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AFFINE COMPLETENESS AND LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF ABELIAN LATTICE ORDERED GROUPS

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Abstract. In this paper it is proved that an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product is never affine complete.

Keywords: Abelian lattice ordered group, lexicographic product decomposition, affine completeness

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1. Introduction

The problem proposed by Kaarli and Pixley (cf. [5, Problem 5.6.19]) on the existence of a nontrivial affine complete lattice ordered group remains open. We remark that this problem was formulated already in [2].

Let $\mathscr{G}_0$ be the class of all nonzero lattice ordered groups. In order to arrive nearer to the solution of the problem mentioned it seems to be useful to describe "large" areas S in $\mathscr{G}_0$ such that no affine complete lattice ordered group can exist in S.

Some types of lattice ordered groups which fail to be affine complete have been described by Kaarli and Pixley [5], the author and Csontóová [4], and the author [2], [3].

In [2] it has was proved that if G is an abelian lattice ordered group which can be expressed as a nontrivial direct product, then G is not affine complete. In [3], this result was generalized to lattice ordered groups which need not be abelian. The corresponding result of [5] also deals with a certain form of direct product decompositions. Let G be a nonzero lattice ordered group; in [2] it was shown that if

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G is complete, then it is not affine complete. An analogous result was proved in [4] for the case when G is abelian and projectable.

Assume that G is an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product. In this paper we define the notion of a regular ℓ-subgroup of G. We prove that if H is a regular ℓ-subgroup of G, then H is not affine complete. In particular, G is not affine complete.

2. Preliminaries

Throughout the paper, G denotes an abelian lattice ordered group. Let $P(G)$ be the set of all polynomials over G. If for each mapping $f\colon G^n \to G$ such that $n \in \mathbb{N}$ and f is compatible with all congruence relations on G we have $f \in P(G)$, then G is called affine complete.

For the sake of completeness and for fixing the notation we recall the definition of the lexicographic product decomposition of G (cf., e.g., Fuchs [1]).

Let I be a linearly ordered set and for each $i \in I$ let G_i be a lattice ordered group such that, whenever i fails to be the greatest element of I, then G_i is linearly ordered. The direct product of groups G_i will be denoted by G^0. The elements of G^0 are written in the form $g = (g_i)_{i \in I}$; g_i is the *component* of g in G_i. We put

$$s(g) = \{i \in I\colon\ g_i \neq 0\}.$$

If $s(g) \neq \emptyset$, then $s(g)$ is linearly ordered (as a subset of I).

Let G be the set of all $g \in G^0$ such that either $g = 0$ or the linearly ordered set $s(g)$ is well-ordered. Then G is a subgroup of the group G^0.

For $g \in G$ we put $g > 0$ if $g \neq 0$ and $g_{i(0)} > 0$, where $i(0)$ is the least element of $s(g)$. Then G turns out to be a lattice ordered group. We denote

$$G = \Gamma_{i \in I} G_i. \tag{1}$$

G is said to be the lexicographic product of lattice ordered groups G_i.

Assume that $G \neq \{0\}$. Then all G_i with $G_i = \{0\}$ can be omitted in (1). Hence without loss of generality we can suppose that $G_i \neq \{0\}$ for each $i \in I$. If this is satisfied and $\operatorname{card} I > 1$, then we say that the lexicographic product decomposition (1) of G is nontrivial.

Let $i(1) \in I$ and let $g^{i(1)}$ be a fixed element of $G_{i(1)}$. We denote by $\bar{g}^{i(1)}$ the element of G^0 such that

$$(\bar{g}^{i(1)})_i = \begin{cases} g^{i(1)} & \text{if } i = i(1), \\ 0 & \text{otherwise.} \end{cases}$$

Then we clearly have $\bar{g}^{i(1)} \in G$.

Let H be an ℓ-subgroup of G such that $\overline{g}^{i(1)} \in H$ whenever $i(1) \in I$ and $g^{i(1)} \in G_{i(1)}$. Under this assumption we say that H is a regular ℓ-subgroup of G.

2.1. Lemma (Cf. [4, Lemma 1.1])**.** *Let G be an abelian lattice ordered group and let $p(x) \in P(G)$ be such that $p(x)$ fails to be a constant. There exist $a, x_0 \in G$ and an integer n such that, whenever $x_1 \in G$ and $x_1 \geqslant x_0$, then $p(x_1) = a + nx_1$.*

3. Regular ℓ-subgroups

In this section we assume that $G \neq \{0\}$ is an abelian lattice ordered group. Further, we suppose that the relation (1) from Section 2 is valid and that H is a regular ℓ-subgroup of G.

3.1. Lemma. *Let $0 \neq x \in H$, $i(0) = \min s(x)$. Then*

(1) $$-2|x| < \overline{x}_{i(0)} < 2|x|.$$

Proof. a) At first suppose that $i(0)$ is not the maximal element of I. Then either $\overline{x}_{i(0)} > 0$ or $\overline{x}_{i(0)} < 0$.

Assume that the first of these possibilities is valid. Then $x > 0$, whence $|x| = x$ and $2|x| = 2x$. Further, $i(0) \in \min s(2x)$ and $\overline{2x}_{i(0)} = 2\overline{x}_{i(0)}$. Since

$$-2\overline{x}_{i(0)} < \overline{x}_{i(0)} < 2\overline{x}_{i(0)}, \quad (\overline{x}_{i(0)})_{i(0)} = x_{i(0)},$$

we get

$$-2|x| < \overline{x}_{i(0)} < 2|x|.$$

The case $\overline{x}_{i(0)} < 0$ can be treated analogously.

b) Now suppose that $i(0)$ is the greatest element of I. Then we have $\overline{x}_{i(0)} = x$ and then it suffices to apply the well-known relation

$$-2|x| < x < 2|x|.$$

□

3.2. Lemma. *Let $0 \neq x \in H$, $i(1) \in I$. Then*

$$-2|x| < \bar{x}_{i(1)} < 2|x|, \tag{2}$$

$$-2|x| < -\bar{x}_{i(1)} < 2|x|. \tag{3}$$

Proof. If $i(1) = i(0) = \min s(x)$, then from (1) we conclude that both (2) and (3) are valid.

