption that $S(\alpha) \neq \emptyset$. If $S(\alpha) = \emptyset$, then we put $v(\alpha) = 0$, hence $v'(\alpha) = u$; in this case (2) remains valid.

Theorem 5.1. *Let $\mathscr{A} \in \mathscr{C}$ and let f be a decreasing cardinal property on $\mathscr{C}$ satisfying the conditions (γ_1) and (γ_2). Let $\alpha \in K$. Then there exists an internal direct product decomposition $\mathscr{A} = \mathscr{A}_1^\alpha \times \mathscr{A}_2^\alpha$ such that*
(i) *$f(\mathscr{A}_1) \geqslant \alpha$;*
(ii) *if b is a nonzero element of A_2^α, then $f(\mathscr{A}_b) < \alpha$.*

Proof. Put $\mathscr{A}_1^\alpha = \mathscr{A}_{v(\alpha)}$, $\mathscr{A}_2^\alpha = \mathscr{A}_{v'(\alpha)}$. We have already verified that $f(\mathscr{A}_{v(\alpha)}) \geqslant \alpha$.

Let $b \in A_2^\alpha$, $b > 0$. By way of contradiction, assume that $f(\mathscr{A}_b) \geqslant \alpha$. Then there is an ordinal $i < m$ such that $b = a^i$. In view of the construction of $v(\alpha)$ we have $b \leqslant v(\alpha)$. Hence $b \in A_{v(\alpha)} = A_1^\alpha$. Thus $b \in A_1^\alpha \cap A_2^\alpha = \{0\}$, which is a contradiction. □

6. Homogeneous direct factors

Assume that $\mathscr{A}$ and f are as in 5.1. Our considerations would be trivial if $A = \{0\}$; therefore in the sequel we suppose that A fails to be a one-element set.

Put $K_0 = \{\alpha \in K \colon \mathscr{S}(\alpha) \neq \emptyset\}$. Then K_0 is a set.

If $\alpha = \infty \in K_0$, then we put $\mathscr{A}_{01}^\alpha = \mathscr{A}_1^\alpha$, where $\mathscr{A}_1^\alpha$ is as in 5.1.

Let $\alpha \in K_0$, $\alpha \neq \infty$. Denote $\beta = \alpha^+$ (the first cardinal larger than α). We obviously have $\mathscr{A}_1^\beta \subseteq \mathscr{A}_1^\alpha$. Since both $\mathscr{A}_1^\beta$ and $\mathscr{A}_1^\alpha$ are internal direct factors of $\mathscr{A}$, we conclude that $\mathscr{A}_1^\beta$ is an internal direct factor of $\mathscr{A}_1^\alpha$. Hence $\mathscr{A}_1^\alpha$ can be written in the form

$$\mathscr{A}_1^\alpha = \mathscr{A}_{01}^\alpha \times \mathscr{A}_1^\beta. \tag{+}$$

Proposition 6.1. *For each $\alpha \in K_0$, the MV-algebra $\mathscr{A}_{01}^{\alpha}$ is homogeneous with respect to f.*

P r o o f. Let $\alpha \in K_0$. For $\alpha = \infty$, the assertion is obvious. Assume that $\alpha \neq \infty$. Let $0 < b \in A_{01}^{\alpha}$. In view of the results of Section 5, we have $f(\mathscr{A}_b) \geqslant \alpha$. By the same method as in the proof of 5.1 we can verify that $f(\mathscr{A}_b) \gneqq \alpha^+$. Hence $f(\mathscr{A}_b) = \alpha$ for each $0 < b \in A_{01}^{\alpha}$. $\square$

For each $\alpha \in K_0$ let p_α be the greatest element of $\mathscr{A}_{01}^{\alpha}$. Consider the systems

$$S_1 = (p_\alpha)_{\alpha \in K_0}, \quad S_2 = (\mathscr{A}_{01}^{\alpha})_{\alpha \in K_0}.$$

The following two assertions are immediate consequences of the definitions of S_1 and S_2.

Lemma 6.2. *The system S_1 is orthogonal.*

Lemma 6.3. *Each element of S_2 is an internal direct factor of $\mathscr{A}$.*

Lemma 6.4. *Let $x \in A$. Assume that $x \wedge t = 0$ whenever t is a member of S_1. Then $x = 0$.*

P r o o f. By way of contradiction, assume that $x > 0$. If $f(\mathscr{A}_x) = \infty$, then $x \leqslant p_\infty \in S_1$, which is a contradiction.

