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ESTIMATES OF THE REMAINDER IN TAYLOR'S THEOREM USING THE HENSTOCK-KURZWEIL INTEGRAL

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Abstract. When a real-valued function of one variable is approximated by its nth degree Taylor polynomial, the remainder is estimated using the Alexiewicz and Lebesgue p-norms in cases where $f^{(n)}$ or $f^{(n+1)}$ are Henstock-Kurzweil integrable. When the only assumption is that $f^{(n)}$ is Henstock-Kurzweil integrable then a modified form of the nth degree Taylor polynomial is used. When the only assumption is that $f^{(n)} \in C^0$ then the remainder is estimated by applying the Alexiewicz norm to Schwartz distributions of order 1.

Keywords: Taylor's theorem, Henstock-Kurzweil integral, Alexiewicz norm

MSC 2000: 26A24, 26A39

1. Introduction

In this paper the Henstock-Kurzweil integral is used to give various estimates of the remainder in Taylor's theorem in terms of Alexiewicz and Lebesgue p-norms. Let $[a,b]$ be a compact interval in $\mathbb{R}$ and let $f\colon [a,b] \to \mathbb{R}$. Let n be a positive integer. When f is approximated by its nth degree Taylor polynomial about a, the remainder is estimated using the Alexiewicz norm of $f^{(n+1)}$ and p-norms of $f^{(n)}$ in the case when $f^{(n+1)}$ is Henstock-Kurzweil integrable (Theorem 4). When the only assumption is that $f^{(n)}$ is Henstock-Kurzweil integrable then $f^{(n)}$ need not exist at a. In this case we use a modified form of the Taylor polynomial where $f^{(n)}$ is evaluated at a point $x_0 \in [a,b]$ at which $f^{(n)}$ exists. The resulting modified remainder is then estimated in the Alexiewicz and p-norms (Theorem 3). And, when the only assumption is that $f^{(n)} \in C^0$, the Alexiewicz norm is used in the space of Schwartz distributions of

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order 1 to estimate the remainder (Theorem 6). The results extend those in [1]. See also [7] for another form of the remainder.

For the Henstock-Kurzweil integral we have the following version of Taylor's theorem.

Theorem 1. *Let $f\colon [a,b]\to\mathbb{R}$ and let n be a positive integer. If $f^{(n)}\in ACG_*$ then for all $x\in[a,b]$ we have $f(x)=P_n(x)+R_n(x)$ where*

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)(x-a)^k}{k!}\tag{1}$$

and

$$R_n(x)=\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,\mathrm{d}t.\tag{2}$$

A proof under the assumption that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}$ exists nearly everywhere on (a,b) is given in [6]. The general case follows from the Fundamental Theorem: If $F\colon [a,b]\to\mathbb{R}$ and $F\in ACG_*$ then F' exists almost everywhere, F' is Henstock-Kurzweil integrable and $\int_a^x F'=F(x)-F(a)$ for all $x\in[a,b]$. For the wide Denjoy integral the corresponding function space is ACG. If $F\in ACG$ then $\int_a^x F'_{\mathrm{ap}}=F(x)-F(a)$ for all $x\in[a,b]$. Here the integral is the wide Denjoy integral and the approximate derivative is used. All the results proved in the paper have a suitable extension to the wide Denjoy integral. The function spaces ACG_* and ACG are defined in [2] and [4]. Note that $AC\subsetneq ACG_*\subsetneq ACG\subsetneq C^0$. The set of continuous functions that are differentiable nearly everywhere is a proper subset of ACG_*, which is itself a proper subset of the set of continuous functions that are differentiable almost everywhere. The other half of the Fundamental Theorem says that if $f\in\mathcal{HK}$ then $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f=f(x)$ almost everywhere (and certainly at points of continuity of f).

The Alexiewicz norm of an integrable function f is defined

$$\|f\|=\sup_{a\leqslant x\leqslant b}\left|\int_a^x f\right|.\tag{3}$$

We will write $\|f\|_I$ when it needs to be made clear the norm is over interval I. See [5] for a discussion of the Alexiewicz norm and the Henstock-Kurzweil integral. An equivalent norm is $\sup\limits_{(c,d)\subset(a,b)}\left|\int_c^d f\right|$. It leads to similar estimates to those obtained for the Alexiewicz norm in Theorems 3, 4 and 6.

Denote the space of Henstock-Kurzweil integrable functions on $[a,b]$ by $\mathcal{HK}$. Note that if $f^{(n)}\in ACG_*$ then $f^{(n+1)}\in\mathcal{HK}$.

