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SIGNED DOMINATION NUMBERS OF DIRECTED GRAPHS

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Abstract. The concept of signed domination number of an undirected graph (introduced by J. E. Dunbar, S. T. Hedetniemi, M. A. Henning and P. J. Slater) is transferred to directed graphs. Exact values are found for particular types of tournaments. It is proved that for digraphs with a directed Hamiltonian cycle the signed domination number may be arbitrarily small.

Keywords: signed dominating function, signed domination number, directed graph, tournament, directed Hamiltonian cycle

MSC 2000: 05C20, 05C69, 05C45

In [1], J. E. Dunbar, S. T. Hedetniemi, M. A. Henning and P. J. Slater have introduced the concept of signed domination number of an undirected graph. Here we transfer this concept to directed graphs (shortly digraphs).

We consider finite digraphs without loops and without pairs of arcs joining the same pair of vertices and equally directed.

Let D be a finite digraph with the vertex set $V(D)$ and the arc set $A(D)$. Let $|V(D)| = n$. For each vertex $v \in V(D)$ let $N_D^-[v]$ (or shortly $N^-[v]$) be the set consisting of v and of all vertices of D from which arcs go into v. If f is a mapping of $V(D)$ onto a set of numbers and $S \subseteq V(D)$, then $f(S) = \sum\limits_{x\in S} f(x)$.

Consider a function $f\colon V(D) \to \{-1, 1\}$. If $f(N_D^-[v]) \geqslant 1$ for each vertex $v \in V(D)$, then f is called a signed dominating function (shortly SDF) on D. Denote $w(f) = f(V(D))$ and call it the weight of f. The minimum of weights of all SDF on D is the signed domination number $\gamma_S(D)$ of D.

First we state three lemmas.

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Lemma 1. *Always* $\gamma_S(D) \equiv n \pmod 2$.

P r o o f. Let n^+ (or n^-) be the number of vertices v of D such that $f(v) = 1$ (or $f(v) = -1$, respectively). Then $n^+ + n^- = n$, $n^+ - n^- = \gamma_S(D)$, therefore $n - \gamma_S(D) = 2n^-$ and the assertion follows. □

Lemma 2. *Let u be a source of D. Let f be a* SDF *on D. Then $f(u) = 1$.*

P r o o f. We have $N^-[u] = \{u\}$ and thus $f(u) = f(N^-[u]) \geqslant 1$, which implies $f(u) = 1$. □

Lemma 3. *Let u be a vertex of indegree* 1 *in D, and let v be the unique vertex from which an arc goes into u in D. Let f be a* SDF *on D. Then $f(u) = f(v) = 1$.*

P r o o f. We have $N^-[u] = \{u, v\}$ and thus $f(u) + f(v) = f(N^-[u])$. This is possible only if $f(u) = f(v) = 1$. □

These lemmas imply the following assertion.

Theorem 1. *Let D be a digraph with n vertices in which the indegrees of vertices do not exceed* 1*. Then $\gamma_S(D) = n$.*

Corollary. *Let C_n (or P_n) be the directed cycle (or directed path, respectively) with n vertices. Then $\gamma_S(C_n) = \gamma_S(P_n) = n$.*

Now we turn our attention to tournaments.

We shall consider two particular types of tournaments.

The acyclic tournament $\mathrm{AT}(n)$ with n vertices has the vertex set $V(\mathrm{AT}(n)) = \{u_1, u_2, \ldots, u_n\}$. An arc goes from u_i into u_j if and only if $i < j$.

Now let n be an odd positive integer. We have $n = 2k + 1$, where k is a positive integer. We define the circulant tournament $\mathrm{CT}(n)$ with n vertices. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs go from u_i to the vertices $u_{i+1}, \ldots, u_{i+k}$, the sums being taken modulo n.

Theorem 2. *Let* $\mathrm{AT}(n)$ *for $n \geqslant 3$ be an acyclic tournament. If n is even, then* $\gamma_S(\mathrm{AT}(n)) = 2$*. If n is odd, then* $\gamma_S(\mathrm{AT}(n)) = 1$.

P r o o f. We have $V(\mathrm{AT}(n)) = N^-[u_n]$. If f is a SDF on $\mathrm{AT}(n)$, then $w(f) = f(V(\mathrm{AT}(n)) = f(N^-[u_n]) \geqslant 1$. Therefore $\gamma_S(\mathrm{AT}(n)) \geqslant 1$. If n is even, then $\gamma_S(\mathrm{AT}(n))$ must also be even (by Lemma 4) and thus $\gamma_S(\mathrm{AT}(n)) \geqslant 2$.

In the case when n is even, consider the mapping $f\colon V(\mathrm{AT}(n)) \to \{-1, 1\}$ such that $f(u_i) = 1$ for $1 \leqslant i \leqslant \frac{1}{2}n + 1$ and $f(u_i) = -1$ for $\frac{1}{2}n + 2 \leqslant i \leqslant n$. Then for $1 \leqslant i \leqslant \frac{1}{2}n + 1$ we have $f(N^-[u]) = i \geqslant 1$ and for $\frac{1}{2}n + 2 \leqslant i \leqslant n$ we have

$f(N^-[u]) = n+2-i \geqslant 1$. The function f is a SDF. We have $w(f) = 2$. Therefore $\gamma_S(\mathrm{AT}(n)) \leqslant 2$, which implies $\gamma_S(\mathrm{AT}(n)) = 2$.

