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RADIAL SOLUTIONS OF A CLASS OF ITERATED PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We give some expansion formulas and the Kelvin principle for solutions of a class of iterated equations of elliptic type

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1. Introduction

Consider the class of equations

$$Lu = \sum_{i=1}^{n} \Big(\frac{r}{x_i}\Big)^p \Big[x_i^2 \frac{\partial^2 u}{\partial x_i^2} + \alpha_i x_i \frac{\partial u}{\partial x_i}\Big] + \lambda u = 0 \tag{1}$$

where λ, α_i $(i = 1, 2, \ldots, n)$ are real parameters, p (> 0) is a real constant and r is defined by

$$r^p = x_1^p + x_2^p + \ldots + x_n^p. \tag{2}$$

The domain of the operator L is the set of all real valued functions $u(x)$ of the class $C^2(D)$, where $x = (x_1, x_2, \ldots, x_n)$ denotes points in $\mathbb{R}^n$ and D is the regularity domain of u in $\mathbb{R}^n$. Note that (1) includes the Laplace equation and an equidimensional (Euler) equation as special cases.

In [1], Almansi gave an expansion formula for the solutions of the Laplace equation. In [2], Altın generalized the idea to a wide range of a class of singular partial differential equations and obtained Lord Kelvin principle for this class of equations.

In this study, we obtain expansion formulas and the Kelvin principle for the iterates of the equation (1).

2. Solutions for the iterated equation

We first give some properties of the operator L. By direct computation, it can be shown that

$$L(r^m) = [m(m+\varphi)+\lambda]r^m \tag{3}$$

where

$$\varphi = -p + n(p-1) + \sum_{i=1}^{n} \alpha_i. \tag{4}$$

Let L^k denote, as usual, the successive applications of the operator L onto itself, that is $L^k u = L(L^{k-1}u)$, where k is a positive integer.

The proof of the following lemma can be done easily by using induction argument on k. For a special case of the lemma see [3], [7].

Lemma 1. *For any real parameter m,*

$$L^k(r^m) = [m(m+\varphi)+\lambda]^k r^m \tag{5}$$

where the integer k is the iteration number.

Let $u, v \in C^2(D)$ be any two functions. It can be shown that

$$L(uv) = uLv + vLu - \lambda uv + 2\sum_{i=1}^{n} \left(\frac{r}{x_i}\right)^p \left(x_i^2 \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right). \tag{6}$$

By replacing v by r^m in (6) and by using (3), we get

$$L(r^m u) = r^m m(m+\varphi+2T^*)u + r^m Lu \tag{7}$$

where

$$T^* = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

If u is a solution of the equation $Lu = 0$, then by (7),

$$L(r^m u) = r^m m(m+\varphi+2T^*)u. \tag{8}$$

By direct computation, one can show that

$$LT^* = T^*L. \tag{9}$$

In fact, for any integer k we have, by induction on k,

$$L(T^*)^k = (T^*)^k L, \tag{10}$$

where $(T^*)^k$ denotes the successive iterations of the operator T^*, k times onto itself.

Now we are ready to give the following lemma.

Lemma 2. *Let u be a solution of the equation $Lu = 0$. Then for any positive integer k and for any real number m,*

$$L^k(r^m u) = r^m m^k (m + \varphi + 2T^*)^k u. \tag{11}$$

P r o o f. We give the proof by induction on k. It is clear by (8) that, the equality (11) is true for $k = 1$. Now, let us assume that the equality is valid for $k-1$, that is,

$$L^{k-1}(r^m u) = r^m m^{k-1}(m + \varphi + 2T^*)^{k-1} u.$$

By applying the operator L to both sides of the above equality, we obtain

$$\begin{aligned} L^k(r^m u) &= L[r^m m^{k-1}(m + \varphi + 2T^*)^{k-1} u] \\ &= m^{k-1} L[r^m (m + \varphi + 2T^*)^{k-1} u]. \end{aligned}$$

Since u is a solution of $Lu = 0$, by using (10) it can be shown by induction that the function $(m + \varphi + 2T^*)^{k-1} u$ is also a solution of the same equation. Hence, by replacing u by $(m + \varphi + 2T^*)^{k-1} u$ in (8), we get

$$\begin{aligned} L^k(r^m u) &= m^{k-1} L[r^m (m + \varphi + 2T^*)^{k-1} u] \\ &= m^{k-1} r^m m (m + \varphi + 2T^*)(m + \varphi + 2T^*)^{k-1} u \\ &= r^m m^k (m + \varphi + 2T^*)^k u. \end{aligned}$$

Hence the proof is complete. □

The following theorem, which is a generalization of Almansi's expansion, states a class of solutions for the iterated equations.

