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STRONG PROJECTABILITY OF LATTICE ORDERED GROUPS

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Abstract. In this paper we prove that the lateral completion of a projectable lattice ordered group is strongly projectable. Further, we deal with some properties of Specker lattice ordered groups which are related to lateral completeness and strong projectability.

Keywords: Lattice ordered group, projectability, strong projectability, lateral completion, orthocompletion, Specker lattice ordered group

MSC 2000: 06F20

1. Introduction

The lateral completion of a lattice ordered group has been investigated by Bernau [1], [2], Byrd and Lloyd [5], Conrad [7] and the author [13]–[16].

The orthocompletion of a lattice ordered group G has been dealt with by Bleier [4]. The strongly projectable hull of G has been studied by Bleier [3], Chambless [6], Conrad [9].

Some related notions (lateral σ-completeness, σ-orthocompleteness) have been studied by Rotkovich [20].

In the present paper we prove that if G is a projectable lattice ordered group, then its lateral completion G^L is strongly projectable. The strongly projectable hull G^{SP} of G need not coincide with G^L.

Specker lattice ordered groups have been investigated by Conrad [10], Darnel [12], Conrad and Darnel [14] and by the author [17].

We show that if G is a Specker lattice ordered group, then it is projectable, whence G^L is strongly projectable. Further, G^L is a complete lattice ordered group. We investigate the conditions under which G^L is equal to the Dedekind completion $G^\wedge$ of G.

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2. Preliminaries

For lattice ordered groups we apply the notation as in Conrad [8].

Let G be a lattice ordered group and $X \subseteq G$. We put

$$X^\delta = \{g \in G\colon\ |g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

The set X^δ is a *polar* of G. Each polar is a convex ℓ-subgroup of G. If $x \in G$, then $\{x\}^{\delta\delta}$ is a *principal polar* of G generated by the element x.

A lattice ordered group is *projectable* (*strongly projectable*) if each principal polar (or each polar, respectively) is a direct factor of G.

For each lattice ordered group G there exists a strongly projectable hull G^{SP} of G.

An indexed system $(x_i)_{i\in I}$ of elements of G^+ is called *orthogonal* (or *disjoint*) if $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I. If each nonempty orthogonal indexed system of elements of G has the join in G, then G is said to be *laterally complete.*

For each lattice ordered group G there exists a lateral completion G^L of G.

An element $0 < x \in G$ is *singular* if the interval $[0, x]$ of G is a Boolean algebra. G is a *Specker lattice ordered group* if it is generated as a group by its singular elements.

G is a *complete lattice ordered group* if each nonempty upper bounded subset of G possesses the supremum in G. In such case the corresponding dual condition is also satisfied. (Instead of 'complete' the term 'Dedekind complete' is also used in literature.)

For each archimedean lattice ordered group G there exists its Dedekind completion which will be denoted by $G^\wedge$.

We denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the set of all integers, rationals and reals, respectively; in each of these sets we consider the usual operation $+$ and the usual linear order.

3. Auxiliary results

In this section we assume that G is a projectable lattice ordered group. Thus we can construct the lateral completion G^L of G by the method described in [16].

We recall that G is an ℓ-subgroup of G^L; moreover, we have

3.0. Lemma (cf. [9], [16]). *For each $0 \leqslant y \in G^L$ there exists an orthogonal indexed system $(x_i)_{i\in I}$ of elements of G such that the relation*

$$y = \bigvee_{i\in I} x_i$$

is valid in G^L.

Further, in view of the definition of G^L, for each orthogonal indexed system $(z_j)_{j\in J}$ of elements G^L there exists $y' \in G^L$ such that

$$y' = \bigvee_{j\in J} z_j$$

holds in G^L.

For $y_1, y_2 \in G^L$ we write $y_1 \perp y_2$ if $|y_1| \wedge |y_2| = 0$. Further, for nonempty subsets Y_1, Y_2 of G^L we put $Y_1 \perp Y_2$ if $y_1 \perp y_2$ for each $y_1 \in Y_1$, $y_2 \in Y$. Next, we put

$$Y_1^{\delta_1} = \{h \in G^L\colon\ \{h\} \perp Y_1\}.$$

The following result is easy to verify.

3.1. Lemma. *Let* $\emptyset \neq X_i \subseteq G$ $(i = 1, 2)$, $X_1 \perp X_2$. *Then* $X_1^{\delta\delta} \perp X_2^{\delta\delta}$.

Similarly we have

3.2. Lemma. *Let* $\emptyset \neq Y_i \subseteq G^L$ $(i = 1, 2)$, $Y_1 \perp Y_2$. *Then* $Y_1^{\delta_1\delta_1} \perp Y_2^{\delta_1\delta_1}$.

