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H. Çalışıcı; A. Pancar

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$\oplus$-COFINITELY SUPPLEMENTED MODULES

H. Çalışıcı, Amasya, and A. Pancar, Samsun

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Abstract. Let R be a ring and M a right R-module. M is called *$\oplus$-cofinitely supplemented* if every submodule N of M with M/N finitely generated has a supplement that is a direct summand of M. In this paper various properties of the $\oplus$-cofinitely supplemented modules are given. It is shown that (1) Arbitrary direct sum of $\oplus$-cofinitely supplemented modules is $\oplus$-cofinitely supplemented. (2) A ring R is semiperfect if and only if every free R-module is $\oplus$-cofinitely supplemented. In addition, if M has the summand sum property, then M is $\oplus$-cofinitely supplemented iff every maximal submodule has a supplement that is a direct summand of M.

Keywords: cofinite submodule, $\oplus$-cofinitely supplemented module

MSC 2000: 16D99

1. Introduction and preliminaries

Throughout this paper we assume that R is an associative ring with identity and all modules are unital right R-modules, unless otherwise specified. Let M be an R-module. By $N \leqslant M$ we mean that N is a submodule of M. A submodule N is called *superfluous* if $N + L \neq M$ for every proper submodule L of M. $N \ll M$ means that N is superfluous submodule of M. $\operatorname{Rad} M$ indicates the Jacobson radical of M. Let N and K be submodules of M. K is called a *supplement* of N in M if it is minimal in the collection of submodules L of M such that $M = N + L$, equivalently $M = N + K$ and $N \cap K \ll K$. For any ring R, an R-module M is called *supplemented* if every submodule of M has a supplement in M. In addition, for any ring R, any finite sum of supplemented R-modules is supplemented [6, 41.2].

Mohamed and Müller [5] call an R-module M *$\oplus$-supplemented* if every submodule of M has a supplement that is a direct summand of M. An R-module M is called *local* if the sum of all proper submodules is a proper submodule of M and is called *hollow* if every proper submodule of M is superfluous in M. Every local module is hollow. Note that hollow modules are $\oplus$-supplemented so that local modules are also $\oplus$-supplemented. Clearly $\oplus$-supplemented modules are supplemented. In addition, it was shown in [3, Theorem 1.4] that any finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented, but it is not generally true that any infinite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented. Let R be a semiperfect ring not right perfect. Then the R-module R_R is $\oplus$-supplemented by [4, Theorem 2.1], but the R-module $R^{(N)}$ is not $\oplus$-supplemented by [4, Theorem 2.10].

For characterizations of supplemented modules and $\oplus$-supplemented modules we refer to [5] and [6].

2. Semiperfect rings

It is known that a ring R is right perfect if and only if every free right R-module is $\oplus$-supplemented [4, Corollary 2.11]. In this section, we will find an analogous characterization for semiperfect rings.

Let R be an arbitrary ring and M be an R-module. A submodule N of M is called *cofinite* in M if the factor module M/N is finitely generated. In [1], an R-module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. In addition, it was shown in [1, Theorem 2.8] that an R-module M is cofinitely supplemented if and only if every maximal submodule of M has a supplement in M. Clearly supplemented modules are cofinitely supplemented.

An R-module M is called *$\oplus$-cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M. Note that $\oplus$-supplemented modules are $\oplus$-cofinitely supplemented. Also, finitely generated $\oplus$-cofinitely supplemented modules are $\oplus$-supplemented. If every maximal submodule of M is a direct summand of M then M is $\oplus$-cofinitely supplemented (see, [1, Lemma 2.7]).

In general it is not true that $\oplus$-cofinitely supplemented module is $\oplus$-supplemented. The $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers has not any proper cofinite submodule. Thus $\mathbb{Q}$ is $\oplus$-cofinitely supplemented, but the $\mathbb{Z}$-module Q is not torsion, so it is not supplemented by [7].

