Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 2, 471–477

Persistent URL: `http://dml.cz/dmlcz/127993`

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GROUPS ASSOCIATED WITH MINIMAL FLOWS

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(Received August 26, 2002)

Abstract. Let S be topological semigroup, we consider an appropriate semigroup compactification $\hat{S}$ of S. In this paper we study the connection between subgroups of a maximal group in a minimal left ideal of $\hat{S}$, which arise as equivalence classes of some closed left congruence, and the minimal flow characterized by the left congruence. A particular topology is defined on a maximal group and it is shown that a closed subgroup under this topology is precisely the intersection of an equivalence class with the maximal group for some left congruence on $\hat{S}$.

Keywords: flow, dynamical system, left congruence, maximal group

MSC 2000: 54H20, 54H15

1. Introduction

The classical theory of dynamical systems arose in the context of the study of differential equations. In recent years the study of these systems has been extended beyond discrete or continuous (real) phase groups or semigroups to the theory of flows of more general topological groups or semigroups.

In papers [4], [5], and [6] we developed a highly semigroup theoretic approach to the study of flows of an arbitrary topological semigroup S on compact phase spaces. This approach involved realizing transitive S-flows as quotients of an appropriate semigroup compactification $\hat{S}$ of S, where the quotient equivalence relations are precisely the closed left congruences on $\hat{S}$. In the case that the S-flow is a minimal flow, the flow may be recovered from the restriction of the corresponding congruence to an arbitrarily chosen minimal left ideal of $\hat{S}$, and such a restricted left congruence factors (by results of [6]) into a product of equivalence relations, called the proximal and distal components. The distal component is uniquely determined by the equivalence class (with respect to the original congruence) of any idempotent in the minimal left ideal. These developments provide motivation for the focus of this paper, the study of those subgroups of a maximal subgroup in a minimal left ideal of $\hat{S}$ which arise as equivalence classes of some closed left congruence.

2. The LUCky compactification

Throughout the paper S denotes a topological semigroup (a Hausdorff space equipped with a continuous associative multiplication). An *action* of S on a compact space X (we will always assume that the phase space is compact Hausdorff) is a continuous map $\pi\colon S \times X \to X$ satisfying $\pi(st, x) = \pi(s, \pi(t, x))$ for all $s, t \in S$, $x \in X$; we further require that $\pi(1, x) = x$ for all $x \in X$ if S has an identity 1. The triple (S, X, π) is called an *S-flow*; we often denote the flow simply by X. We also often write $\pi(s, x)$ as sx. A *(flow) homomorphism* is a continuous map $f\colon X \to Y$ between S-flows which is equivariant ($f(sx) = s(f(x))$ for all $s \in S$, $x \in X$).

Definition 2.1. A *(monoidal) compactification* of a topological semigroup S is a pair (T, j) such that

(1) T is a compact Hausdorff right topological (all right translations are continuous) semigroup with identity;
(2) j is a continuous homomorphism from S to T such that $(s, t) \mapsto j(s)t\colon S \times T \to T$ is continuous;
(3) $j(S) \cup \{1\}$ is dense in T;
(4) j carries the identity of S to the identity of T, provided S has an identity.

It is well known that for a given semigroup S there exists a *universal monoidal compactification* of S which is a monoidal compactification $(\hat{S}, j)$ such that if (T, i) is a monoidal compactification of S, then there exists a unique continuous identity-preserving homomorphism $F\colon \hat{S} \to T$ with $i = F \circ j$. In [2] it is shown (with the proviso that one may need to attach a discrete identity in our setting) that this semigroup compactification arises as the compactification associated to the space $\mathrm{LUC}(S)$ of left norm continuous functions on S; we thus refer to $\hat{S}$ as the LUC*ky compactification* of S.

