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SOME SOLUTIONS FOR A CLASS OF SINGULAR EQUATIONS

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Abstract. In this paper we obtain all solutions which depend only on r for a class of partial differential equations of higher order with singular coefficients.

Keywords: higher order equation, solution of type r^m, singular equation, iterated form

MSC 2000: 35G99

1. Introduction

This paper deals with some particular solutions for a class of singular partial differential equations of even order which include some well-known classical equations such as Laplace equation, GASPT (Generalized Axially Symmetric Potential Theory) equation and their iterated forms. There are numerous studies about these equations and equations involving these equations which have a lot of applications in potential theory. In [3], Elderly discussed singularities of the solutions which are regular in some region and are even functions of y for the partial differential equation

$$L(u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}=0 \tag{1.1}$$

where $x=x_1$, $y=(x_2^2+\ldots+x_n^2)^{\frac12}$ and $k=n-2$. Then, in [4], Weinstein showed that the stream function $\psi(x,y)$ which is defined by the equations

$$y^p\varphi_x=\psi_y,\quad y^p\varphi_y=-\psi_x \qquad \text{(Generalized Stokes-Beltrami Equations)}$$

satisfies equation (1.1) for $k=-p$. In [5], Weinstein studied a class of partial differential equation of the type

$$L_{\alpha_1}L_{\alpha_2}\ldots L_{\alpha_m}u=0 \tag{1.2}$$

where

$$L_{\alpha_k} = \frac{\partial^2}{\partial y^2} + \frac{\alpha_k}{y}\frac{\partial}{\partial y} \pm \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}, \quad \alpha_k = \text{constant}.$$

He showed that if $\alpha_{m-k} - 2k \neq \alpha_{m-l} - 2l$ for any $l \neq k$ then the general solution of (1.2) is given by $u = \sum_{i=0}^{m-1} u^{\alpha_{m-i}-2i}$ where u^{α_k} denotes a solution of $L_{\alpha_k}(u) = 0$. Later, in [8], Payne supposed that all α_i are equal in (1.2) and showed that the m times iterated equation $L_\alpha^m(u) = 0$ admits the solutions $u = \sum_{i=0}^{m-1} y^{2i}u^{\alpha+2i}$ and, provided $\alpha + 2j - 1 \neq 0$ for any integer j in the range $0 \leqslant j \leqslant m-2$, the solution $u = \sum_{i=0}^{m-2} y^{2i}u^{\alpha+2(i-1)} + y^{2(m-1)}u^{\alpha+2(m-1)}$. In [6], Weinstein proved that the equation which is known as the Weinstein or GASPT equation

$$L(u) = \sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{k_i}{x_i}\frac{\partial u}{\partial x_i}\right) = 0, \quad -\infty < k_i < \infty \tag{1.3}$$

admits the solution

$$r^{2-n-\sum_{i=1}^{n} k_i} u\left(\frac{x_1}{r^2}, \ldots, \frac{x_n}{r^2}\right)$$

where $r^2 = \sum_{i=1}^{n} x_i^2$ and $u(x_1, \ldots, x_n)$ is itself a solution of (1.3). In [7], Weinstein was interested in some solutions of the equation

$$L_k(u) = u_{tt} + \frac{k}{t}u_t + X(u) = 0 \tag{1.4}$$

where k is real or sometimes complex, $t \geqslant 0$ and $X(u)$ denotes a sufficiently regular linear differential operator which vanishes for $u = 0$. More recently , Altın in [1] and [2] obtained a solution of a class of partial differential equations

$$\left(L_1^{q_1} \ldots L_p^{q_p}\right) u = 0 \tag{1.5}$$

where

$$L_j = \sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2} + \frac{\alpha_i^{(j)}}{x_i}\frac{\partial}{\partial x_i}\right) \pm \sum_{i=1}^{s}\left(\frac{\partial^2}{\partial y_i^2} + \frac{\beta_i^{(j)}}{y_i}\frac{\partial}{\partial y_i}\right) + \frac{\gamma_j}{r^2} \tag{1.6}$$

and

$$L_j = \sum_{i=1}^{n}\left[\frac{\partial^2}{\partial x_i^2} + \frac{1}{x_i}\left(a_j r^2 \frac{\partial^3}{\partial x_i^3} + b_i^{(j)}\frac{\partial}{\partial x_i}\right)\right] + \frac{c_j}{r^2}, \tag{1.7}$$

respectively. In (1.6), $\alpha_i^{(j)}, \beta_i^{(j)}$ and γ_j are real parameters and $r^2 = \sum_{i=1}^{n} x_i^2 \pm \sum_{i=1}^{s} y_i^2$. In (1.7), a_j, $b_i^{(j)}$ and c_j are arbitrary real parameters and r is the Euclidean distance.