Lemma 2.1. *Let M be cofinitely supplemented. Then $M/\operatorname{Rad} M$ is $\oplus$-cofinitely supplemented.*

P r o o f. It follows from [1, Lemma 2.6]. □

Recall from Garcia [2] that a module M is said to have the *Summand Sum Property* (*SSP*) if the sum of two direct summands of M is again a direct summand of M.

Let $\{L_\lambda\}_{\lambda\in\Lambda}$ be the family of local submodules of M such that each of them is a direct summand of M. $\operatorname{Loc}^{\oplus} M$ will denote the sum of L_λs for all $\lambda \in \Lambda$. That is $\operatorname{Loc}^{\oplus} M = \sum\limits_{\lambda\in\Lambda} L_\lambda$. Note that 0 is a local submodule of M.

Lemma 2.2. *Let R be a ring and M be an R-module. Then every maximal submodule of M has a supplement that is a direct summand of M if and only if $M/\operatorname{Loc}^{\oplus} M$ does not contain a maximal submodule.*

Proof. ($\Rightarrow$) Suppose that $M/\operatorname{Loc}^{\oplus} M$ contains a maximal submodule $Q/\operatorname{Loc}^{\oplus} M$. Then Q is a maximal submodule of M. By assumption, there exist L, L' submodules of M such that $Q + L = M$, $Q \cap L \ll L$ and $M = L \oplus L'$. L is a local by [6, 41.1]. Therefore $L \leqslant \operatorname{Loc}^{\oplus} M \leqslant Q$ which is a contradiction.

($\Leftarrow$) Let P be a maximal submodule of M. By assumption, P does not contain $\operatorname{Loc}^{\oplus} M$. Hence there exists a local submodule L that is direct summand of M such that it is not a submodule of P. Since P is maximal, $P + L = M$, and $P \cap L \neq L$ so that $P \cap L \ll L$. $\square$

Theorem 2.3. *Let R be any ring and M be an R-module with* SSP. *Then the following statements are equivalent.*

1. *M is $\oplus$-cofinitely supplemented.*
2. *Every maximal submodule of M has a supplement that is a direct summand of M.*
3. *$M/\operatorname{Loc}^{\oplus} M$ does not contain a maximal submodule.*

Proof. (2) $\Leftrightarrow$ (3) is proved in Lemma 2.2.

(1) $\Rightarrow$ (2) If P is maximal submodule of M then M/P is simple so that it is cyclic.

(3) $\Rightarrow$ (1) Let N be a cofinite submodule of M. Then $N + \operatorname{Loc}^{\oplus} M$ is a cofinite submodule of M and by (3), $M = N + \operatorname{Loc}^{\oplus} M$. Because M/N is finitely generated, there exist local submodules $L_{\lambda_i} \in \{L_\lambda\}_{\lambda\in\Lambda}$, $1 \leqslant i \leqslant n$ for some positive integer n, such that $M = N + L_{\lambda_1} + \ldots + L_{\lambda_n}$. Clearly $N + L_{\lambda_1} + \ldots + L_{\lambda_n}$ has supplement 0 in M. By [1, Lemma 2.9], there exists a subset J of $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\sum\limits_{j\in J} L_j$ is a supplement of N in M. By hypothesis, $\sum\limits_{j\in J} L_j$ is a direct summand of M. Thus M is $\oplus$-cofinitely supplemented. $\square$

Let R be a ring and M an R-module. We consider the following condition.

(D3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

If M is a $\oplus$-supplemented module with (D3) then M is *completely $\oplus$-supplemented* (i.e. every direct summand of M is $\oplus$-supplemented) (see, [3, Proposition 2.3]). Now, we prove an analogue of this fact.