Definition 2.2. An action $\pi\colon S \times X \to X$ is said to *extend* to the LUCky compactification $(\hat{S}, j)$ if there is a function (called an *extended action*) $\hat{\pi}\colon \hat{S} \times X \to X$ satisfying

(1) $\hat{\pi}(j(s), x) = \pi(s, x)$ for all $s \in S$;
(2) the identity of $\hat{S}$ acts as the identity on X;
(3) $\hat{\pi}(st, x) = \hat{\pi}(s, \hat{\pi}(t, x))$ for all $s, t \in \hat{S}$, $x \in X$;
(4) $s \mapsto \hat{\pi}(s, x)\colon \hat{S} \to X$ is continuous for all $x \in X$ (i.e., the extended action is right continuous).

It is standard from the universal properties of $\hat{S}$ that an action of S on a compact space X extends uniquely to an extended action $\hat{\pi}$ of $\hat{S}$ on X (see e.g. [6]). It follows from the standard theory of compact semigroups (see e.g. [2]) that $\hat{S}$ has a unique smallest ideal, called the minimal ideal, and that this ideal decomposes into a disjoint union of closed minimal left ideals. Furthermore, each minimal left ideal may be written uniquely as a disjoint union of maximal subgroups of $\hat{S}$, all of the form $e\hat{S}e$, where e ranges over the idempotents in the minimal left ideal (but these subgroups are in general no longer closed).

Let X be a minimal S-flow (one for which there are no non-trivial invariant closed subsets) and let $x \in X$. Extend the action to $\hat{S}$ and define an equivalence relation on $\hat{S}$ by $s \sim t$ if $sx = tx$. From the right continuity of the action it follows that $\sim$ is closed (as a subset of $\hat{S} \times \hat{S}$). Also $a \sim b$ implies $sa \sim sb$ for $s, a, b \in \hat{S}$, so $\sim$ is a closed left congruence. Let L be a closed minimal left ideal in $\hat{S}$. Then Lx is a closed invariant subset of X and hence by minimality $Lx = X$. The mapping $s \mapsto sx \colon L \to X$ is easily verified to be a surjective flow-homomorphism, and it then follows quickly that the S-flow $L/{\sim}$ is flow isomorphic to X. Thus (up to flow-isomorphism) the minimal S-flows all arise by fixing a minimal left ideal L in $\hat{S}$, restricting all closed left congruences on $\hat{S}$ to L, and forming the corresponding quotient flows (with elements of the phase space being the congruence classes).

3. Group topologies

Let S be a topological semigroup with LUCky compactification $\hat{S}$. For $s \in S$, we typically denote its image $j(s) \in \hat{S}$ simply by s (the notational simplification outweighing the potential resulting confusion).

Given an S-flow on a compact space X, there is a naturally associated S-flow given by $(s, A) \mapsto sA$ on the space 2^X of non-empty compact subsets equipped with the Vietoris topology. One verifies directly that this action is indeed continuous. There is then a (unique) extended flow $\hat{S} \times 2^X \to 2^X$. We denote this flow by $(u, A) \mapsto u \circ A$. As a matter of convenience we extend this action to the set $\mathscr{P}(X)$ of all non-empty subsets by defining $u \circ A := u \circ \overline{A}$. We develop basic features of the extended flow on the non-empty subsets of a given S-flow X.

Lemma 3.1. *For all $s \in S$ and $A \in \mathscr{P}(X)$, $s \circ A = s\overline{A} = \overline{sA}$.*

Proof. Note that by compactness and continuity of $\circ$ we have $s\overline{A} = \overline{sA}$, and thus $s \circ A = s \circ \overline{A} = \overline{sA}$ since $\circ$ is a flow extension. $\square$

Proposition 3.2. *For $u \in \hat{S}$ and $\emptyset \neq A \subseteq X$,*

$$u \circ A = \bigcap\{\overline{VA}\colon\ u \in V,\ \text{an open subset of } \hat{S}\}$$
$$= \bigcap\{\overline{(V \cap S)A}\colon\ u \in V,\ \text{an open subset of } \hat{S}\}.$$