Theorem 1. *Let $u_i(x)$, $i = 0, 1, \dots, k-1$ be any k solutions of the equation $Lu = 0$. Then the function*

$$w = \sum_{i=0}^{k-1} (\ln r)^i u_i(x) \tag{12}$$

is a solution of the iterated equation $L^k u = 0$.

P r o o f. By the hypothesis and by Lemma 2, for $m = 0$ we already have that

$$L^k[u_i(x)] = 0, \quad i = 0, 1, \dots, k-1.$$

Now, let us take the derivative on both sides of the equality (11) with respect to the parameter m. Then

$$\frac{\partial}{\partial m}[L^k(r^m u_i)] = \frac{\partial}{\partial m}[m^k r^m (m + \varphi + 2T^*)^k u_i].$$

Setting $\beta_0(m) = r^m(m + \varphi + 2T^*)^k u_i$ we have

$$L^k\Big[\frac{\partial}{\partial m}(r^m)u_i\Big] = \frac{\partial}{\partial m}[m^k \beta_0(m)],$$

which, in turn, gives

$$L^k[r^m(\ln r)u_i] = km^{k-1}\beta_0(m) + m^k\beta_0'(m)$$

or

$$L^k[r^m(\ln r)u_i] = m^{k-1}\beta_1(m) \tag{13}$$

where $\beta_1(m) = k\beta_0(m) + m\beta_0'(m)$. Now, taking $m = 0$ we obtain

$$L^k[(\ln r)u_i] = 0, \quad i = 0, 1, \dots, k-1.$$

Once again, the differentiation of (13) with respect to m gives

$$L^k[r^m(\ln r)^2 u_i] = m^{k-2}\beta_2(m),$$

where $\beta_2(m) = (k-1)\beta_1(m) + m\beta_1'(m)$. Thus, for $m = 0$, we have

$$L^k[(\ln r)^2 u_i] = 0, \quad i = 0, 1, \dots, k-1.$$

Proceeding in this way, by taking the derivative with respect to m, $(k-2)$ times in (13), we finally get

$$L^k[r^m(\ln r)^{k-1}u_i] = m\beta_{(k-1)}(m),$$

where $\beta_{(k-1)}(m) = 2\beta_{(k-2)}(m) + m\beta'_{(k-2)}(m)$. Thus, for $m = 0$ we have

$$L^k[(\ln r)^{k-1}u_i] = 0, \quad i = 0, 1, \dots, k-1.$$

Hence, we conclude that the functions

$$(\ln r)^j u_i(x), \quad i, j = 0, 1, \dots, k-1$$

are solutions of $L^k u = 0$. Therefore, by the principle of superposition,

$$L^k w = 0.$$

The proof is complete. □

Remark 1. It is clear that under the hypotheses of Theorem 1, the function

$$w = \sum_{i,j=0}^{k-1} (\ln r)^j u_i(x)$$

is in fact also a solution of $L^k u = 0$. In addition, if u is any solution of the equation $Lu = 0$, then we conclude that for any nonnegative integer i, $(\ln r)^i u$ is also a solution of the iterated equation $L^k u = 0$ $(k \geqslant 2)$.

The following theorem gives an expansion formula for the homogeneous solutions.

Theorem 2. *Let $u_\nu(x)$, $\nu = 0, 1, \dots, k-1$, be homogeneous (of degree λ_ν, respectively) solutions of the equation $Lu = 0$. Then the function*

$$w = \sum_{\nu=0}^{k-1} r^{-\varphi-2\lambda_\nu}(\ln r)^\nu u_\nu(x) \tag{14}$$

is a solution of the iterated equation $L^k u = 0$.