Proposition 2.4. *Let M be a $\oplus$-cofinitely supplemented module with* (D3). *Then every cofinite direct summand of M is $\oplus$-cofinitely supplemented.*

Proof. Let N be a cofinite direct summand of M. Then there exists a submodule N' of M such that $M = N \oplus N'$ and N' is finitely generated. Let U be a cofinite submodule of N. Note that $M/U = (N \oplus N')/U \cong N/U \oplus N'$ is finitely generated so that U is also cofinite submodule of M. Since M is $\oplus$-cofinitely supplemented, there exists a direct summand V of M such that $M = U + V$ and $U \cap V \ll V$. Hence $N = U + V \cap N$. Since M has (D3), $V \cap N$ is a direct summand of M. Furthermore $V \cap N$ is a direct summand of N because N is a direct summand of M. Then $U \cap (N \cap V) = U \cap V$ is superfluous in $V \cap N$ by [6, 19.3]. Hence N is $\oplus$-cofinitely supplemented. □

Lemma 2.5. *Let M be an R-module and N, U be submodules of M such that N is cofinitely supplemented, U cofinite and $N + U$ has a supplement A in M. Then $N \cap (U + A)$ has a supplement B in N, and $A + B$ is a supplement of U in M.*

Proof. Let A be a supplement of $N + U$ in M. Then $M = (N + U) + A$ and $(N + U) \cap A$ is superfluous in A. Now

$$\frac{N}{N \cap (U + A)} \cong \frac{N + U + A}{U + A} = \frac{M}{U + A} \cong \frac{M/U}{(U + A)/U}.$$

Since U is a cofinite submodule of M, $N \cap (U + A)$ is a cofinite submodule of N. Because N is cofinitely supplemented, $N \cap (U + A)$ has a supplement B in N. Note that $(U + A) \cap B$ is superfluous in B. Then

$$M = (N + U) + A = U + A + B$$

and by [6, 19.3],

$$\begin{aligned} U \cap (A + B) &\leqslant A \cap (U + B) + B \cap (U + A) \\ &\leqslant A \cap (N + U) + B \cap (U + A) \ll A + B. \end{aligned}$$

Therefore $A + B$ is a supplement of U in M. □

Theorem 2.6. *For any ring R, arbitrary direct sum of $\oplus$-cofinitely supplemented R-modules is $\oplus$-cofinitely supplemented.*

Proof. Let R be any ring and M_i $(i \in I)$ be any collection of $\oplus$-cofinitely supplemented R-modules. Let $M = \bigoplus_{i \in I} M_i$ and N be a cofinite submodule of M.

Then M/N is generated by some finite set $\{x_1+N, x_2+N, \ldots, x_k+N\}$ and therefore $M = x_1R+x_2R+\ldots+x_kR+N$. Since each x_i is contained in the direct sum $\bigoplus_{j\in F_i} M_j$ for some finite subset F_i of I, $x_1R + x_2R + \ldots + x_kR \leqslant \bigoplus_{j\in F} M_j$ for some finite subset $F = \{i_1, i_2, \ldots, i_r\}$ of I. Then $M = \bigoplus_{t=1}^{r} M_{i_t} + N$. Clearly $M = M_{i_1} + \left(\bigoplus_{t=2}^{r} M_{i_t} + N\right)$ has trivial supplement 0 in M. Since M_{i_1} is $\oplus$-cofinitely supplemented, $M_{i_1} \cap \left(\bigoplus_{t=2}^{r} M_{i_t} + N\right)$ has a supplement S_{i_1} in M_{i_1} such that S_{i_1} is a direct summand of M_{i_1}. By Lemma 2.5, S_{i_1} is a supplement of $\bigoplus_{t=2}^{r} M_{i_t} + N$ in M. Note that since M_{i_1} is a direct summand of M, S_{i_1} is also a direct summand of M. Continuing in this way, since the set J is finite at the end we will obtain that N has a supplement $S_{i_1} + S_{i_2} + \ldots + S_{i_r}$ in M such that every S_{i_t} $(1 \leqslant t \leqslant r)$ is a direct summand of M_{i_t}. Since every M_{i_t} is a direct summand of M, it follows that $\sum_{t=1}^{r} S_{i_t} = \bigoplus_{t=1}^{r} S_{i_t}$ is a direct summand of M. □

Corollary 2.7. *Any direct sum of $\oplus$-supplemented modules is $\oplus$-cofinitely supplemented.*

Therefore any direct sum of local (hollow) modules is $\oplus$-cofinitely supplemented.