In the case then when n is odd, consider the mapping $f\colon V(\mathrm{AT}(n)) \to \{-1,1\}$ such that $f(u_i) = 1$ for $1 \leqslant i \leqslant \frac{1}{2}(n+1)$ and $f(u) = -1$ for $\frac{1}{2}(n+1) \leqslant i \leqslant n$. We have $f(N^-[u_i]) = i \geqslant 1$ for $1 \leqslant i \leqslant \frac{1}{2}(n+1)$ and $f(N^-[u_i]) = n+1-i \geqslant 1$ for $\frac{1}{2}(n+1) \leqslant i \leqslant n$. The function f is again a SDF. We have then $w(f) = 1$. Therefore $\gamma_S(\mathrm{AT}(n)) \leqslant 1$, which implies $\gamma_S(\mathrm{AT}(n)) = 1$. □

Theorem 3. *Let* $\mathrm{CT}(n)$ *for odd* $n \geqslant 3$ *be a circulant tournament. Then* $\gamma_S(\mathrm{CT}(n)) = 2$.

P r o o f. Let f be a SDF on $\mathrm{CT}(n)$. If $f(x) = 1$ for each $x \in V(\mathrm{CT}(n))$, then $w(f) = n \geqslant 3$. If it is not so, then without loss of generality we may suppose that $f(u_0) = -1$. Consider the sets $N^-[u_0] = \{u_{k+1}, \ldots, u_{r-1}, u_0\}$ and $N^-[u_k] = \{u_0, \ldots, u_k\}$. As f is a SDF, we have $f(N^-[u_0]) \geqslant 1$, $f(N^-[u_k]) \geqslant 1$. Further $N^-[u_0] \cup N^-[u_k] = V(\mathrm{CT}(n))$, $N^-[u_0] \cap N^-[u_k] = \{u_0\}$. Therefore $w(f) = f(V(\mathrm{CT}(n))) = f(N^-[u_0]) + f(N^-[u_k]) - f(u_0) = f(N^-[u_0] + f(N^-[u_k]) + 1 \geqslant 3$. This implies that $\gamma_S(\mathrm{CT}(n)) \geqslant 2$.

If k is even, then let $s = \frac{1}{2}k - 1$. Let $V^- = \{u_1, u_2, \ldots, u_S, u_{k+1}, u_{k+2}, \ldots, u_{k+s}\}$, $V^+ = V(\mathrm{CT}(n)) - V^-$. Define the function f such that $f(v) = 1$ for $v \in V^+$ and $f(v) = -1$ for $v \in V^-$. For any vertex $v \in V(\mathrm{CT}(n))$ we have $|N^-[v]| = k+1$, $|N^-[v] \cap V^-| \leqslant s$. Therefore $f(N^-[v]) \geqslant k+1-2s = 3 \geqslant 1$ and f is a SDF. We have $|V^+| = n-2s$, $|V^-| = 2s$, $w(f) = |V^+| - |V^-| = n-4s = 3$ and thus $\gamma_S(\mathrm{CT}(n)) = 3$.

If k is odd, then let $t = \frac{1}{2}(k-1)$. Let $V^- = \{u_1, \ldots, u_t, u_{k+1}, \ldots, u_{k+t-1}\}$, $V^+ = V(\mathrm{CT}(n)) - V^-$. Analogously as in the preceding case we define the function f which is a SDF and $w(f) = 3$. Therefore again $\gamma_S(\mathrm{CT}(n)) = 3$. □

Now we shall show that $\gamma_S(D)$ can be arbitrarily small.

Theorem 4. *Let q be a positive integer. Then there exists a digraph D with $q+8$ vertices having a Hamiltonian directed cycle and such that $\gamma_S(D) \leqslant -q$.*

P r o o f. Let $V(D) = \{u_1, u_2, u_3, u_4, v_1, v_2, \ldots, v_{q+4}\}$. Consider the directed cycle H such that $V(H) = V(D)$ and the arcs of H are u_1u_2, u_2u_3, u_3u_4, u_4v_1, $v_1v_2, \ldots, v_{1+3}v_{q+4}, v_{q+4}u_1$. From the cycle H we construct the digraph D by adding the edge u_4u_1 and the edges u_2v_i, u_3v_i for $i \in \{1, \ldots, q+4\}$, and u_4v_i for $i \in \{2, \ldots, q+4\}$. Let $f\colon V(D) \to \{-1,1\}$ be such that $f(u_i) = 1$ for $i \in \{1,2,3,4\}$ and $f(v_i) = -1$ for $i \in \{1, \ldots, q+4\}$. Then $f(N^-[u_1]) = 1$, $f(N^-[u_2]) = 2$ for $i \in \{2,3,4\}$, $f(N^-[v_1]) = 2$, $f(N^-[v_i]) = 1$ for $i \in \{2, \ldots, q+4\}$. The function f is a SDF and $w(f) = -q$. Therefore $\gamma_S(D) \leqslant -q$. The cycle H is Hamiltonian in D. □

The domination number $\gamma(D)$ may be defined also for digraphs D. A subset $S \subseteq V(D)$ is called dominating in D if each vertex of D either is in S, or is the terminal vertex of an arc outgoing from a vertex of S in D. The minimum number of vertices of a dominating set in D is the domination number $\gamma(D)$ of D.

Proposition. *There are digraphs D with $\gamma(D) < \gamma_S(D)$ and also digraphs D with $\gamma_S < \gamma(D)$.*

Proof. The acyclic tournament $\mathrm{AT}(n)$ with n even has $\gamma(\mathrm{AT}(n)) = 1$, because it has a dominating set $\{u_1\}$, and $\gamma_S(\mathrm{AT}(n)) = 2$, by Theorem 2; therefore $\gamma(\mathrm{AT}(n)) < \gamma_S(\mathrm{AT}(n))$. On the other hand, the digraph D from Theorem 4 has $\gamma(D) = 2$, because it has a dominating set $\{u_2, u_4\}$, and $\gamma_S(D) = -q$, therefore $\gamma_S(D) < \gamma(D)$. $\square$

References

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