Tunay Bilgin

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LACUNARY STRONG (A_σ, p)-CONVERGENCE

Tunay Bilgin, Van

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Abstract. The definition of lacunary strongly convergence is extended to the definition of lacunary strong (A_σ, p)-convergence with respect to invariant mean when A is an infinite matrix and $p = (p_i)$ is a strictly positive sequence. We study some properties and inclusion relations.

Keywords: lacunary sequence, invariant convergence, infinite matrix

MSC 2000: 40A05, 40F05

1. Introduction

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if (i) $\varphi(x) \geqslant 0$ when the sequence $x = (x_n)$ satisfies $x_n \geqslant 0$ for all n, (ii) $\varphi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and (iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in l_\infty$. The mapping φ is assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus φ extends the limit functional to c, the space of convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x \in c$.

Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequence all of whose σ-means are equal. If σ is the translation mapping $n \to n + 1$, a σ-mean is often called a Banach limit (see [1]) and V_σ is the set of almost convergent sequences (see [7]). $[V_\sigma]$ denotes the set of all strongly σ-convergent sequences (see [8]). It is easy to see that $[V_\sigma]$ is a proper closed linear subspace of V_σ. Furthermore, $[V_\sigma]$ contains c.

Several authors (e.g. in [2], [8], [11], [12]) have studied invariant convergent sequences.

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$ we mean an increasing sequence of positive integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. The sequence space of lacunary strongly convergent sequences N_θ was defined in [6] as follows:

$$N_\theta = \left\{ x = (x_i) \colon \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

Lacunary convergent sequences have been studied most recently in [3], [4], [5], [10], and [11].

The purpose of this paper is to introduce and study a concept of lacunary strong (A_σ, p)-convergence.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax_{\sigma^i(n)} = (A_{\sigma^i(n)}(x))$ if $A_{\sigma^i(n)}(x) = \sum\limits_{k=1}^{\infty} a_{ik} x_{\sigma^k(n)}$ converges for all n and i.

For $\sigma(n) = n+1$ we write $Ax_{i+n} = (A_{i+n}(x))$ and $A_{i+n}(x) = \sum\limits_{k=1}^{\infty} a_{ik} x_{n+k}$ for $Ax_{\sigma^i(n)} = (A_{\sigma^i(n)}(x))$ and $A_{\sigma^i(n)}(x) = \sum\limits_{k=1}^{\infty} a_{ik} x_{\sigma^k(n)}$, respectively.

2. Lacunary strongly (A_σ, p)-convergent sequences

We now introduce the generalizations of lacunary strongly convergent sequences and investigate some inclusion relations.

Definition 1. Let $A = (a_{ik})$ be an infinite matrix of complex numbers and $p = (p_i)$ a sequence of strictly positive real numbers. We define spaces

$$\sigma_\theta[A,p] = \left\{ x = (x_i) \colon \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |Ax_{\sigma^i(n)} - s|^{p_i} = 0, \text{uniformly in } n, \text{ for some } s \right\},$$

$$\sigma_\theta^0[A,p] = \left\{ x = (x_i) \colon \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |Ax_{\sigma^i(n)}|^{p_i} = 0, \text{uniformly in } n \right\}.$$

A sequence x of real or complex numbers is said to be lacunary strongly (A_σ, p)-convergent to the value s if $x \in \sigma_\theta[A,p]$. Then we write $x_k \to s[\sigma_\theta[A,p]]$.

If $p_i = 1$ for all i, we write $\sigma_\theta[A]$ and $\sigma_\theta^0[A]$ for $\sigma_\theta[A,p]$ and $\sigma_\theta^0[A,p]$, respectively. If A is a unit matrix we write σ_θ and σ_θ^0 for $\sigma_\theta[A]$ and $\sigma_\theta^0[A]$, respectively. Hence σ_θ is the same as the space L_θ of [1]. For $\sigma(n) = n+1$, the space σ_θ is the same as M_θ, the space of lacunary strongly almost convergent sequences (see [2]).

The following inequality will be used frequently throughout the paper:

$$|a_i + b_i|^{p_i} \leqslant \max(1, 2^{H-1})(|a_i|^{p_i} + |b_i|^{p_i}) \tag{1}$$

where a_i, b_i are complex numbers and $H = \sup p_i < \infty$.

It is easy to see that $\sigma_\theta[A,p]$ and $\sigma_\theta^0[A,p]$ are linear spaces. We consider only $\sigma_\theta[A,p]$.