As we remarked at the beginning of this section, a ring R is right perfect if and only if every free right R-module is $\oplus$-supplemented. Now we prove an analogue for semiperfect rings. Firstly we need the following lemma.

Lemma 2.8. *Let R be a ring with identity. Then the R-module R_R is $\oplus$-cofinitely supplemented if and only if every free R-module is $\oplus$-cofinitely supplemented.*

P r o o f. ($\Leftarrow$) Clear.

($\Rightarrow$) Let M be a free R-module and $A = \{a_i\}_{i\in I}$ be a basis of M. Then, it is well known that $M = \bigoplus_{i\in I} a_iR$ and $R \cong a_iR$ for all $i \in I$. By assumption, every cyclic R-module a_iR $(i \in I)$ is $\oplus$-cofinitely supplemented and M is $\oplus$-cofinitely supplemented by Theorem 2.6. □

Theorem 2.9. *The following statements are equivalent for a ring with identity.*

1. *R is semiperfect.*
2. *Every finitely generated free R-module is $\oplus$-supplemented.*
3. *R_R is $\oplus$-supplemented.*
4. *R_R is $\oplus$-cofinitely supplemented.*
5. *Every free R-module is $\oplus$-cofinitely supplemented.*

P r o o f. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is proved in [4, Theorem 2.1].

(3) $\Rightarrow$ (4) Clear from the definition.

(4) $\Rightarrow$ (5) It follows from Lemma 2.8.

(5) $\Rightarrow$ (2) Let M be a finitely generated free R-module. By hypothesis, M is $\oplus$-cofinitely supplemented. Since M is finitely generated, it follows that M is $\oplus$-supplemented. □

Corollary 2.10. *If R is a semiperfect division ring then every R-module is $\oplus$-cofinitely supplemented.*

P r o o f. Let R be a semiperfect division ring. By [6, 20.10], every R-module is free. Then by Theorem 2.9, we have the result. □

We give examples of modules, which are $\oplus$-cofinitely supplemented but not $\oplus$-supplemented. The R-module $R^{(N)}$ mentioned at the end of the first section is $\oplus$-cofinitely supplemented. In addition, if the $\mathbb{Z}$-module M is a direct sum of an infinite number of copies of the Prüfer p-group $\mathbb{Z}(p^\infty)$ then M is a direct sum of infinite number of $\oplus$-supplemented modules but is not supplemented. Note that M is $\oplus$-cofinitely supplemented by Corollary 2.7.

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References

[1] *R. Alizade, G. Bilhan and P. F. Smith*: Modules whose maximal submodules have supplements. Comm. Algebra *29* (2001), 2389–2405.

[2] *J. L. Garcia*: Properties of direct summands of modules. Comm. Algebra *17* (1989), 73–92.

[3] *A. Harmanci, D. Keskin and P. F. Smith*: On $\oplus$-supplemented modules. Acta Math. Hungar. *83* (1999), 161–169.

[4] *D. Keskin, P. F. Smith and W. Xue*: Rings whose modules are $\oplus$-supplemented. J. Algebra *218* (1999), 470–487.

[5] *S. H. Mohamed B. J. Müller*: Continuous and Discrete Modules. London Math. Soc. LNS Vol. 147. Cambridge Univ. Press, Cambridge, 1990.

[6] *R. Wisbauer*: Foundations of Module and Ring Theory. Gordon and Breach, Philadelphia, 1991.

[7] *H. Zöschinger*: Komplementierte Moduln über Dedekindringen. J. Algebra *29* (1974), 42–56.

Authors' addresses: H. Ç a l ı ş ı c ı, Department of Mathematics, Faculty of Education, Ondokuz Mayıs University, 05189, Amasya-Turkey, e-mail: hcalisici@omu.edu.tr; A. P a n c a r, Department of Mathematics, Faculty of Arts and Science, Ondokuz Mayıs University, 55139, Samsun-Turkey.