Now, let $\emptyset \neq X \subseteq G$. Then we obviously have

$$X^{\delta} = X^{\delta_1} \cap G. \tag{1}$$

Put $Y = X^{\delta}$. In view of (1) we get $Y^{\delta} = Y^{\delta_1} \cap G$, whence

$$X^{\delta\delta} = X^{\delta\delta_1} \cap G. \tag{2}$$

Also, (1) yields

$$X^{\delta\delta_1} = (X^{\delta_1} \cap G)^{\delta_1}. \tag{3}$$

3.3. Lemma. *Let* $\emptyset \neq X \subseteq G$. *Then*

$$(X^{\delta_1} \cap G)^{\delta_1} = X^{\delta_1\delta_1}.$$

P r o o f. a) Let $t \in (X^{\delta_1} \cap G)^{\delta_1}$. Hence $t \perp y$ for each $y \in X^{\delta_1} \cap G = X^{\delta}$.

We have to verify that $t \perp z$ for each $z \in X^{\delta_1}$. Without loss of generality it suffices to consider the case $z \geqslant 0$. Then there exists an orthogonal subset $\{y_i\}_{i\in I}$ of G^+ such that

$$z = \bigvee_{i\in I} y_i.$$

For each $i \in I$ we have $y_i \in X^{\delta_1}$, whence $y_i \in X^{\delta}$ and thus $t \perp y_i$. Each lattice ordered group is infinitely distributive, therefore

$$|t| \wedge z = \bigvee_{i\in I}(|t| \wedge y_i) = 0;$$

hence $t \perp z$.

b) Let $t \in X^{\delta_1\delta_1}$. Then $t \perp X^{\delta_1}$ and so $t \perp X^{\delta_1} \cap G$, yielding that $t \in (X^{\delta_1} \cap G)^{\delta_1}$. □

3.4. Lemma. *Let $\emptyset \neq X \subseteq G$. Then $X^{\delta\delta} = X^{\delta_1\delta_1} \cap G$.*

P r o o f. In view of (2) and (3) we have

$$X^{\delta\delta} = (X^{\delta_1} \cap G)^{\delta_1} \cap G,$$

hence 3.3 yields $X^{\delta\delta} = X^{\delta_1\delta_1} \cap G$. □

4. The lateral completion

In this section we continue to assume that G is a projectable lattice ordered group.

Let $\emptyset \neq Y \subseteq G^L$, $A = Y^{\delta_1\delta_1}$. In accordance with the terminology from Section 2 we say that A is a polar in G^L. Our aim is to verify that A is a direct factor of G^L.

The following result is well-known.

4.1. Lemma. *Let H be a lattice ordered group and let A_1 be a convex ℓ-subgroup of H. The following conditions are equivalent:*

(i) *A_1 is a direct factor of H.*

(ii) *Whenever $0 \leqslant h \in H$, then the set*

$$\{x \in A_1 \colon\ x \leqslant h\}$$

has a maximal element.

For each element $y \in G$ we denote $\{y\}^{\delta\delta} = [y]$. Since G is projectable, $[y]$ is a direct factor of G. If $g \in G$, then the component of g in the direct factor $[y]$ will be denoted by $g[y]$.

Let A be as above. The case $A = \{0\}$ is trivial for our purposes; hence we can assume that $A \neq \{0\}$. Then by applying the Axiom of Choice we conclude that there exists an orthogonal set $\{a_i\}_{i\in I}$ of elements of G such that

(i) $0 < a_i \in A$ for each $i \in I$;

(ii) if $b \in G \cap A$ and $b \wedge a_i = 0$ for each $i \in I$, then $b = 0$.

Let $0 \leqslant g \in G$. For each $i \in I$ we put

$$g_i = g[a_i].$$

Then in view of 3.1 we obtain that $(g_i)_{i\in I}$ is an orthogonal indexed system of elements of G. Thus there exists $g^0 \in G^L$ such that

$$g^0 = \bigvee_{i\in I} g_i.$$

For each $i \in I$ we have $g_i \leqslant g$, hence $g^0 \leqslant g$.