P r o o f. Since $u_\nu(x)$ is a homogeneous function of degree λ_ν, the Euler theorem on homogeneous functions yields

$$T^* u_\nu(x) = \sum_{i=1}^{n} x_i \frac{\partial u_\nu(x)}{\partial x_i} = \lambda_\nu u_\nu(x).$$

On the other hand, since u_ν satisfies the equation $Lu = 0$, Lemma 2 implies for each ν

$$L^k[r^m u_\nu(x)] = r^m m^k (m + \varphi + 2\lambda_\nu)^k u_\nu(x), \tag{15}$$

which yields

$$L^k[r^{-\varphi-2\lambda_\nu} u_\nu(x)] = 0, \quad \nu = 0, 1, \dots, k-1.$$

Now, let us take the derivative on both sides of the equality (15) with respect to the parameter m. Then

$$\frac{\partial}{\partial m}[L^k(r^m u_\nu)] = \frac{\partial}{\partial m}[m^k r^m (m + \varphi + 2\lambda_\nu)^k u_\nu.$$

Setting $\theta_0(m) = m^k r^m u_\nu$, we obtain

$$L^k\left[\frac{\partial}{\partial m}(r^m)u_\nu\right] = \frac{\partial}{\partial m}[(m + \varphi + 2\lambda_\nu)^k \theta_0(m)],$$

which, after differentiation gives

$$L^k[r^m(\ln r)u_\nu] = k(m + \varphi + 2\lambda_\nu)^{k-1}\theta_0(m) + (m + \varphi + 2\lambda_\nu)^k \theta_0'(m)$$

or

$$L^k[r^m(\ln r)u_\nu] = (m + \varphi + 2\lambda_\nu)^{k-1}\theta_1(m)$$

where $\theta_1(m) = k\theta_0(m) + (m+\varphi+2\lambda_\nu)\theta_0'(m)$. Now, taking $m = -\varphi - 2\lambda_\nu$ we obtain

$$L^k[r^{-\varphi-2\lambda_\nu}(\ln r)u_\nu] = 0, \quad \nu = 0, 1, \dots, k-1.$$

By taking the successive derivatives with respect to m and proceeding in a way similar to the proof of Theorem 1, we conclude that

$$L^k[r^{-\varphi-2\lambda_\nu}(\ln r)^j u_\nu] = 0, \quad j, \nu = 0, 1, \dots, k-1.$$

Hence, for $j, \nu = 0, 1, \dots, k-1$, the functions

$$r^{-\varphi-2\lambda_\nu}(\ln r)^j u_\nu(x)$$

are solutions of $L^k u = 0$. Therefore, by the principle of superposition,

$$L^k w = 0,$$

which is the proof. □

3. Solutions generated by Kelvin inversion

The standard Kelvin inversion principle states that if a solution $u(x_1, x_2, \dots, x_n)$ of the n-dimensional Laplace equation $\Delta u = 0$ is known, then the function $v = r^{2-n} u(x_1/r^2, x_2/r^2, \dots, x_n/r^2)$ is also a solution of the Laplace equation, where $r^2 = x_1^2 + x_2^2 + \dots + x_n^2$. The Kelvin principle is studied by several authors. (See, for example Altın [2], Çelebi [4], Weinstein [5], Özalp and Çetinkaya [6]).

In this section we state a generalized Kelvin principle for the solutions of the equation (1) in the following theorem.

Theorem 3 (Kelvin principle). *Let $u(x) = u(x_1, x_2, \dots, x_n)$ be any solution of the equation* (1). *Then the function*

$$v = r^{p+n(1-p)-\sum\limits_{i=1}^{n}\alpha_i} u\Big(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \dots, \frac{x_n}{r^2}\Big) \tag{16}$$

is also a solution of the same equation, where r is as defined in (2).

P r o o f. From (7) we already have

$$L(r^m u) = r^m m(m + \varphi + 2T^*)u + r^m Lu.$$

Now, let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, where $\xi_i = x_i/r^2$, $i = 1, 2, \dots, n$. Then for $\varrho^p = \xi_1^p + \xi_2^p + \dots + \xi_n^p$, clearly, $r^p \varrho^p = 1$. By making the change of variables, a rather lengthy computations yields

$$T^* u(\xi) = \sum_{i=1}^{n} x_i \frac{\partial u(\xi)}{\partial x_i} = -\sum_{i=1}^{n} \xi_i \frac{\partial u(\xi)}{\partial \xi_i} = -T^*_{(\xi)} u(\xi) \tag{17}$$