Assume that $f(\mathscr{A}_x) = \alpha < \infty$. Then $x \in \mathscr{A}_1^{\alpha}$. Consider the relation $(+)$. If $x \in \mathscr{A}_1^{\beta}$, then $f(\mathscr{A}_x) \geqslant \beta$, which is impossible. Hence $x \notin \mathscr{A}_1^{\beta}$ and so $x(\mathscr{A}_{01}^{\alpha}) = x_1 > 0$. We have $x \wedge p_\alpha \geqslant x_1 \wedge p_\alpha = x_1 > 0$; we have arrived at a contradiction. $\square$

Theorem 6.5. *Let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$. Let f be a decreasing generalized cardinal property on $\mathscr{C}$ satisfying the conditions (γ_1) and (γ_2). Then $\mathscr{A}$ is a direct product of the MV-algebras of the system S_2 and all these direct factors are homogeneous with respect to f. For each $\alpha \in K_0$ we have $f(A_{01}^{\alpha}) = \alpha$.*

P r o o f. This is a consequence of the definition of K_0 and of 2.2, 6.1, 6.2, 6.3 and 6.4. $\square$

7. The α-completeness

In the present section we deal with the generalized cardinal property f_1 which was defined in Section 2. We have already remarked above that f_1 is decreasing.

Proposition 7.1. *f_1 satisfies the conditions (γ_1) and (γ_2).*

Proof. a) Let $\mathscr{A} \in \mathscr{M}$, $\mathscr{A} = \prod_{i\in I} \mathscr{A}_i$, $\alpha \in K$. Assume that $f_1(\mathscr{A}_i) \geqslant \alpha$ for each $i \in I$.

First suppose that $\alpha = \infty$. Hence all $\mathscr{A}_i$ are completely distributive. Then $\mathscr{A}$ is completely distributive as well; consequently, $f(A) = \infty$.

Further, suppose that $\alpha < \infty$. In view of the definition of f_1, if α_1 is a cardinal with $\alpha_1 < \alpha$, then for each $i \in I$, $\mathscr{A}_i$ is α_1-distributive. This yields that $\mathscr{A}$ is α_1-distributive, whence $f_1(\mathscr{A}) \geqslant \alpha$.

We have verified that (γ_1) is valid for f_1.

b) Let $\mathscr{A} \in \mathscr{M}$, $\mathscr{A} = \Gamma(G, u)$. Let n be a positive integer and $\mathscr{A}_1 = \Gamma(G, nu)$. Put $f_1(\mathscr{A}) = \alpha$.

Suppose that $\alpha = \infty$. Hence the lattice $\mathscr{A}$ is complete. Then in view of [12], the lattice ordered group G is conditionally complete. This yields that the lattice $\ell(\Gamma(G, nu))$ is complete. Hence $f_1(\mathscr{A}_1) = \infty$.

Further, suppose that $\alpha < \infty$. The MV-algbra $\mathscr{A}$ is a substructure of $\mathscr{A}_1$; since f_1 is decreasing, we obtain $f(\mathscr{A}_1) \leqslant f(\mathscr{A})$. By way of contradiction, assume that $f(\mathscr{A}_1) < f(\mathscr{A}) = \alpha$. Then $\mathscr{A}_1$ is α-complete. By using [12] again we get that G is conditionally α-complete. Thus $\mathscr{A}$ is α-complete, which is a contradiction. Hence $f(\mathscr{A}_1) = f(\mathscr{A})$ and hence (γ_2) is satisfied. □

Let $\mathscr{A} \in \mathscr{M}$. Assume that a_1, a_2 are elements of A with $a_1 < a_2$. Put $a = a_2 - a_1$. Thus the intervals $[0, a]$ and $[a_1, a_2]$ of the lattice $\ell(\mathscr{A})$ are isomorphic. Thus from the definition of f_1 we immediately obtain that the MV-algebra $\mathscr{A}$ is homogeneous with respect to f_1 if and only if the following condition is satisfied:

(∗) Whenever a_1, a_2, b_1, b_2 are elements of A with $a_1 < a_2$, $b_1 < b_2$ and α is a cardinal such that the interval $[a_1, a_2]$ is α-complete then the interval $[b_1, b_2]$ is α-complete as well.

Let $\mathscr{A} \in \mathscr{C}$. In view of 7.1, the assertion of 6.5 is valid for the generalized cardinal property f_1. Let us apply the notation as in 6.5. We will show that in this case, the set $\mathscr{S}_2$ has at most two elements.

Theorem 7.2 (cf. [13])**.** *Let $\mathscr{B}$ be an MV-algebra. Then the following conditions are equivalent:*

(i) *$\mathscr{B}$ is complete.*

(ii) *$\mathscr{B}$ is σ-complete and orthogonally complete.*

Corollary 7.3. *Let $\mathscr{A} \in \mathscr{C}$. If $\mathscr{A}$ is σ-complete, then it is complete.*

Theorem 7.4. *Let A be an MV-algebra belonging to the class $\mathscr{C}$. Then $\mathscr{A}$ can be expressed as a direct product $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ such that*

(i) *$\mathscr{A}_1$ is complete;*

(ii) *if $0 < x \in A_2$, then the interval $[0, x]$ fails to be σ-complete.*

Proof. Consider the direct product decomposition from 6.5 for the case $f = f_1$. It suffices to verify that $K_0 \subseteq \{\aleph_0, \infty\}$. By way of contradiction, assume that there exists $\alpha \in K_0$ such that $\aleph_0 < \alpha < \infty$. Then $\mathscr{A}_{01}^{\alpha} \neq \{0\}$. Since $\alpha > \aleph_0$ we get that $\mathscr{A}_{01}^{\alpha}$ is σ-complete, thus in view of 7.3 it is complete, whence $f_1(A_{01}^{\alpha}) = \infty$, which is a contradiction. $\square$

Recall that in defining the class $\mathscr{C}$ we have used the following conditions for an element $\mathscr{A}$ of $\mathscr{C}$: (i) $\mathscr{A}$ is semisimple; (ii) $\mathscr{A}$ is projectable; (iii) $\mathscr{A}$ is orthogonally complete.