According to 3.4,

$$[a_i] = \{a_i\}^{\delta\delta} = \{a_i\}^{\delta_1\delta_1} \cap G.$$

Since $a_i \in A$, we get

$$\{a_i\}^{\delta_1\delta_1} \subseteq A^{\delta_1\delta_1} = A.$$

From $g_i \in [a_i]$ we get $g_i \in A$ for each $i \in I$. It is well-known that each polar is a closed sublattice of the corresponding lattice ordered group; therefore

(1) $$g^0 \in A.$$

Assume that there exists $g^1 \in A$ such that

$$g^0 < g^1 \leqslant g.$$

Denote $-g^0 + g^1 = g^2$. Then $0 < g^2 \in A$. There exists an orthogonal set $\{z_k\}_{k\in K}$ of elements of G such that

$$g^2 = \bigvee_{k\in K} z_k.$$

There is $k(0) \in K$ with $z_{k(0)} > 0$. Clearly $z_{k(0)} \in A$. Hence there exists $i(0) \in I$ such that

$$z_{k(0)} \wedge a_{i(0)} = \overline{a}_{i(0)} > 0.$$

We also have $\overline{a}_{i(0)} \in G \cap A$. Further, $\overline{a}_{i(0)} \in [a_{i(0)}]$ and

$$g^0 < g^0 + \overline{a}_{i(0)} \leqslant g^1 \leqslant g.$$

Then we have

$$g_{i(0)} = g[a_{i(0)}] \geqslant g^0[a_{i(0)}] + \overline{a}_{i(0)}[a_{i(0)}] \geqslant g^0[a_{i(0)}].$$

If $i \in I$, $i \neq i(0)$, then

$$g_i[a_{i(0)}] = 0.$$

From this we easily obtain that

$$g^0[a_{i(0)}] = g_{i(0)}.$$

Further, from the relation $\overline{a}_{i(0)} \in [a_{i(0)}]$ we get

$$\overline{a}_{i(0)}[a_{i(0)}] = \overline{a}_{i(0)}.$$

Thus we have

$$g_{i(0)} \geqslant g_{i(0)} + \overline{a}_{i(0)} > g_{i(0)},$$

which is a contradiction. Hence we have proved

4.2. Lemma. *Let A be a polar of G^L, $0 \leqslant g \in G$. Then the set*

$$\{x \in A\colon\ x \leqslant g\}$$

possesses the greatest element.

Now let A be as above and $0 \leqslant h \in G^L$. Then h can be expressed in the form

$$h = \bigvee_{t \in T} h_t,$$

where $\{h_t\}_{t \in T}$ is an orthogonal subset of elements of G.

Let $t \in T$. In view of 4.2, the set

$$\{x \in A\colon\ x \leqslant h_t\}$$

possesses the greatest element, which will be denoted by h_t^0. Then $(h_t^0)_{t \in T}$ is a disjoint indexed system of elements of G^L. Hence there exists $h^0 \in G^L$ with

$$h^0 = \bigvee_{t \in T} h_t^0.$$

Because A is a closed sublattice of G^L and all h_t^0 belong to A, we conclude that $h^0 \in A$. Further, $h_t^0 \leqslant h_t$ for $t \in T$, thus $h^0 \leqslant h$.

Assume that there exists $h^1 \in A$ with

(2) $$h^0 < h^1 \leqslant h.$$

Denote $h^2 = -h^0 + h^1$. Then we have $0 < h^2 \leqslant h$, hence

$$h^2 = h^2 \wedge h = \bigvee_{t\in T} (h^2 \wedge h_t).$$

Put $h^2 \wedge h_t = \overline{h}_t$ for each $t \in T$. Since $h^2 \in A$, we get $\overline{h}_t \in A$ for each $t \in T$.

There exists $t_{t(0)} \in T$ such that $\overline{h}_{t(0)} > 0$. Then

$$h^0_{t(0)} < h^0_{t(0)} + \overline{h}_{t(0)} \leqslant h^0 + h^2 = h^1 \leqslant h$$

and $h^0_{t(0)} + \overline{h}_{t(0)} \in A$.

For $t \in T$, $t \neq t_0$ we have $h_{t(0)} \perp h_t$. Since $h^0_{t(0)} \leqslant h_{t(0)}$ and $\overline{h}_{t(0)} \leqslant h_{t(0)}$, we get

$$h^0_{t(0)} \perp h_t, \quad \overline{h}_{t(0)} \perp h_t,$$

thus

$$h^0_{t(0)} + \overline{h}_{t(0)} \perp h_t.$$

Then we obtain

$$h^0_{t(0)} + \overline{t}_{t(0)} = (h^0_{t(0)} + \overline{h}_{t(0)}) \wedge h = \bigvee_{t\in t} (h^0_{t(0)} + \overline{h}_{t(0)}) \wedge h_t = (h^0_{t(0)} + \overline{h}_{t(0)}) \wedge h_{t(0)},$$

whence

$$h^0_{t(0)} < h^0_{t(0)} + \overline{h}_{t(0)} \leqslant h_{t(0)}.$$

This relation contradicts the definition of $h^0_{t(0)}$. Therefore the element h^1 with the property as in (2) cannot exist. Hence we have