P r o o f. By definition, $u \circ A = u \circ \overline{A}$. Let s_α be a net in S (identified with $j(S)$) with $s_\alpha \to u$. Then $s_\alpha \circ \overline{A} \to u \circ \overline{A}$. Let $V \subseteq \hat{S}$ be open with $u \in V$ and pick β such that $s_\alpha \in V$ for $\alpha \geqslant \beta$. Then since convergence in the Vietoris topology is liminf-limsup convergence,

$$u \circ A \subseteq \overline{\bigcup_{\alpha \geqslant \beta} s_\alpha \circ \overline{A}} = \overline{\bigcup_{\alpha \geqslant \beta} s_\alpha \overline{A}} \subseteq \overline{(V \cap S)A} \subseteq \overline{VA}.$$

This establishes that $u \circ A$ is contained in the second intersection, which in turn is contained in the first intersection.

Conversely, suppose $y \in \bigcap\{\overline{VA}\colon\ u \in V$ and $V \subseteq \hat{S}$ is open$\}$. Consider the directed set of all $\alpha := (V, W)$ where V and W are open neighborhoods of u and y respectively ordered by coordinatewise reverse inclusion. For each $\alpha = (V, W)$, there exists $v_\alpha \in V$ and $a_\alpha \in A$ such that $v_\alpha a_\alpha \in W$. Also, by right continuity, there is $s_\alpha \in V \cap S$ such that $s_\alpha a_\alpha \in W$. Note that $s_\alpha \to u$ and $s_\alpha a_\alpha \to y$. Now, $s_\alpha \circ \overline{A} \to u \circ \overline{A}$ and $s_\alpha \circ \overline{A} = s_\alpha \overline{A}$. Since $s_\alpha a_\alpha \in \overline{s_\alpha A} = s_\alpha \circ \overline{A} \to u \circ \overline{A}$ and $s_\alpha a_\alpha \to y$, we conclude that $y \in u \circ \overline{A} = u \circ A$. Thus $\bigcap \overline{VA} \subseteq u \circ A$. □

Corollary 3.3. *For $u \in \hat{S}$ and $\emptyset \neq A \subseteq \hat{X}$,*

$$y \in u \circ A \Leftrightarrow \exists\, u_\alpha \to u,\ a_\alpha \in A \text{ such that } u_\alpha a_\alpha \to y$$
$$\Leftrightarrow \exists\,\{s_\alpha\} \subseteq S,\ s_\alpha \to u,\ a_\alpha \in A \text{ such that } s_\alpha a_\alpha \to y.$$

Proposition 3.4. *For all $u \in \hat{S}$ and for all non-empty $A, B \subseteq X$:*

(1) $u \circ \{x\} = \{ux\}$;
(2) $A \subseteq B \Rightarrow u \circ A \subseteq u \circ B$;
(3) $uA \subseteq u \circ A$;
(4) *for all* $r, s \in \hat{S}$, $r(s \circ A) \subseteq r \circ (s \circ A) = (rs) \circ A$;
(5) $u \circ (A \cup B) = u \circ A \cup u \circ B$.

P r o o f. (1) By the preceding corollary, $y \in u \circ \{x\}$ if and only if there is $u_\alpha \to u$ and $a_\alpha \in \{x\}$ such that $u_\alpha a_\alpha \to y$, that is, $u_\alpha x \to y$. By right continuity $u_\alpha x \to ux$, and hence $y = ux$.

(2) For $V \subseteq \hat{S}$ open and $u \in V$, $A \subseteq B$ implies $\overline{VA} \subseteq \overline{VB}$. Thus by Proposition 3.2, $u \circ A \subseteq u \circ B$.

(3) Note by (1) and (2) that $\{x\} \subseteq A$ implies $\{ux\} = u \circ \{x\} \subseteq u \circ A$, and thus $uA \subseteq u \circ A$.

(4) By (3), $r(s \circ A) \subseteq r \circ (s \circ A)$, and since $\circ$ is an action, $r \circ (s \circ A) = (rs) \circ A$.

(5) By (2), $(u \circ A) \cup (u \circ B) \subseteq u \circ (A \cup B)$.