Suppose that $x_i \to s^1$ and $y_i \to s^2$ in $\sigma_\theta[A,p]$ and that a, b are in C, the set of complex numbers. Then there exist integers T_a and T_b such that $|a| \leqslant T_a$ and $|b| \leqslant T_b$. From (1) we have

$$\begin{aligned} h_r^{-1}\sum_{i\in I_r}|aAx_{\sigma^i(n)}+bAy_{\sigma^i(n)}-(as^1+bs^2)|^{p_i} &\leqslant \max\{1,2^{H-1}\}\\ &\times\Bigl[(T_a)^H h_r^{-1}\sum_{i\in I_r}|Ax_{\sigma^i(n)}-s^1|^{p_i}+(T_b)^H h_r^{-1}\sum_{i\in I_r}|Ay_{\sigma^i(n)}-s^2|^{p_i}\Bigr]. \end{aligned}$$

This implies that $ax+by\in\sigma_\theta[A,p]$.

The lacunary sequence $\theta^! = (k_r^!)$ is called a lacunary refinement of $\theta=(k_r)$ if $(k_r)\subset(k_r^!)$ (see [4]).

Now we establish the inclusion relations among $\sigma_\theta[A,p]$ for different θ.

Theorem 2. (i) *Let $\theta^!$ be a refinement of θ. Then $\sigma_{\theta^!}[A,p]\subset\sigma_\theta[A,p]$.*

(ii) *Let L be a set of lacunary sequences closed under arbitrary union and intersection. Writing $\alpha=\bigcup_{\theta\in L}\theta$ and $\gamma=\bigcap_{\theta\in L}\theta$ we have $\sigma_\alpha[A,p]\subset\sigma_\theta[A,p]\subset\sigma_\gamma[A,p]$ for all $\theta\in L$.*

Proof. Let $x\in\sigma_{\theta^!}[A,p]$ and let us suppose that there is a finite number of points $\theta^!=(k_r^!)$ in the interval $I_r=(k_{r-1},k_r]$. We assume for simplicity that there is exactly one point $k_r^!$ of $\theta^!$ in the interval I_r, that is, $k_{r-1}=k_{j-1}^!<k_j^!<k_{j+1}^!=k_r$. Now, from (1) we have

$$\begin{aligned} h_r^{-1}\sum_{i\in I_r}|Ax_{\sigma^i(n)}-s|^{p_i} &\leqslant \max\{1,2^{H-1}\}\\ &\times\Bigl[(h_r^{-1}h_r^1)h_r^{-1}\sum_{i\in I_r^1}|Ax_{\sigma^i(n)}-s|^{p_i}+(h_r^{-1}h_r^2)h_r^{-2}\sum_{i\in I_r^2}|Ax_{\sigma^i(n)}-s|^{p_i}\Bigr], \end{aligned}$$

where $I_r^1=(k_{r-1},k_j]$, $I_r^2=(k_j,k_r]$, $h_r^1=k_j-k_{r-1}$ and $h_r^2=k_r-k_j$. Since $x\in\sigma_{\theta^!}(A,p)$, $0<h_r^{-1}h_r^1\leqslant 1$ and $0<h_r^{-1}h_r^2\leqslant 1$, we have $x\in\sigma_\theta[A,p]$. This proves (i).

(ii) follows from the fact that α is a refinement of θ and θ is a refinement of γ for every $\theta\in L$. $\square$

For $\sigma(n)=n+1$ we write $M_\theta[A,p]$ for $\sigma_\theta[A,p]$. We now give a lemma to be used later.

Lemma 3. *Suppose that for a given $\varepsilon > 0$ there exist k_0 and n_0 such that*

$$k^{-1}\sum_{i=0}^{k-1}|Ax_{i+n}-s|^{p_i} < \varepsilon$$

for all $k \geqslant k_0$ and $n \geqslant n_0$. Then $x \in w[A,p]$, where

$$w[A,p] = \Big\{x=(x_i)\colon \lim_{k\to\infty} k^{-1}\sum_{i=0}^{k-1}|Ax_{i+n}-s|^{p_i} = 0 \textit{ uniformly in } n, \textit{ for some } s\Big\}.$$