4.3. Lemma. *Let A be a polar of G^L, $0 \leqslant h \in G^L$. Then the set $\{x \in A\colon\ x \leqslant h\}$ possesses the greatest element.*

Now, 4.1 and 4.3 yield

4.4. Corollary. *Each polar of G^L is a direct factor of G^L.*

Thus we have

4.5. Theorem. *Let G be a projectable lattice ordered group. Then G^L is strongly projectable.*

If H is an ℓ-subgroup of G^L with $G \subseteq H \subset G^L$, then H fails to be laterally complete, hence H is not orthocomplete. Thus we obtain

4.6. Proposition. *Let G be a projectable lattice ordered group. Then G^L is the orthocompletion of G.*

4.7.1. Example. A strongly projectable lattice ordered group need not be laterally complete. Let G_1 be the set of all bounded, integer valued functions on the set $\mathbb{N}$ of all positive integers. Under pointwise operations, G_1 is a lattice ordered group. Moreover, G_1 is strongly projectable, hence $G_1^{\mathrm{SP}} = G_1$. For each $n \in \mathbb{N}$ we define $f_n \in G_1$ by

$$f_n(m) = \begin{cases} m & \text{if} \quad m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(n)_{n\in\mathbb{N}}$ is an orthogonal indexed system of elements of G_1 which has no join in G_1. Thus G_1 fails to be orthogonally complete and so we have $G_1 \subset G_1^L$.

4.7.2. Example. A laterally complete lattice ordered group need not be projectable. Let $G_1 = \mathbb{Z} \times \mathbb{Z}$, $G_2 = \mathbb{R}$ and let G be the lexicographic product $G_2 \circ G_1$. Then G is laterally complete. Let us denote the elements of G as triples (r, z_1, z_2), where $r \in \mathbb{R}$ and $(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$. Let A be the set of all elements of the form $(0, 0, z_2)$, where z_2 runs over the set $\mathbb{Z}$. Then A is a principal polar of G (generated by the element $(0, 0, 1)$), but A fails to be a direct factor of G. Hence G is not projectable.

5. Specker lattice ordered groups

In the present section we assume that G is a Specker lattice ordered group. We denote by $B(G)$ the set of all $x \in G$ such that either $x = 0$ or x is a singular element of G.

5.1. Proposition (cf. [10], [12]). *For any $0 \neq g \in G$, there exists a set of mutually disjoint singular elements $\{s_1, s_2, \ldots, s_n\} \subseteq G$ and non-zero integers $m_1, \ldots, m_n$ such that $g = m_1 s_1 + \ldots m_n s_n$.*

The following assertion is easy to verify (by applying the well-known properties of disjoint elements); the proof will be omitted.

5.2. Lemma. *Let $0 \neq g \in G$ and let us apply the notation as in* 5.1. *Then $g > 0$ if and only if $m_i > 0$ for $i = 1, 2, \ldots, n$.*

Let $0 < g$ be as in 5.1. We want to describe the polars $\{g\}^\delta$ and $\{g\}^{\delta\delta}$. Since each polar is uniquely determined by its positive cone, it suffices to characterize the elements of these polars which belong to G^+.

If $x, y \in G$ and if k_1, k_2 are positive integers, then

$$x \wedge y = 0 \Leftrightarrow k_1 x \wedge k_2 y = 0.$$

In view of the mutual orthogonality of the elements $s_1, s_2, \ldots, s_n$ we have

$$g = m_1 s_1 \vee m_2 s_2 \vee \ldots \vee m_n s_n.$$

Put $s = s_1 \vee \ldots \vee s_n$.

Hence for $0 \leqslant g' \in G$ we have

$$g' \perp g \Leftrightarrow g' \perp s_i \ \ (i = 1, \ldots, n) \Leftrightarrow g' \perp s.$$

Thus we obtain

5.3. Lemma. *Let $g' \in G^+$. Then g' belongs to $\{g\}^\delta$ if and only if $g' \perp s$ and if and only if $g' \perp ks$ for some positive integer k.*

Assume that $0 < g' \in G$. Then g' has an analogous representation as the element g in 5.1; let us use the notation

$$(*) \qquad g' = m'_1 s'_1 + \ldots + m'_{n'} s'_{n'}.$$

Put $I = \{1, 2, \ldots, n'\}$.