2. Estimates when $f^{(n-1)} \in ACG_*$

If $f^{(n-1)} \in ACG_*$ (i.e., $f^{(n)} \in \mathcal{HK}$) then $f^{(n)}$ need only exist almost everywhere on (a,b). If $f^{(n)}$ exists at $x_0 \in [a,b]$ then we can modify the Taylor polynomial (1) so that the nth derivative is evaluated at x_0 and then obtain estimates on the resulting remainder term. The following lemma is used.

Lemma 2. *If $f\colon [a,b] \to \mathbb{R}$ and $f^{(n-1)} \in ACG_*$ then let $x_0 \in [a,b]$ such that $f^{(n)}$ exists at x_0. Define the modified Taylor polynomial by*

$$P_{n,x_0}(x) = P_{n-1}(x) + \frac{f^{(n)}(x_0)(x-a)^n}{n!} \tag{4}$$

and define the modified remainder by $R_{n,x_0}(x) = f(x) - P_{n,x_0}(x)$. Then for all $x \in [a,b]$

$$R_{n,x_0}(x) = \frac{1}{(n-1)!}\int_a^x [f^{(n)}(t) - f^{(n)}(x_0)](x-t)^{n-1}\,\mathrm{d}t. \tag{5}$$

The proof is the same as for Lemma 1 in [1] (which is false without the proviso that $f^{(n)}$ exists at x_0). Of course we can take x_0 as close to a as we like. See [3] for remainder estimates based on estimates of $|f^{(n)}(t) - f^{(n)}(x_0)|$.

Theorem 2 in [1] gives pointwise estimates of R_n in terms of p-norms of $f^{(n)}$ when $f^{(n-1)} \in AC$ and $f^{(n)} \in L^p$ $(1 \leqslant p \leqslant \infty)$. We have the following analogue when $f^{(n-1)} \in ACG_*$. Note that if $f^{(n-1)} \in ACG_* \setminus AC$ (i.e., $f^{(n)} \in \mathcal{HK} \setminus L^1$) then for each $1 \leqslant p \leqslant \infty$ we have $f^{(n)} \notin L^p$. However, we can use the Alexiewicz norm to estimate R_{n,x_0}.

Theorem 3. *With the notation and assumptions of Lemma 2,*

$$\|R_{n,x_0}\| \leqslant \frac{(b-a)^n}{n!}\|f^{(n)}(\cdot) - f^{(n)}(x_0)\|. \tag{6}$$

For all $x \in [a,b]$,

$$|R_{n,x_0}(x)| \leqslant \frac{(x-a)^{n-1}}{(n-1)!}\|f^{(n)}(\cdot) - f^{(n)}(x_0)\|_{[a,x]}. \tag{7}$$

And,

$$\|R_{n,x_0}\|_p \leqslant \begin{cases} \dfrac{(b-a)^{n-1+1/p}}{(n-1)!\,[(n-1)p+1]^{1/p}}\|f^{(n)}(\cdot) - f^{(n)}(x_0)\|, & 1 \leqslant p < \infty, \\ \dfrac{(b-a)^{n-1}}{(n-1)!}\|f^{(n)}(\cdot) - f^{(n)}(x_0)\|, & p = \infty. \end{cases} \tag{8}$$

P r o o f. Let $a \leqslant c \leqslant b$. Using Lemma 2 and the reversal of integrals criterion in [2, Theorem 57, p. 58],

$$\begin{aligned} \int_a^c R_{n,x_0} &= \frac{1}{(n-1)!}\int_a^c [f^{(n)}(t) - f^{(n)}(x_0)] \int_t^c (x-t)^{n-1}\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{(c-a)^n}{n!}\int_a^\xi [f^{(n)}(t) - f^{(n)}(x_0)]\,\mathrm{d}t \quad \text{for some } \xi \in [a,c]. \end{aligned} \tag{9}$$

Equation (9) comes from the second mean value theorem for integrals [2]. Taking the supremum over $c \in [a,b]$ now gives (6).

Similarly,

$$\begin{aligned} |R_{n,x_0}(x)| &= \frac{(x-a)^{n-1}}{(n-1)!}\left|\int_a^\xi [f^{(n)}(t) - f^{(n)}(x_0)]\,\mathrm{d}t\right| \quad \text{for some } \xi \in [a,x] \\ &\leqslant \frac{(x-a)^{n-1}}{(n-1)!}\|f^{(n)}(\cdot) - f^{(n)}(x_0)\|. \end{aligned} \tag{10}$$

This gives (7). The other estimates in (8) follow from this. □

An alternative approach in Theorem 3 is to assume $f^{(n-1)}$ is ACG_* on $[a,b]$ and $f^{(n)}$ exists at a. Then equation (6) is replaced with $\|R_n\| \leqslant (b-a)^n\|f^{(n)}(\cdot) - f^{(n)}(a)\|/n!$ with similar changes in (7) and (8). This is Young's expansion theorem [8, p. 16] but with a more precise error estimate.