and

$$\begin{aligned} Lu(\xi) &= \Big\{\sum_{i=1}^{n} r^p \Big(x_i^{2-p} \frac{\partial^2}{\partial x_i^2} + \alpha_i x_i^{1-p} \frac{\partial}{\partial x_i}\Big) + \lambda\Big\} u(\xi) \\ &= \Big\{\sum_{i=1}^{n} \varrho^p \Big(\xi_i^{2-p} \frac{\partial^2}{\partial \xi_i^2} + \alpha_i \xi_i^{1-p} \frac{\partial}{\partial \xi_i}\Big) + \lambda\Big\} u(\xi) \\ &\quad - 2\Big[-p + n(p-1) + \sum_{i=1}^{n} \alpha_i\Big] \sum_{i=1}^{n} \xi_i \frac{\partial u(\xi)}{\partial \xi_i} \\ &= L_{(\xi)} u(\xi) - 2\varphi T^*_{(\xi)} u(\xi), \end{aligned} \tag{18}$$

where we use the notation $T^*_{(\xi)}$ and $L_{(\xi)}$, respectively, for the operators T^* and L with x replaced by ξ.

Now substituting (17) and (18) in (7), we obtain

$$L[r^m u(\xi)] = r^m m(m + \varphi - 2T^*_{(\xi)})u(\xi) + r^m L_{(\xi)} u(\xi) - 2\varphi r^m T^*_{(\xi)} u(\xi). \tag{19}$$

Since $u(x)$ is a solution of $Lu = 0$, the equation (19) becomes

$$L[r^m u(\xi)] = r^m m(m + \varphi - 2T^*_{(\xi)})u(\xi) - 2\varphi r^m T^*_{(\xi)} u(\xi)$$

or simply

$$L[r^m u(\xi)] = r^m (m + \varphi)(m - 2T^*_{(\xi)})u(\xi). \tag{20}$$

Hence, by setting $m = -\varphi$ in (20), we get

$$L[r^{-\varphi} u(\xi)] = 0$$

or, explicitly,

$$L\left[r^{p+n(1-p)-\sum\limits_{i=1}^{n}\alpha_i} u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \dots, \frac{x_n}{r^2}\right)\right] = 0,$$

which completes the proof. □

The Kelvin principle roughly tells us that if a solution of the equation (1) is known, then one can obtain another solution just by using the transformation mentioned above. The following simple example clears out the case:

An Example of the Inversion: Let, in (1), $n = 3$, $\alpha_1 = -3$, $\alpha_2 = -2$, $\alpha_3 = 1$, $\lambda = -6$, $p = 4$ and thus $r = (x_1^4 + x_2^4 + x_3^4)^{1/4}$. By (3), we easily conclude that $u(x_1, x_2, x_3) = r^2 = (x_1^4 + x_2^4 + x_3^4)^{1/2}$ is a solution of (1) with the given parameters. Hence, by using the Kelvin principle, we obtain that

$$v = r^{-1} u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \frac{x_3}{r^2}\right) = r^{-1}\left[\left(\frac{x_1}{r^2}\right)^4 + \left(\frac{x_2}{r^2}\right)^4 + \left(\frac{x_3}{r^2}\right)^4\right]^{1/2} = r^{-3}$$

is also a solution of the same equation. If we used $u = r^{-3}$ as a solution, then we should get $v = r^2$ as another solution under the inversion.

The following result states that the Kelvin principle also holds for the iterated equation $L^k u = 0$.

Theorem 4. *Let $u_i(x)$, $i = 0, 1, \ldots, k-1$ be any k solutions of the equation $Lu = 0$. Then the function*

$$w = \sum_{i=0}^{k-1} (\ln r)^i r^{-\varphi} u_i(\xi) = r^{-\varphi} \sum_{i=0}^{k-1} (\ln r)^i u_i\Big(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \ldots, \frac{x_n}{r^2}\Big)$$

is a solution of the iterated equation $L^k u = 0$, where φ is given by (4).

Proof. By the hypothesis, since each $u_i(x)$ is a solution of $Lu = 0$, Theorem 3 implies that

$$v_i(x) = r^{-\varphi} u_i\Big(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \ldots, \frac{x_n}{r^2}\Big)$$

is also a solution of the same equation. Hence, by Theorem 1,

$$w = \sum_{i=0}^{k-1} (\ln r)^i v_i(x)$$

is a solution of $L^k u = 0$. Thus, the proof is complete. $\square$

Lemma 4. *Let $u(x)$ be a homogeneous function of degree μ. Then*

$$T^*(r^{-\varphi} u(\xi)) = -r^{-\varphi}(\varphi + \mu) u(\xi).$$

Proof. Since $u(x)$ is a function of degree μ, it is clear that $T^* u(\xi) = -T^*_{(\xi)} u(\xi) = -\mu u(\xi)$ and $T^*(r^{-\varphi}) = -\varphi r^{-\varphi}$. Thus,

$$\begin{aligned} T^*(r^{-\varphi} u(\xi)) &= r^{-\varphi} T^*(u(\xi)) + u(\xi) T^*(r^{-\varphi}) \\ &= -r^{-\varphi}(\varphi + \mu) u(\xi). \end{aligned}$$