Proof. Let $\varepsilon > 0$ be given. Choose k_1 and n_0 such that

$$k^{-1}\sum_{i=0}^{k-1}|Ax_{i+n}-s|^{p_i} < \frac{\varepsilon}{2}$$

for all $k \geqslant k_1$, $n \geqslant n_0$. It is enough to prove that there is k_2 such that $k \geqslant k_2$, $0 \leqslant n \leqslant n_0$,

$$k^{-1}\sum_{i=0}^{k-1}|Ax_{i+n}-s|^{p_i} < \varepsilon. \tag{2}$$

Taking $k_0 = \max(k_1,k_2)$, (2) will hold for $k \geqslant k_0$ and for all n, which gives the result. Once n_0 has been chosen, n_0 is fixed, so

$$\sum_{i=0}^{n_0-1}|Ax_{i+n}-s|^{p_i} = R(n) \quad \text{(say)}.$$

Now, taking $0 \leqslant n \leqslant n_0$ and $k > n_0$, we have

$$k^{-1}\sum_{i=0}^{k-1}|Ax_{i+n}-s|^{p_i} = k^{-1}\Big(\sum_{i=0}^{n_0-1}+\sum_{i=n_0}^{k-1}\Big)|Ax_{i+n}-s|^{p_i} \leqslant \frac{R(n)}{k}+\frac{\varepsilon}{2}.$$

Taking k sufficiently large, we have (2) and hence the result. □

Theorem 4. *$w[A,p] = M_\theta[A,p]$ for every lacunary sequence θ, where*

$$M_\theta[A,p] = \Bigl\{x = (x_i)\colon \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |Ax_{i+n} - s|^{p_i} = 0 \text{ uniformly in } n, \text{ for some } s\Bigr\}.$$

Proof. Let $x \in M_\theta[A,p]$. Then given $\varepsilon > 0$, there exist r_0 and s such that

$$h_r^{-1} \sum_{i=0}^{h_r-1} |Ax_{i+n} - s|^{p_i} < \varepsilon$$

for $r \geqslant r_0$ and $n = k_{r-1} + 1 + u$, $u \geqslant 0$. Let $k \geqslant h_r$, write $k = mh_r + p$, where $0 \leqslant p \leqslant h_r$, m is an integer. Since $k \geqslant h_r$, $m \geqslant 1$, we have

$$\begin{aligned} k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} &\leqslant k^{-1} \sum_{i=0}^{(m+1)h_r-1} |Ax_{i+n} - s|^{p_i} = k^{-1} \sum_{j=0}^{m} \sum_{i=jh_r}^{(j+1)h_r-1} |Ax_{i+n} - s|^{p_i} \\ &\leqslant (m+1)k^{-1}h_r\varepsilon \leqslant 2mk^{-1}h_r\varepsilon \end{aligned}$$

for $k^{-1}h_r \leqslant n$, and since $mk^{-1}h_r \leqslant 1$ we conclude

$$k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} \leqslant 2\varepsilon.$$

Then by Lemma 3, $M_\theta[A,p] \subset w[A,p]$. It is easy to see that $w[A,p] \subset M_\theta[A,p]$ for every θ. $\square$

Theorem 5. *Let A be a strongly regular matrix and $0 < \inf p_i$, then $x_i \to s$ implies $x_i \to s[M_\theta[A,p]]$.*

Proof. Suppose that $x_i \to s$ as $i \to \infty$. This implies $Ax_{i+n} \to s$ as $i \to \infty$ uniformly in n. Since $0 < h = \inf p_i$ we have $\lim\limits_{i\to\infty} |Ax_{i+n} - s|^h = 0$ uniformly in n. So for $0 < \varepsilon < 1$, there is $i_0 \in \mathbb{N}$ such that for all $i > i_0$ and for all n, $|Ax_{i+n} - s|^h < \varepsilon < 1$ and since $p_i > h$ for all i,

$$|Ax_{i+n} - s|^{p_i} \leqslant |Ax_{i+n} - s|^h < \varepsilon.$$

Then $\lim\limits_{i\to\infty} |Ax_{i+n} - s|^{p_i} = 0$ uniformly in n and therefore $\lim\limits_{k\to\infty} k^{-1} \sum\limits_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} = 0$. From Theorem 4 we have $\lim\limits_{r\to\infty} h_r^{-1} \sum\limits_{i\in I_r} |Ax_{i+n} - s|^{p_i} = 0$ uniformly in n. So $x_i \to s[M_\theta[A,p]]$. $\square$

Theorem 6. *Let A be a limitation method, $x \in \ell_\infty$ and $p_i = p$ for all i. Then $w(Ap) \equiv M_\theta(Ap)$ for every lacunary sequence θ, where*

$$w(Ap) = \left\{x = (x_i)\colon \lim_{k\to\infty} k^{-1}\sum_{i=0}^{k-1}(Ax_{i+n} - s)^p = 0 \text{ uniformly in } n, \text{ for some } s\right\}$$

and

$$M_\theta(Ap) = \left\{x = (x_i)\colon \lim_{r\to\infty} h_r^{-1}\sum_{i\in I_r}(Ax_{i+n} - s)^p = 0 \text{ uniformly in } n, \text{ for some } s\right\}.$$

In order to prove this theorem, we need the following lemma.