3. Estimates when $f^{(n)} \in ACG_*$

Corollary 1 in [1] gives a pointwise estimate of R_n in terms of p-norms of $f^{(n+1)}$ when $f^{(n)} \in AC$ and $f^{(n+1)} \in L^p$. When $f^{(n)} \in ACG_* \setminus AC$ (i.e., $f^{(n+1)} \in \mathcal{HK} \setminus L^1$) then for no $1 \leqslant p \leqslant \infty$ do we have $f^{(n+1)} \in L^p$. But, we can estimate R_n using the Alexiewicz norm of $f^{(n+1)}$ and p-norms of $f^{(n)}$.

Theorem 4. *If $f\colon [a,b] \to \mathbb{R}$ such that $f^{(n)} \in ACG_*$ then*

$$\|R_n\| \leqslant \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n+1)}\|. \tag{11}$$

For all $x \in [a,b]$,

$$|R_n(x)| \leqslant \frac{(x-a)^n}{n!}\|f^{(n+1)}\|_{[a,x]}. \tag{12}$$

And,

$$\|R_n\|_p \leqslant \begin{cases} \dfrac{(b-a)^{n+1/p}}{n!\,(np+1)^{1/p}}\|f^{(n+1)}\|, & 1 \leqslant p < \infty, \\ \dfrac{(b-a)^n}{n!}\|f^{(n+1)}\|, & p = \infty. \end{cases} \tag{13}$$

Also,

$$\|R_n\|_p \leqslant \begin{cases} \dfrac{(b-a)^{n+1/p}}{n!\,(np+1)^{1/p}}|f^{(n)}(a)| + A_i, & 1 \leqslant p < \infty, \\ \dfrac{(b-a)^n}{n!}|f^{(n)}(a)| + \dfrac{(b-a)^n}{n!}\|f^{(n)}\|_\infty, & p = \infty, \end{cases} \tag{14}$$

where $1/\alpha + 1/\beta = 1$ *and*

$$\begin{aligned} A_1 &= \frac{(b-a)^{n-1+1/p}\|f^{(n)}\|}{(n-1)!\,[(n-1)p+1]^{1/p}}, \quad \textit{for } n \geqslant 1, \\ A_1 &= \|f(\cdot) - f(a)\|_p, \quad \textit{for } n = 1, \\ A_2 &= \frac{(b-a)^{n-1+1/p+1/\beta}\|f^{(n)}\|_\alpha}{(n-1)!\,[(n-1)\beta+1]^{1/\beta}[(n-1+1/\beta)p+1]^{1/p}}, \quad \textit{for } 1 < \alpha \leqslant \infty, \\ A_2 &= \frac{(b-a)^{n-1+1/p}\|f^{(n)}\|_1}{(n-1)!\,[(n-1)p+1]^{1/p}}, \quad \textit{for } \alpha = 1, \\ A_3 &= \frac{(b-a)^{1-1/p}}{(n-1)!\,[(n-1)p+1]^{1/p}}\left(\int_a^b |f^{(n)}(t)|^p(b-t)^{(n-1)p+1}\,dt\right)^{1/p}, \\ A_4 &= \frac{(b-a)^{n(1-1/p)}}{n!}\left(\int_a^b |f^{(n)}(t)|^p(b-t)^n\,\mathrm{d}t\right)^{1/p}. \end{aligned}$$

P r o o f. The proof of (11) is very similar to the proof of (6), except that we begin with the remainder in the form of (2).

Using the second mean value theorem, we have

$$|R_n(x)| = \frac{(x-a)^n}{n!}\left|\int_a^\xi f^{(n+1)}(t)\,\mathrm{d}t\right| \quad \text{for some } \xi \in [a, x]. \tag{15}$$

The estimates in (12) and (13) now follow directly.

Integrate (2) by parts to get

$$R_n(x) = -\frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t. \tag{16}$$

Then

$$\|R_n\|_p \leqslant \frac{|f^{(n)}(a)|}{n!}\, I_1^{1/p} + \frac{1}{(n-1)!}\, I_2^{1/p} \tag{17}$$

where

$$I_1 = \int_a^b (x-a)^{np}\,\mathrm{d}x = \frac{(b-a)^{np+1}}{np+1} \tag{18}$$

and

$$I_2 = \int_a^b \left| \int_a^x f^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t \right|^p \mathrm{d}x. \tag{19}$$

A_1 is obtained from I_2 using the second mean value theorem and A_2 using Hölder's inequality.