Let $i \in I$. Consider the elements s and s'_i. Put

$$u = s \wedge s'_i, \quad v = s \vee s'_i.$$

Since s and s'_i belong to $B(G)$, the elements u and v belong to $B(G)$ as well. Thus the interval $[0, v]$ of G is a Boolean algebra. Let t be the complement of the element u in $[0, v]$. We denote

$$s_i^0 = s \wedge t, \quad t_i = s'_i \wedge t.$$

Then we have

$$s = u \vee s_i^0, \quad u \wedge s_i^0 = 0,$$
$$s'_i = u \vee t_i, \quad u \wedge t_i = 0, \quad s_i^0 \wedge t_i = 0.$$

Let k be a positive integer. We get

$$ks = k(u \vee s_i^0) = k(u + s_i^0) = ku + ks_i^0 = ku \vee ks_i^0,$$

and analogously

$$m_i' s_i' = m_i' u \vee m_i' t_i.$$

In view of the relations

$$k s_i^0 \wedge m_i' t_i = 0, \quad m_i' u \wedge k s_i^0 = 0, \quad k u \wedge m_i' t_i = 0$$

we obtain

(1) $$k s \wedge m_i' s_i' = k u \wedge m_i' u = \min(k, m_i') u.$$

Applying $(*)$ and (1) we get

(2) $$g' \wedge k s = \min(k, m_1')(s \wedge s_1') \vee \ldots \vee \min(k, m_{n'}')(s \wedge s_{n'}').$$

In the proof of the following lemma we apply the notation as above with a slight modification: when considering the elements s and s_i, we write u_i instead of u.

5.4. Lemma. *Let $0 < g' \in G$. Then g' belongs to $\{g\}^{\delta\delta}$ if and only if there is a positive integer k_0 such that $g' \leqslant k_0 s$.*

Proof. Let k be any positive integer. Expressing the element g' as above we obtain

$$g' = m_1' u_1 \vee m_1' t_1 \vee \ldots \vee m_{n'}' u_{n'} \vee m_{n'}' t_{n'}.$$

a) Assume that g' belongs to $\{g\}^{\delta\delta}$. Suppose that there exists $i \in I$ with $t_i > 0$. We have $t_i \wedge s = 0$, thus in view of 5.3 we get $t_i \in \{g\}^{\delta}$. Further, $t_i = g' \wedge t_i$, whence g' cannot belong to $\{g\}^{\delta\delta}$, which is a contradiction. Thus $t_i = 0$ for each $i \in I$. Then we have

$$g' = m_1' u_1 \vee \ldots \vee m_{n'}' u_{n'} = m_1' u_1 + \ldots + m_{n'}' u_{n'}.$$

Put $k_0 = m_1' + \ldots + m_{n'}'$. Since $u_i \leqslant s$ for each $i \in I$, we get $g' \leqslant k_0 s$.

b) Let $g' \leqslant k_0 s$ for some positive integer k_0. Then in view of 5.3, $g' \perp x$ for each $x \in \{g\}^{\delta}$, whence $g' \in \{g\}^{\delta\delta}$. □

Again, let $0 < g' \in G$. Under the notation as above we put

$$\overline{g} = m_1'(s \wedge s_1') + \ldots + m_{n'}'(s \wedge s_{n'}').$$

Then in view of 5.4 we have $\overline{g} \in \{g\}^{\delta\delta}$. Clearly $0 \leqslant \overline{g} \leqslant g'$.

Assume that $0 \leqslant g'' \in \{g\}^{\delta\delta}$, $g'' \leqslant g'$. In view of 5.4 there exists a positive integer k_1 with $g'' \leqslant k_1 s$. Put

$$k = \max\{k_1, m_1', \ldots, m_{n'}'\}.$$

Thus $g'' \leqslant ks$, whence $g'' \leqslant ks \wedge g'$. According to (2) we have $ks \wedge g' = \overline{g}$, thus $g'' \leqslant \overline{g}$. We obtain

5.5. Lemma. *Let* $0 < g' \in G$ *and let* $\overline{g}$ *be as above. Then*

$$\overline{g} = \max\{h \in \{g\}^{\delta\delta}\colon\ 0 \leqslant h \leqslant g'\}.$$

As a corollary, we get

5.6. Theorem. *Each Specker group is projectable.*

Thus in view of 4.5 we have

5.7. Theorem. *If* G *is a Specker group, then* G^L *is strongly projectable.*

6. Dedekind completeness

We start by remarking that, in general, lateral completeness of a lattice ordered group does not imply its Dedekind completeness. E.g., the linearly ordered group $\mathbb{Q}$ is laterally complete, but it is not Dedekind complete. Thus, in general, G^L need not be Dedekind complete.

It is well-known that a lattice ordered group G is Dedekind complete if and only if for each $0 < g \in G$, the interval $[0, g]$ of G is a complete lattice.