Let $i(1) < i(0)$. Then $\bar{x}_{i(1)} = 0$, whence (2) and (3) hold. Finally, let $i(1) > i(0)$. Then we have

$$-|\bar{x}_{i(0)}| < \bar{x}_{i(1)} < |\bar{x}_{i(0)}|,$$
$$-|\bar{x}_{i(0)}| < -\bar{x}_{i(1)} < |\bar{x}_{i(0)}|,$$

whence according to 3.1, the relations (2) and (3) are satisfied. □

3.3. Lemma. *Let A be an ℓ-ideal of H and $x \in A$. Then for each $i(1) \in I$ we have $\bar{x}_{i(1)} \in A$.*

Proof. From $x \in A$ we obtain $|x| \in A$ and $2|x| \in A$, $-2|x| \in A$. Since A is a convex subset of H, in view of 3.2 we conclude that $\bar{x}_{i(1)} \in A$. □

Let $i(1) \in I$. Define a mapping $f\colon H \to H$ by putting

$$f(x) = \bar{x}_{i(1)} \quad \text{for each } x \in H.$$

3.4. Lemma. *The mapping f is compatible with all congruence relations on H.*

Proof. Let $\equiv$ be a congruence relation on H. There exists an ℓ-ideal A of H such that $\equiv$ is determined by A.

Let $x, y \in H$. Suppose that $x \equiv y$. This means that $x - y \in A$. Put $x - y = z$. In view of 3.3 we get $\bar{z}_{i(1)} \in A$. Clearly

$$\bar{z}_{i(1)} = \overline{(x-y)}_{i(1)} = \bar{x}_{i(1)} - \bar{y}_{i(1)} = f(x) - f(y).$$

Hence $f(x) - f(y) \in A$ and thus $f(x) \equiv f(y)$. □

3.5. Theorem. *Let $i(1) \in I$ and let $f(x)$ be as above. Then $f(x) \notin P(H)$.*

Proof. By way of contradiction, assume that there exists $p(x) \in P(H)$ such that $p(x) = f(x)$. Then $p(x)$ fails to be a constant. We apply Lemma 2.1 for H. Let a, x_0 and n be as in 2.1.

Since the lexicographic product decomposition (1) is nontrivial, there exists $i(2) \in I$ with $i(2) \neq i(1)$. It is easy to verify that there exists $z \in G$ with $z_{i(2)} > 0$; then $0 < \overline{z}_{i(2)} \in H$.

Further, choose $x_1 \in H$ with $x_1 \geqslant x_0 \vee 0$. If $(x_1)_{i(2)} = 0$, then we replace x_1 by the element $x_2 = x_1 + \overline{z_{i(2)}}$. We have $x_1 < x_2 \in H$ and $(x_2)_{i(2)} \neq 0$.

Thus without loss of generality we can suppose that $(x_1)_{i(2)} \neq 0$. We obtain

$$\overline{(x_1)}_{i(1)} = a + nx_1.$$

Similarly, taking $2x_1$ instead of x_1 we get

$$\overline{(2x_1)}_{i(1)} = a + n \cdot 2x_1.$$

Since $\overline{(2x_1)}_{i(1)} = 2\overline{(x_1)}_{i(1)}$, we have

$$\overline{(x_1)}_{i(1)} = nx_1.$$

But

$$0 = \left(\overline{(x_1)}_{i(1)}\right)_{i(2)}, \quad 0 \neq (nx_1)_{i(2)},$$

and thus we have arrived at a contradiction. □

3.6. Theorem. *Let G be an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product. Assume that H is a regular ℓ-subgroup of G. Then H fails to be affine complete.*

Proof. This is a consequence of 3.4 and 3.5. □

3.7. Corollary. *Let G be an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product. Then G is not affine complete.*

We conclude by remarking that if G is a nontrivial lexicographic product, then

(i) G is not complete;

(ii) G is not projectable;

(iii) G is directly indecomposable.

Hence the affine incompleteness of G does not follow from the results of [2], [3], [4].

The condition applied in [5] when investigating affine incompleteness of a lattice ordered group G was as follows:

(α) G is a direct product of a nonzero subdirectly irreducible lattice ordered group and any lattice ordered group.

It is easy to construct a lattice ordered group G such that G is a nontrivial lexicographic product and G fails to be subdirectly irreducible. Therefore 3.7 is not a consequence of the mentioned result of [5].

References

[1] *L. Fuchs*: Partially Ordered Algebraic Systems. Pergamon Press, Oxford, 1963.

[2] *J. Jakubík*: Affine completeness of complete lattice ordered groups. Czechoslovak Math. J. *45* (1995), 571–576.

[3] *J. Jakubík*: On the affine completeness of lattice ordered groups. Czechoslovak Math. J. *54* (2004), 423–429.

[4] *J. Jakubík and M. Csontóová*: Affine completeness of projectable lattice ordered groups. Czechoslovak Math. J. *48* (1998), 359–363.

[5] *K. Kaarli and A. F. Pixley*: Polynomial Completeness in Algebraic Systems. Chapman-Hall, London-New York-Washington, 2000.

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