Writing

$$I_2 \leqslant \int_a^b \left(\int_a^x |f^{(n)}(t)|(x-t)^{n-1} \frac{\mathrm{d}t}{x-a} \right)^p (x-a)^p\,\mathrm{d}x, \tag{20}$$

Jensen's inequality and Fubini's theorem give

$$I_2 \leqslant \int_a^b |f^{(n)}(t)|^p \int_t^b (x-t)^{(n-1)p}(x-a)^{p-1}\,\mathrm{d}x\,\mathrm{d}t \tag{21}$$

$$\leqslant \frac{(b-a)^{p-1}}{(n-1)p+1} \int_a^b |f^{(n)}(t)|^p (b-t)^{(n-1)p+1}\,\mathrm{d}t. \tag{22}$$

From this we obtain A_3.

Using Jensen's inequality in the form

$$I_2 \leqslant \int_a^b \left(\int_a^x |f^{(n)}(t)| \frac{(x-t)^{n-1} n\,\mathrm{d}t}{(x-a)^n} \right)^p \frac{(x-a)^{np}}{n^p}\,\mathrm{d}x \tag{23}$$

$$\leqslant \frac{1}{n^{p-1}} \int_a^b \int_a^x |f^{(n)}(t)|^p (x-t)^{n-1}(x-a)^{n(p-1)}\,\mathrm{d}t\,\mathrm{d}x, \tag{24}$$

we can apply Fubini's theorem to get

$$I_2 \leqslant \frac{(b-a)^{n(p-1)}}{n^p} \int_a^b |f^{(n)}(t)|^p (b-t)^n\,\mathrm{d}t, \tag{25}$$

which gives A_4. The case $p = \infty$ follows directly from (16). □

Note that A_2, A_3 and A_4 all lead to estimates of form $A_k \leqslant C_{n,p}(b-a)^n\|f^{(n)}\|_p$ $(k=2,3,4)$ where $C_{n,p}$ is independent of a, b and f.

The integral over x in (21) can be evaluated using hypergeometric functions. However, this does not markedly improve the estimate for A_3. Similarly with A_4.

In (12) we have $R_n(x) = o[(x-a)^n]$ as $x \to a$.

Note that if $f^{(n)} \in ACG_*$ then $\|f^{(n+1)}\| = \max_{a\leqslant x\leqslant b} |f^{(n)}(x) - f^{(n)}(a)|$. This affects how the remainder is written in Theorems 3 and 4.

Example 5. Let $0 < a_n \leqslant 1$ be a sequence that decreases to 0. Let b_n be a positive sequence that decreases to 0 such that the intervals $(a_n - b_n, a_n + b_n)$ are disjoint. For this it suffices to take $b_n \leqslant \min([a_{n-1} - a_n]/2, [a_n - a_{n+1}]/2)$. Let $f_n(x) = (x - a_n + b_n)^2(x - a_n - b_n)^2$ for $|x - a_n| \leqslant b_n$ and 0, otherwise. Let $\alpha > 0$ and define $f(x) = \sum n^\alpha f_n(x)$. For $|x - a_n| < b_n$ we have $f_n'(x) = 4(x - a_n + b_n)(x - a_n - b_n)(x - a_n)$ and $f_n''(x) = 4[3(x - a_n)^2 - b_n^2]$. Suppose $b_n = o(a_n)$ as $n \to \infty$ and $n^\alpha b_n^3 \to 0$. Then $\max_{|x-a_n|\leqslant b_n} |f_n'(x)| = O(b_n^3)$ so $f \in C^1_{[0,1]}$ but $f \notin C^2_{[0,1]}$. If $\sum n^\alpha b_n^3 = \infty$ then $f'' \in \mathcal{HK} \setminus L^1$. Let $a = 0$ in Theorems 3 and 4.