6.1. Lemma. *Let* G *be a lattice ordered group,* $0 \leqslant a_i \in G$ $(i = 1, 2, \ldots, n)$. *Assume that all intervals* $[0, a_i]$ *are complete lattices. Then* $[0, a_1 + a_2 + \ldots + a_n]$ *is a complete lattice.*

P r o o f. By induction we need only to prove the assertion for $n = 2$. Assume that $[0, a]$ and $[0, b]$ are complete. The interval $[a, a+b]$ is isomorphic to $[0, b]$, whence it is complete as well. Let $\emptyset \neq X \subseteq [0, a+b]$. For each $x \in X$ we put $x_1 = a \wedge x$, $x_2 = a \vee x$. Then we have

$$x_2 - a = x - x_1,$$

whence $x_2 - a + x_1 = x$. In view of the assumption, there exists $u = \sup\{x_1\colon\ x \in X\}$ in $[0, a]$ and $v = \sup\{x_2\colon\ x \in X\}$ in $[a, a+b]$. Put

$$v - a + u = x^0.$$

Then we have $x \leqslant x^0$ for each $x \in X$. If $y \in [0, a+b]$, $x \leqslant y$ for each $x \in X$, then we put $y_1 = y \wedge a$, $y_2 = y \vee a$. We get $y_1 \geqslant x_1$, $y_2 \geqslant x_2$ for each x, thus $x^0 \leqslant y$. Therefore $\sup X = x^0$ in $[0, a+b]$. Analogously we verify that $\inf X$ exists in $[0, a+b]$. □

6.2. Lemma. *Let B be a Boolean algebra. Then the following conditions are equivalent:*

(i) *B is Dedekind complete.*

(ii) *B is orthogonally complete.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is obvious. The relation (ii) $\Rightarrow$ (i) is a consequence of Theorem 20.1 in Sikorski [21]. (We remark that Sikorski attributes the corresponding result to Smith and Tarski [22].) $\square$

6.3. Proposition. *Let G be a Specker lattice ordered group. Then the following conditions are equivalent:*

(i) *G is Dedekind complete.*

(ii) *Each interval $[0, x]$ with $x \in B(G)$ is complete.*

(iii) *Each interval $[0, x]$ of the lattice $B(G)$ is orthogonally complete.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is obvious. Let (ii) be valid and let $0 < g \in G$. Let us express the element g as in 5.1. Then all intervals $[0, s_i]$ $(i = 1, 2, \ldots, n)$ are complete. According to 6.2, the interval $[0, g]$ is complete. The relation (ii) $\Rightarrow$ (iii) follows from 6.2. $\square$

The orthogonality of elements of a Boolean algebra B is defined analogously as in the case of lattice ordered groups; also, the orthogonal completeness of B is defined in a similar way.

Let G be a Specker lattice ordered group. In view of 5.6, G is projectable. Hence G^L has the properties as in Section 3.

For $g_1, g_2 \in G$ with $g_1 \leqslant g_2$ we have to distinguish between the interval in G with the endpoints g_1, g_2 (this will be denoted by $[g_1, g_2]^1$) and the interval in G^L with the same endpoints (which we denote by $[g_1, g_2]^2$).

6.4. Lemma. *Let $x \in B(G)$. Then $[0, x]^2$ is a Boolean algebra.*

P r o o f. Let $y \in [0, x]^2$. We have to verify that y has a complement in the interval $[0, x]^2$.

According to Section 3, there exists an orthogonal subset $\{x_i\}_{i \in I}$ of elements of $[0, x]^1$ such that the relation

$$y = \bigvee_{i \in I} x_i$$

is valid in $[0, x]^2$. Each element x_i has a complement in the interval $[0, x]^1$ which will be denoted by x_i'. If $i(1)$ and $i(2)$ are distinct elements of I, then

$$x'_{i(1)} \vee x'_{i(2)} = x. \tag{1}$$

For each $i \in I$ we have $-x'_i \in [-x, 0]$ and if $i(1) \neq i(2)$, then

$$(-x'_{i(1)}) \wedge (-x'_{i(2)}) = -x. \tag{2}$$

It is obvious that the interval $[-x, 0]^2$ is isomorphic with the interval $[0, x]^2$ which is orthogonally complete. Hence in view of (2) there exists $z_1 \in [-x, 0]^2$ with

$$\bigvee_{i \in I} (-x'_i) = z_1.$$

Then we have

$$-\Big(\bigvee_{i \in I} (-x'_i)\Big) = \bigwedge_{i \in I} x'_i = -z_1.$$

Put $-z_1 = z$.