To see the other inclusion, let $y \in u \circ (A \cup B)$ and let $\{u_\alpha\} \subseteq \hat{S}$ with $u_\alpha \to u$ and $c_\alpha \in A \cup B$ such that $u_\alpha c_\alpha \to y$. Then cofinally $c_\alpha \in A$ or $c_\alpha \in B$. It follows that $y \in u \circ A$ or $y \in u \circ B$. □

We consider the S-flow on $\hat{S}$ itself given by $(s,t) \mapsto j(s)t\colon\ S \times \hat{S} \to \hat{S}$, and denote the extended flow $\hat{S} \times 2^{\hat{S}} \to 2^{\hat{S}}$ by $(u, A) \mapsto u \circ A$.

Defnition 3.5. Let e be an idempotent in $\hat{S}$. We define a topology ν on $e\hat{S}e$ as follows. For all $A \subseteq e\hat{S}e$,

$$A \text{ is } \nu\text{-closed} \Leftrightarrow A = e \circ A \cap e\hat{S}e \Leftrightarrow A = \bigcap\{\overline{VA}\colon\ V \subseteq \hat{S} \text{ open and } e \in V\} \cap e\hat{S}e.$$

Proposition 3.6. *The ν-closed sets form the closed sets for a topology. Furthermore, for any nonempty $A \subseteq e\hat{S}e$, the ν-closure of A is equal to $e \circ A \cap e\hat{S}e$.*

P r o o f. The equality $e \circ (A \cup B) = e \circ A \cup e \circ B$ implies closure under finite unions. Suppose $A = \bigcap A_\alpha$ where each A_α is ν-closed. Then $A = eA \subseteq e \circ A \subseteq \bigcap \overline{VA}$, where $e \in V$ and V is open in $\hat{S}$. Hence

$$A \subseteq \bigcap \overline{VA} \cap e\hat{S}e \subseteq \bigcap \overline{VA_\alpha} \cap e\hat{S}e = A_\alpha$$

since A_α is ν-closed. Thus $A \subseteq \bigcap \overline{VA} \cap e\hat{S}e \subseteq \bigcap A_\alpha = A$, implying that $A = \bigcap \overline{VA} \cap e\hat{S}e$, and hence that A is ν-closed.

Let A be a non-empty subset of $e\hat{S}e$, and let $B := e \circ A \cap e\hat{S}e$. Then $B = eB \subseteq e \circ B \cap e\hat{S}e$. Conversely,

$$e \circ B \cap e\hat{S}e \subseteq e \circ (e \circ A) \cap e\hat{S}e = e^2 \circ A \cap e\hat{S}e = B.$$

Thus B is ν-closed. Note that $A = eA \subseteq e \circ A \cap e\hat{S}e = B$. Thus $\overline{A}^\nu \subseteq B$. On the other hand,

$$B = e \circ A \cap e\hat{S}e \subseteq e \circ \overline{A}^\nu \cap e\hat{S}e = \overline{A}^\nu,$$

where the last equality follows from the fact that $\overline{A}^\nu$ is closed. Thus $B = \overline{A}^\nu$. □

Proposition 3.7. *Let $e = e^2 \in \hat{S}$ and let τ denote the topology on $\hat{S}$.*

(i) *If $A \subseteq e\hat{S}e$ and y is in the τ-closure of A, then ey is in the ν-closure of A. Thus a net x_α in A which τ-converges to some $x \in \hat{S}$ also ν-converges to $ex \in e\hat{S}e$.*

(ii) *The space $(e\hat{S}e, \nu)$ is compact and T_1.*

(iii) *The identity map from $(e\hat{S}e, \tau)$ to $(e\hat{S}e, \nu)$ is continuous.*

(iv) *For $t \in e\hat{S}e$, the right translation $x \mapsto xt\colon (e\hat{S}e, \nu) \to (e\hat{S}e, \nu)$ is continuous.*

Proof. (i) We have $y \in \hat{S}e$, since the latter is compact, hence closed, since multiplication is right continuous. Thus $ey \in e\hat{S}e$. We have also $ey \in e\overline{A} \subseteq e \circ \overline{A} = e \circ A$. Thus $ey \in e \circ A \cap e\hat{S}e$, which by the previous proposition is the ν-closure of A. The second assertion is a straightforward topological consequence of the first assertion.