(i) If $n^\alpha b_n^3/a_n \not\to 0$ then $f''(0)$ does not exist. An example of this case is $a_n = 1/n$, $b_n = c\,n^{-\beta}$ for $\beta \geqslant 2$ and small enough c, and $3\beta - 1 \leqslant \alpha < 3\beta$. Let $x_0 \in (0,1)\setminus\{a_n \pm b_n\}_{n\in\mathbb{N}}$. Then $f''(x_0)$ exists and we have the modified second degree Taylor polynomial $P_{2,x_0}(x) = f(0) + f'(0)x + f''(x_0)x^2/2 = f''(x_0)x^2/2$. The modified remainder is $R_{2,x_0}(x) = \int_0^x [f''(t) - f''(x_0)](x-t)\,\mathrm{d}t$. If we take $x_0 = a_n \pm b_n/\sqrt{3}$ for some n then $f''(x_0) = 0$ and we have $\|f''(\cdot) - f''(x_0)\|_{[0,x]} \leqslant \max_{0\leqslant y\leqslant 1/\lfloor 1/x\rfloor} |f'(y)| = \sup_{n\geqslant\lfloor 1/x\rfloor} n^\alpha |f_n'(a_n \pm b_n/\sqrt{3})| = (8c^3/3\sqrt{3}) \sup_{n\geqslant\lfloor 1/x\rfloor} n^{\alpha-3\beta} = (8c^3/3\sqrt{3})\lfloor 1/x\rfloor^{\alpha-3\beta}$. This allows us to obtain all of the estimates in Theorem 3. Note that $|R_{2,x_0}(x)| = O(x\lfloor 1/x\rfloor^{\alpha-3\beta}) = O(x^{1-\alpha+3\beta})$ as $x \to 0$ (and $1 < 1 - \alpha + 3\beta \leqslant 2$).

(ii) If $n^\alpha b_n^3/a_n \to 0$ then $f''(0)$ exists. An example is $0 < \alpha < 3\beta - 1$ with a_n and b_n as in (i). This gives $f'' \in L^1$. Now $P_{2,0}(x) = 0$ and $R_{2,0}(x) = \int_0^x f''(t)(x-t)\,\mathrm{d}t$. The quantity $\|f''\|_{[0,x]}$ can be estimated as in (i).

(iii) We can use Theorem 4 with $n = 1$ to get $P_1(x) = 0$ and $R_1(x) = \int_0^x f''(t)(x-t)\,\mathrm{d}t$. This leads to the same estimates for $\|f''\|_{[0,x]}$ as in (i) in the case $x_0 = a_n \pm b_n/\sqrt{3}$.

4. Estimates when $f^{(n)} \in C^0$

We now show that (11) continues to hold when the only assumption is that $f^{(n)} \in C^0$. Under the Alexiewicz norm, the space of Henstock-Kurzweil integrable functions is not complete. It's completion is the subspace of distributions that are the distributional derivative of a continuous function, i.e., distributions of order 1 (see [5]). Thus, if f is in the completion of $\mathcal{HK}$ then $f \in \mathscr{D}'$ (Schwartz distributions) and there is a continuous function F, vanishing at a, such that $F'(\varphi) = -F(\varphi') = -\int_a^b F\varphi' = f(\varphi)$ for all test functions $\varphi \in \mathscr{D} = \{\varphi\colon [a,b] \to \mathbb{R}\colon \varphi \in C^\infty$ and $\operatorname{supp}(\varphi) \subset (a,b)\}$. And, we can compute the Alexiewicz norm of f via $\|f\| = \max_{x\in[a,b]} |F(x)| = \|F\|_\infty$.

Theorem 6. *If $f\colon [a,b] \to \mathbb{R}$ such that $f^{(n)} \in C^0$ then*

$$\|R_n\| \leqslant \frac{(b-a)^{n+1}}{(n+1)!}\|f^{(n+1)}\| = \frac{(b-a)^{n+1}}{(n+1)!} \max_{a\leqslant x\leqslant b} |f^{(n)}(x) - f^{(n)}(a)|. \tag{26}$$

Proof. From Lemma 2, $R_n(x) = (n-1)!^{-1}\int_a^x [f^{(n)}(t) - f^{(n)}(a)](x-t)^{n-1}\,\mathrm{d}t$. Integrating x from a to y and reversing orders of integration gives $\int_a^y R_n = n!^{-1}\int_a^y [f^{(n)}(t) - f^{(n)}(a)](y-t)^n\,\mathrm{d}t$. Equation (26) now follows. □

Example 7. Let g be continuous and nowhere differentiable on $[0,1]$. Let $n \geqslant 1$ and define $f(x) = (n-1)!^{-1}\int_0^x g(t)(x-t)^{n-1}\,\mathrm{d}t$. By differentiating under the integral and then using the Fundamental Theorem, $f^{(n)} = g \in C^0$ but $f^{(n)} \notin ACG_*$. We have $R_n(x) = (n-1)!^{-1}\int_0^x [g(t) - g(0)](x-t)^{n-1}\,\mathrm{d}t$ and $\|R_n\| \leqslant (n+1)!^{-1}\max_{0\leqslant x\leqslant 1}|g(x) - g(0)|$.

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