In this study, using a method similar to that given in [1] and [2] we will obtain solutions of type r^m for the equations of the form

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right) u = \left(L_1^{q_1} \dots L_p^{q_p}\right) u = 0 \tag{1.8}$$

where p, $q_1, \dots, q_p$ are positive integers and

$$\begin{aligned} L_j = r^2 \sum_{i=1}^{n} \left\{ \frac{\lambda_j a_i^6 r^2}{(x_i - x_i^0)^2} \frac{\partial^4}{\partial x_i^4} + \left[\frac{\mu_j a_i^4}{x_i - x_i^0} - \frac{\lambda_j a_i^6 r^2}{(x_i - x_i^0)^3} \right] \frac{\partial^3}{\partial x_i^3} \right\} \\ + \sum_{i=1}^{n} \left(a_i^2 \frac{\partial^2}{\partial x_i^2} + \frac{\alpha_i^{(j)}}{x_i - x_i^0} \frac{\partial}{\partial x_i} \right) + \frac{\gamma_j}{r^2}. \end{aligned} \tag{1.9}$$

The domain of the operator L_j is the set of all real-valued functions $u(x_1, \dots, x_n)$ of class $C^4(D)$ where D is a regularity domain of u in $\mathbb{R}^n$. The iterated operators $L_j^{q_j}$ are defined by the relations

$$L_j^{k+1}(u) = L_j \left[L_j^k(u)\right], \quad k = 1, \dots, q_j - 1.$$

In (1.9), x_i, $a_i \neq 0$ $(i = 1, \dots, n)$ are any real constants, λ_j, μ_j, γ_j, and $\alpha_i^{(j)}$ $(i = 1, \dots, n,\ j = 1, \dots, p)$ are any real parameters and r is given by

$$r = \left[\sum_{i=1}^{n} \left(\frac{x_i - x_i^0}{a_i}\right)^2\right]^{\frac{1}{2}}, \quad r > 0. \tag{1.10}$$

Note that if we set $a_i = 1$ $(i = 1, \dots, n)$ in (1.10), we get the Euclidean distance. In that case our solutions can be called as radial type solutions.

2. Solution of type r^m

In this section we will find a solution for equation (1.8) of type r^m which means the solution depends on r^m where m is any real or complex parameter.

Now, we first establish the following lemma.

Lemma 1. *Let p and $q_1, \ldots, q_p$ be arbitrary positive integers and m a real or complex parameter. Then*

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) = \prod_{j=1}^{p}\prod_{k=0}^{q_j-1} \varphi_j\left(m - 2\left[Q(p) - Q(j)\right] - 2k\right) r^{m-2Q(p)} \tag{2.1}$$

where $Q(j) = q_1 + \ldots + q_j$, $1 \leqslant j \leqslant p$ and $\varphi_j(m)$ is a fourth degree polynomial given by

$$\begin{aligned} \varphi_j(m) = {} & \lambda_j m^4 + (5n\lambda_j - 12\lambda_j + \mu_j)\, m^3 \\ & + (44\lambda_j - 30n\lambda_j + 3n\mu_j - 6\mu_j + 1)\, m^2 \\ & + (40n\lambda_j - 48\lambda_j - 6n\mu_j + 8\mu_j + n - 2 + \varrho_j)\, m + \gamma_j \end{aligned} \tag{2.2}$$

with $\varrho_j = \sum\limits_{i=1}^{n} \alpha_i^{(j)}/a_i^2$.

P r o o f. From the definitions of L_j and r, for any real or complex parameter m, it is easily seen by direct calculation that

$$L_j(r^m) = \varphi_j(m)\, r^{m-2}. \tag{2.3}$$

Applying the operator L_j repeatedly $q-1$ times on both sides of (2.3), we obtain

$$L_j^q(r^m) = \left\{\prod_{k=0}^{q-1} \varphi_j(m-2k)\right\} r^{m-2q}. \tag{2.4}$$

Replacing q in (2.4) by q_j we then have

$$L_j^{q_j}(r^m) = \left\{\prod_{k=0}^{q_j-1} \varphi_j(m-2k)\right\} r^{m-2q_j}. \tag{2.5}$$

Now we will establish (2.1) by induction on p. Hence, for $j=1$, (2.5) can be written as

$$L_1^{q_1}(r^m) = \left\{\prod_{k=0}^{q_1-1} \varphi_1(m-2k)\right\} r^{m-2q_1} \tag{2.6}$$

and considering the relation $Q(j) = q_1 + \ldots + q_j$, $1 \leqslant j \leqslant p$, we can write (2.6) as

$$\left(\prod_{j=1}^{1} L_j^{q_j}\right)(r^m) = \prod_{j=1}^{1}\prod_{k=0}^{q_j-1} \varphi_j\left(m - 2\left[Q(1) - Q(j)\right] - 2k\right) r^{m-2Q(1)}.$$

Therefore, (2.1) is true for $p=1$. Now assume that (2.1) is valid for $p-1$, that is,

$$\left(\prod_{j=1}^{p-1} L_j^{q_j}\right)(r^m) = \prod_{j=1}^{p-1}\prod_{k=0}^{q_j-1} \varphi_j\left(m-2\left[Q(p-1)-Q(j)\right]-2k\right) r^{m-2Q(p-1)}. \tag{2.7}$$