Now by easy calculation we obtain that the relations

$$y \vee z = x, \quad y \wedge z = 0$$

are valid in $[0, x]^2$. Hence $[0, x]^2$ is a Boolean algebra. □

6.5. Lemma. *Let $x \in B(G)$. Then the interval $[0, x]^2$ is a complete lattice.*

Proof. This is a consequence of 6.2, 6.4 and of the fact that $[0, x]^2$ is orthogonally complete. □

6.6. Theorem. *Let G be a Specker lattice ordered group. Then G^L is a complete lattice ordered group.*

Proof. Let $0 < g \in G$. We express g as in 5.1. In view of 6.5, all intervals $[0, s_i]^2$ $(i = 1, 2, \ldots, m)$ are complete lattices. Thus in view of 6.1, the interval $[0, g]^2$ is a complete lattice. Then G^L is a complete lattice. □

Since each complete lattice ordered group is strongly projectable, from 6.6 we get an alternative method of obtaining Theorem 5.7.

7. A relation between $G^\wedge$ and G^L

In view of 6.1 we can ask under which condition for a Specker lattice ordered group G the lateral completion G^L coincides with the Dedekind completion $G^\wedge$ of G.

7.1. Proposition. *Let $G \neq \{0\}$ be a Specker lattice ordered group. Then the following conditions are equivalent:*

(i) $G^L = G^\wedge$.

(ii) *Each orthogonal subset of G is finite.*

(iii) *Each orthogonal subset of $B(G)$ is finite.*

(iv) *The set $B(G)$ is finite and each strictly positive element of G exceeds some atom of $B(G)$.*

(v) *G is isomorphic to a direct product of a finite number of linearly ordered groups isomorphic to $\mathbb{Z}$.*

(vi) $G = G^L = G^\wedge$.

We need some lemmas.

7.2. Lemma. *Let B be a generalized Boolean algebra, $B \neq \{0\}$. Then the following conditions are equivalent:*

(i) *Each orthogonal subset of B is finite.*

(ii) *For each $0 < b \in B$ there exists $0 < c \in B$ such that $c \leqslant b$ and c is an atom in B; moreover, the set of atoms of B is finite.*

Proof. Assume that (i) is valid. By way of contradiction, suppose that there exists $0 < b \in B$ such that b exceeds no atom of B.

Thus there exists $0 < x_1 < b$. Let y_1 be the complement of x_1 in the interval $[0, x]$. There exists $x_2 \in B$ with $0 < x_2 < y_1$; let y_2 be the complement of x_2 in the interval $[0, y_1]$. Proceeding in this way and applying the obvious induction we obtain an orthogonal set of elements $x_1, x_2, x_3, \ldots$, with $0 < x_n < x$ for each $n \in \mathbb{N}$. Hence we have arrived at a contradiction. Thus each strictly positive element of B exceeds some atom of B.

Let A_0 be the set of all atoms of B. Since $B \neq \{0\}$, we get $A_0 \neq \emptyset$. It is clear that the set A_0 is orthogonal, thus in view of (i) it must be finite. Hence (ii) holds.

Conversely, let (ii) be valid. Let $\{b_i\}_{i \in I}$ be an orthogonal subset of B such that $0 < b_i$ for each $i \in I$. There exists a set $\{a_i\}_{i \in I}$ such that $a_i \in A_0$ and $a_i \leqslant b_i$ for each $i \in I$. Then the set I must be finite, whence (i) is satisfied. □

7.3. Lemma. *Let H be an archimedean lattice ordered group and let $h_1, h_2, \ldots, h_n$ be atoms of the lattice H^+. Then*

(i) *there exist linearly ordered ℓ-subgroups $X_1, X_2, \ldots, X_n$ of H such that $h_i \in X_i$ for $i = 1, 2, \ldots, n$ and G can be expressed as a direct product*

$$G = X_1 \times X_2 \times \ldots \times A_n \times G_0,$$

where $G_0 = (X_1 \cup X_2 \cup \ldots \cup X_n)^\delta$;

(ii) *for each $i \in I$, X_i is isomorphic to $\mathbb{Z}$.*

Proof. (ii) is a consequence of the results of [18]. Let $i \in I$. Then X_i is an archimedean linearly ordered group, whence it is isomorphic to some ℓ-subgroup of $\mathbb{R}$. Since X_i possesses an atom (namely, h_i), it must be isomorphic to $\mathbb{Z}$. □