Lemma 7. *Suppose that for a given $\varepsilon > 0$ there exist k_0 and n_0 such that*

$$k^{-1}\sum_{i=0}^{k-1}(Ax_{i+n} - s)^p < \varepsilon$$

for all $k \geqslant k_0$ and $n \geqslant n_0$. Then $x \in w(Ap)$.

Proof. The proof of Lemma 7 is similar to the proof of Lemma 3. $\square$

Proof of Theorem 6. Let $x \in M_\theta(Ap)$. Then given $\varepsilon > 0$, there exist r_0 and n_0 such that

$$h_r^{-1}\left|\sum_{i=0}^{h_r-1}(Ax_{i+n} - s)^p\right| < \frac{\varepsilon}{2} \tag{3}$$

for $r \geqslant r_0$, $n \geqslant n_0$ and $n = k_{r-1} + 1 + u$, $u \geqslant 0$. Let $k \geqslant h_r$, m be an integer greater than or equal to 1. Then

$$k^{-1}\left|\sum_{i=0}^{k-1}(Ax_{i+n} - s)^p\right| \leqslant k^{-1}\sum_{j=0}^{m}\left|\sum_{i=jh_r}^{(j+1)h_r-1}(Ax_{i+n} - s)^p\right| + k^{-1}\sum_{i=mh_r}^{k-1}|Ax_{i+n} - s|^p. \tag{4}$$

Since A is a limitation method and $x \in \ell_\infty$, let for all i and n, $|Ax_{i+n} - s|^p \leqslant M$ (say). So (3) and (4) imply

$$k^{-1}\left|\sum_{i=0}^{k-1}(Ax_{i+n} - s)^p\right| \leqslant mk^{-1}h_r\,\frac{\varepsilon}{2} + Mk^{-1}h_r.$$

For $k^{-1}h_r \leqslant 1$, since $mk^{-1}h_r \leqslant 1$, we can make $Mk^{-1}h_r$ less than $\varepsilon/2$ by taking k sufficiently large. So

$$k^{-1}\left|\sum_{i=0}^{k-1}(Ax_{i+n} - s)^p\right| < \varepsilon$$

for all $k \geqslant k_0$ and $n \geqslant n_0$. Hence, by Lemma 7, $M_\theta(Ap) \subset w(Ap)$. It is trivial to see that $w(Ap) \subset M_\theta(Ap)$ for every θ. This completes the proof. $\square$

Let $p_k = s$ for all k, $q_k = t$ for all k and $0 < s \leqslant t$. Then it follows from Hölder's inequality that

$$h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma'(n)}|^s \leqslant \left(h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma^i(n)}|^t \right)^{s/t}$$

and therefore $\sigma_\theta^0[A, q] \subseteq \sigma_\theta^0[A, p]$.

We now consider the case that (p_i) and (q_i) are not constant sequences. We are able to prove $\sigma_\theta^0[A, q] \subseteq \sigma_\theta^0[A, p]$ only under additional conditions.

Theorem 8. *Let $0 < p_i \leqslant q_i$ for all k and let (q_i/p_i) be bounded. Then*

$$\sigma_\theta^0[A, q] \subseteq \sigma_\theta^0[A, p] \quad \text{and} \quad \sigma_\theta[A, q] \subseteq \sigma_\theta[A, p].$$

Proof. If we take $t_i = |Ax_{\sigma^i(n)}|^{p_i}$ for all i, then using the same technique as in [9, Theorem 2], it is easy to prove the theorem. □

Result.

(i) *If $0 < \inf p_i \leqslant p_i \leqslant 1$ for all k, then $\sigma_\theta[A] \subseteq \sigma_\theta[A, p]$.*

(ii) $1 \leqslant p_i \leqslant \sup p_i = H < \infty$, *then $\sigma_\theta[A, p] \subseteq \sigma_\theta[A]$.*

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Author's address: Department of Mathematics, University of 100. Yil, Van, Turkey, e-mail: `tbilgin@yyu.edu.tr`.