(ii) Let x_α be a net in $e\hat{S}e$. Since $(\hat{S}, \tau)$ is compact, some subnet converges to some $x \in \hat{S}$. By part i), this subnet then ν-converges to ex. Thus $e\hat{S}e$ is ν-compact. Since for any $y \in e\hat{S}e$, $\{y\} = \{ey\} = e \circ \{y\}$, it follows that the singleton set $\{y\}$ is ν-closed. Hence $e\hat{S}e$ equipped with the ν-topology is T_1.

(iii) This part follows immediately from part i), since $ex = x$ for $x \in e\hat{S}e$.

(iv) Let A be a non-empty subset of $e\hat{S}e$ and let y be in the ν-closure of A, i.e., $y \in e \circ A \cap e\hat{S}e$. By Corollary 3.3 there exists a net $u_\alpha \to e$ and a net a_α in A such that $u_\alpha a_\alpha \to y$. By right continuity, $u_\alpha a_\alpha t \to yt$. Since $a_\alpha t \in At$, we conclude from Corollary 3.3 that $yt \in e \circ (At) \cap e\hat{S}e$, the ν-closure of At. Thus right translation by t is continuous on $e\hat{S}e$. □

The ν-topology on $e\hat{S}e$ has close connections with closed left congruences on $\hat{S}$ and hence with S-flows.

Proposition 3.8. *Let $\sim$ be a closed left congruence on $\hat{S}$. Then the intersection of any congruence class with $e\hat{S}e$ is ν-closed. Conversely, any ν-closed subgroup of $e\hat{S}e$ containing e is such an intersection for some closed left congruence.*

Proof. Let $A = [x]_\sim \cap e\hat{S}e$, where $\sim$ is a closed left congruence relation, $x \in e\hat{S}e$, and $[x]_\sim$ is its equivalence class. It suffices to show that $A = e \circ A \cap e\hat{S}e \subseteq A$, since the other inclusion is automatic. Let $y \in e \circ A \cap e\hat{S}e$. By Corollary 3.3 there exist nets $u_\alpha \to e$ and $a_\alpha \in A$ such that $u_\alpha a_\alpha \to y$. Then $a_a \sim x$ implies $u_\alpha a_\alpha \sim u_\alpha x$, and thus $y \sim ex = x$ since the relation $\sim$ is closed. Thus $y \in [x]_\sim \cap e\hat{S}e = A$.

Conversely, suppose that A is a ν-closed subgroup of the group $e\hat{S}e$ containing e. Let $L = \hat{S}e$ and consider the extended flow $(u, A) \mapsto u \circ A\colon \hat{S} \times 2^L \to 2^L$. Let $\sim$ denote the left congruence defined by $t \sim u$ if and only if $t \circ (e \circ A) = u \circ (e \circ A)$; it is straightforward to check that this is a closed left congruence on $\hat{S}$. Let $s \in A$; $A = sA = s^{-1}A = eA$ since A is a subgroup. Since A is ν-closed, $A = e \circ A \cap e\hat{S}e$.

Then

$$e \circ A = e \circ (sA) \subseteq e \circ (s \circ A) = s \circ A = s \circ (s^{-1}A) \subseteq s \circ (s^{-1} \circ A) = e \circ A;$$

it follows that $e \circ (e \circ A) = e \circ A = s \circ A = (se) \circ A = s \circ (e \circ A)$. Hence $e \sim s$. Therefore, $A \subseteq [e]_\sim \cap e\hat{S}e$. On the other hand, if $t \in [e]_\sim \cap e\hat{S}e$, we see that

$$\{t\} = \{te\} = t \circ \{e\} \subseteq t \circ A \cap e\hat{S}e = e \circ A \cap e\hat{S}e = A,$$

since A is ν-closed. □

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