Proof of 7.1. (i) ⇒ (ii). Let (i) be valid; by way of contradiction, suppose that (ii) fails to hold. Hence there exists an orthogonal subset $\{a_n\}_{n\in\mathbb{N}}$ of G such that $a_n > 0$ for each $n \in \mathbb{N}$. In view of 5.1, for each $n \in \mathbb{N}$ there exists $0 < s_n^0 \in B(G)$ with $s_n^0 \leqslant a_n$. Thus $\{s_n^0\}_{n\in\mathbb{N}}$ is an orthogonal subset of elements of $B(G)$. We put $b_n = ns_n^0$; we get an orthogonal system $\{b_n\}_{n\in\mathbb{N}}$ in G. Hence there exists $b \in G^L$ such that the relation

$$b = \bigvee_{n\in\mathbb{N}} b_n$$

is valid in G^L. In view of (i), the element b belongs to $G^\wedge$. Thus there must exist $g \in G$ with $g \geqslant b$. Then $g \geqslant b_n$ for each $n \in \mathbb{N}$.

Let us express the element g as in 5.1. Further, put

$$s = s_1 \vee s_2 \vee \ldots \vee s_n,$$
$$k = \max\{m_1, \ldots, m_n\}.$$

Thus we have $g \leqslant ks$, whence $ks \geqslant b_n = ns_n^0$ for each $n \in \mathbb{N}$.

In view of the relation (1) in Section 5, whenever s^1 and s^2 are elements of $B(G)$, then for any positive integers k_1, k_2 we have

$$k_1 s^1 \wedge k_0 s^2 = \min(k_1, k_2)(s^1 \wedge s^2).$$

Thus if $s^1 \leqslant k_2 s^2$, then

$$s^1 = s^1 \wedge k_2 s^2 = \min(1, k_2)(s^1 \wedge s^2) = s^1 \wedge s^2,$$

whence $s^1 \leqslant s^2$.

We apply these facts below.

We have $ks \geqslant ns^0 \geqslant s^0$. Thus $ks \wedge ns^0 = ns^0$.

On the other hand, $s \geqslant s^0$ and

$$ks \wedge ns^0 = \min(k, n)(s \wedge s^0) = \min(k, n)s^0.$$

There exists $n(1) \in \mathbb{N}$ with $n(1) > k$ and for $n(1)$ we obtain

$$n(1)s^0 = ks \wedge n(1)s^0 = ks^0,$$

which is a contradiction. Therefore (ii) is valid.

The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Assume that (iii) holds. Then in view of 7.2, the set A_0 of all atoms of $B(G)$ is nonempty and finite; moreover, each strictly positive element of G exceeds an atom of $B(G)$. Put $A_0 = \{a_1, a_2, \ldots, a_n\}$, $a = a_1 \vee a_2 \vee \ldots \vee a_n$. Let $0 < s \in B(G)$, $u = a \wedge s$. We have $u \in B(G)$. Let t be the complement of u in the interval $[0, s]$. Suppose that $0 < t$. Then there is $a^0 \in A_0$ with $a^0 \leqslant t$. Thus we have

$$a^0 \leqslant a \wedge t = a \wedge (t \wedge s) = (a \wedge s) \wedge t = u \wedge t = 0,$$

which is a contradiction. Then $t = 0$, whence $u = s$, yielding that $s \leqslant a$. We obtain

$$s = (s \wedge a_1) \vee \ldots \vee (s \wedge a_n).$$

If $i \in \{1, 2, \ldots, n\}$, then either $s \wedge a_i = 0$ or $s \wedge a_i = a_i$. Thus a is the greatest element of $B(G)$ and then $B(G)$ is a Boolean algebra generated by its set of atoms A_0. Hence $B(G)$ is finite.

(iv) $\Rightarrow$ (v). Assume that (iv) holds. Then according to 7.3, there exist linearly ordered ℓ-subgroups $X_1, X_2, \ldots, X_n$ of G and a direct factor G_0 of G such that

$$(*) \qquad G = X_1 \times X_2 \times \ldots \times X_n \times G_0$$

and $a_i \in X_i$ for $i = 1, 2, \ldots, n$, where $A_0 = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of G. Suppose that $G_0 \neq \{0\}$. Then there is $0 < g_0 \in G_0$. Since each strictly positive element of G exceeds some atom of G, there is $i \in \{1, 2, \ldots, n\}$ such that $a_i \leqslant g_0$, but this contradicts the relation $(*)$. Thus $G_0 = \{0\}$ and we have

$$G = X_0 \times \ldots \times X_n.$$

Moreover, according to 7.3, all X_i are isomorphic to $\mathbb{Z}$.

(v) $\Rightarrow$ (vi). Assume that (v) holds. Then it is clear that G is complete and orthogonally complete, whence

$$G^{\wedge} = G = G^{L}.$$

It is obvious that (vi) $\Rightarrow$ (i). $\square$

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