In connection with 7.4, let us consider two examples dealing with these conditions.

Example 1. Let $G = \mathbb{Z} \circ \mathbb{Z}$ (where $\circ$ denotes the lexicographic product and $\mathbb{Z}$ is the additive group of all integers with the natural linear order). Put $u = (1, 0)$ and $\mathscr{A} = \Gamma(G, u)$. Denote $a_1 = (0, 1)$, $a_2 = u$. The interval $[0, a_1]$ of $\mathscr{A}$ is complete and the interval $[0, a_2] = A$ fails to be σ-complete. The MV-algebra $\mathscr{A}$ is not semisimple, but it is projectable and orthogonally complete. If $\mathscr{A}_1 \neq \{0\}$ is an internal direct factor of $\mathscr{A}$, then $\mathscr{A}_1 = \mathscr{A}$. Hence $\mathscr{A}$ cannot be represented as an internal direct product of direct factors which are homogeneous with respect to f_1.

Example 2. Let $\mathbf{2}$ be a two-element Boolean algebra and let m be an infinite cardinal. Put $\mathscr{B} = \mathbf{2}^m$. The elements of $\mathscr{B}_1$ will be written in the form $x = (x_i)_{i \in I}$, where $x_i \in \mathbf{2}$ and $\operatorname{card} I = m$. Thus $\mathscr{B}_1$ is a complete Boolean algebra. Denote

$$I_0(x) = \{i \in I \colon\ x_i = 0\}, \quad I_1(x) = \{i \in I \colon\ x_i = 1\},$$
$$B^1 = \{x \in \mathscr{B} \colon\ I_0(x) \text{ is finite}\}, \quad B^0 = \{x \in \mathscr{B} \colon\ I_1(x) \text{ is finite}\},$$
$$\mathscr{B}' = B^1 \cup B^0.$$

Hence $\mathscr{B}'$ is a subalgebra of $\mathscr{B}$. Thus there exists a semisimple MV-algebra $\mathscr{A}$ such that $\ell(\mathscr{A}) = \mathscr{B}'$. It is obvious that $\mathscr{A}$ is not orthogonally complete. For $a \in \ell(\mathscr{A})$ let a' be the complement of $\mathscr{A}$. Thus $\ell(\mathscr{A}) = [0, a] \times [0, a']$. By applying [7] we obtain that the MV-algebra $\mathscr{A}$ is projectable.

Suppose that $\mathscr{A}$ can be expressed as an internal direct product of MV-algebras $\mathscr{A}_j$ ($j \in J$) such that each $\mathscr{A}_j$ is homogeneous with respect to f_1. Without loss of generality we can assume that all $\mathscr{A}_j$ are nonzero. We denote by u_j the greatest element of $\mathscr{A}_j$. Thus the underlying set A_j of $\mathscr{A}_j$ is the interval $[0, u_j]$ of $\mathscr{A}$. We have $u_j \neq 0$ for each $j \in J$. Hence for each $j \in J$ there exists an atom a^j in $\ell(\mathscr{A}_j)$.

Since the interval $[0, a^j]$ of $\mathscr{A}_j$ is complete, in view of the homogeneity of $\mathscr{A}_j$ we conclude that $\ell(\mathscr{A}_j)$ is a complete lattice.

If $j \in J$ and $u_j \in B^1$, then it is obvious that $\ell(\mathscr{A}_j)$ is not σ-complete, which is a contradiction. Hence $u_j \in B^0$ for each $j \in J$ and then A_j is finite; therefore $\mathscr{A}_j$ is complete. Thus $\mathscr{A}$ is complete as well. But, in view of the definition of $\mathscr{A}$, we get that $\mathscr{A}$ is not σ-complete; we have arrived at a contradiction.

In Section 1 we have remarked that each complete MV-algebra belongs to $\mathscr{C}$. The following example shows that if $\mathscr{A} \in \mathscr{C}$, then $\mathscr{A}$ need not be complete.

Example 3. Let G be the additive group of all rationals with the natural linear order. Put $u = 1$ and consider the MV-algebra $\mathscr{A} = \Gamma(G, u)$. Then $\mathscr{A}$ belongs to $\mathscr{C}$, but it fails to be complete. Also, if $\mathscr{A}_1$ is a direct product of MV-algebras isomorphic to $\mathscr{A}$, then $\mathscr{A}_1 \in \mathscr{C}$ and $\mathscr{A}_1$ is not complete.

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