Lemma 5. *Let $u(x)$ be a homogeneous function of degree μ. Then*

$$(m + \varphi + 2T^*)^k (r^{-\varphi} u(\xi)) = (m - \varphi - 2\mu)^k r^{-\varphi} u(\xi).$$

Proof. We give the proof by induction. For $k = 1$, Lemma 4 yields

$$\begin{aligned} (m + \varphi + 2T^*)(r^{-\varphi} u(\xi)) &= (m + \varphi)(r^{-\varphi} u(\xi)) + 2T^*(r^{-\varphi} u(\xi)) \\ &= (m + \varphi) r^{-\varphi} u(\xi) - 2(\varphi + \mu) r^{-\varphi} u(\xi) \\ &= (m - \varphi - 2\mu) r^{-\varphi} u(\xi). \end{aligned}$$

Thus, the lemma holds for $k = 1$. Now let the assertion be true for $k - 1$, that is, let

$$(m + \varphi + 2T^*)^{k-1}(r^{-\varphi}u(\xi)) = (m - \varphi - 2\mu)^{k-1}r^{-\varphi}u(\xi)$$

hold. Then

$$\begin{aligned}(m + \varphi + 2T^*)^k(r^{-\varphi}u(\xi)) &= (m + \varphi + 2T^*)^{k-1}(m + \varphi + 2T^*)(r^{-\varphi}u(\xi)) \\ &= (m + \varphi + 2T^*)^{k-1}(m - \varphi - 2\mu)(r^{-\varphi}u(\xi)) \\ &= (m - \varphi - 2\mu)(m + \varphi + 2T^*)^{k-1}(r^{-\varphi}u(\xi)) \\ &= (m - \varphi - 2\mu)(m - \varphi - 2\mu)^{k-1}(r^{-\varphi}u(\xi)) \\ &= (m - \varphi - 2\mu)^k(r^{-\varphi}u(\xi)),\end{aligned}$$

which gives the desired result. □

Theorem 6. *Let $u_\nu(x)$, $\nu = 0, 1, \ldots, k-1$ be homogeneous (of degree λ_ν, respectively) solutions of the equation $Lu = 0$. Then the function*

$$w = \sum_{\nu=0}^{k-1} r^{2\lambda_\nu}(\ln r)^\nu u_\nu(\xi)$$

is a solution of the iterated equation $L^k w = 0$.

Proof. By the hypothesis, since each $u_\nu(x)$ is a solution of $Lu = 0$, Theorem 3 implies that

$$v_\nu(x) = r^{-\varphi}u_\nu\Big(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \ldots, \frac{x_n}{r^2}\Big)$$

is also a solution of the same equation. On the other hand, for each ν, by Lemma 2,

$$L^k(r^m v_\nu(x)) = r^m m^k(m + \varphi + 2T^*)^k v_\nu(x)$$

or

$$L^k(r^m r^{-\varphi}u_\nu(\xi)) = r^m m^k(m + \varphi + 2T^*)^k r^{-\varphi}u_\nu(\xi).$$

Thus, by Lemma 5, we have

$$L^k(r^{m-\varphi}u_\nu(\xi)) = r^{m-\varphi}m^k(m - \varphi - 2\lambda_\nu)^k u_\nu(\xi). \tag{21}$$

Hence for $m = \varphi + 2\lambda_\nu$

$$L^k(r^{2\lambda_\nu}u_\nu(\xi)) = 0,$$

which means that $r^{2\lambda_\nu} u_\nu(\xi)$ is a solution of $L^k u = 0$. Analogously to the proof of Theorem 2, by taking successive derivatives $(k-1)$ times with respect to m in (21), we conclude that

$$L^k(r^{2\lambda_\nu}(\ln r)^i u_\nu(\xi)) = 0, \quad i, \nu = 0, 1, \ldots, k-1,$$

which completes the proof. $\square$

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