We set $j=p$ in (2.5), obtaining

$$L_p^{q_p}(r^m) = \left\{\prod_{k=0}^{q_p-1} \varphi_p(m-2k)\right\} r^{m-2q_p}. \tag{2.8}$$

Applying the linear operator $\prod\limits_{j=1}^{p-1} L_j^{q_j}$ on both sides of (2.8), we then find

$$\left(\prod_{j=1}^{p-1} L_j^{q_j}\right)\left(L_p^{q_p}(r^m)\right) = \prod_{k=0}^{q_p-1} \varphi_p(m-2k)\left(\prod_{j=1}^{p-1} L_j^{q_j}\left(r^{m-2q_p}\right)\right). \tag{2.9}$$

If we replace m in (2.7) by $m-2q_p$, then (2.9) can be written as

$$\begin{aligned}\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) &= \left(\prod_{k=0}^{q_p-1} \varphi_p(m-2k)\right)\\ &\quad\times\left(\prod_{j=1}^{p-1}\prod_{k=0}^{q_j-1} \varphi_j\left(m-2q_p-2\left[Q(p-1)-Q(j)\right]-2k\right)\right) r^{m-2q_p-2Q(p-1)}.\end{aligned} \tag{2.10}$$

Since $2q_p+2Q(p-1)=Q(p)$, (2.10) gives formula (2.1). Thus Lemma is proved. □

We turn to formula (2.1) and write the algebraic polynomial equation

$$\prod_{j=1}^{p}\prod_{k=0}^{q_j-1} \varphi_j\left(m-2\left[Q(p)-Q(j)\right]-2k\right)=0 \tag{2.11}$$

which is of degree $4Q(p)$. The number of real or complex roots of equation (2.11) is $4Q(p)$ for all $\lambda_j \neq 0$ $(1 \leqslant j \leqslant p)$.

Now using Lemma 1, we can prove the following theorem.

Theorem 1. *Let the algebraic polynomial equation* (2.11) *have distinct real roots* $c_1, \ldots, c_M$ *having multiplicity* $\xi_1, \ldots, \xi_M$*, respectively, and distinct complex roots* $\alpha_1 \pm \mathrm{i}\beta_1, \ldots, \alpha_N \pm \mathrm{i}\beta_N$ *having multiplicity* $\tau_1, \ldots, \tau_N$*, respectively. Then solutions of type* r^m *of the equation* (1.8) *are given by the formula*

$$
\begin{aligned}
u(r) = &\sum_{w=1}^{M} \sum_{k_1=0}^{\xi_w - 1} A_{wk_1} r^{c_w} (lnr)^{k_1} \\
&+ \sum_{s=1}^{N} \sum_{k_2=0}^{\tau_s - 1} r^{\alpha_s} (lnr)^{k_2} \left[B_{sk_2} \cos(\beta_s lnr) + C_{sk_2} \sin(\beta_s lnr)\right]
\end{aligned}
\tag{2.12}
$$

where A_{wk_1}, B_{sk_2} *and* C_{sk_2} *are arbitrary constants.*

P r o o f. According to the hypothesis, (2.11) has the following factors which concern its real and complex roots:

$$
\prod_{w=1}^{M} (m - c_w)^{\xi_w} \qquad \text{and} \qquad \prod_{s=1}^{N} \left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\tau_s}.
$$

Therefore, (2.1) can be written as

$$
\begin{aligned}
\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) = &\left\{\prod_{j=1}^{p} \lambda_j^{q_j} \prod_{w=1}^{M} (m - c_w)^{\xi_w} \right. \\
&\left. \times \prod_{s=1}^{N} \left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\tau_s}\right\} r^{m-2Q(p)}
\end{aligned}
\tag{2.13}
$$

where $\sum\limits_{w=1}^{M} \xi_w + 2\sum\limits_{s=1}^{N} \tau_s = 4Q(p)$ is the order of equation (1.8).

On the other hand, the following equalities are well known:

$$
\begin{aligned}
\frac{\partial^k}{\partial m^k}\left[\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m)\right] &= \left(\prod_{j=1}^{p} L_j^{q_j}\right)\left(\frac{\partial^k r^m}{\partial m^k}\right) \\
&= \left(\prod_{j=1}^{p} L_j^{q_j}\right)\left[r^m (lnr)^k\right], \quad k \in N,
\end{aligned}
\tag{2.14}
$$

$$
r^{\alpha_s \pm \mathrm{i}\beta_s} = r^{\alpha_s} r^{\pm \mathrm{i}\beta_s} = r^{\alpha_s} \mathrm{e}^{\pm i\beta_s lnr} = r^{\alpha_s}\left[\cos(\beta_s lnr) \pm \mathrm{i}\sin(\beta_s lnr)\right].
\tag{2.15}
$$

Now consider again (2.13). It is obvious that the right-hand side of (2.13) has factors $(m - c_w)^{\xi_w}$, $w = 1, \ldots, M$ which vanish for $m = c_w$, $w = 1, \ldots, M$ together with their derivatives with respect to m

$$
\frac{\mathrm{d}^{k_1}}{\mathrm{d}m^{k_1}} (m - c_w)^{\xi_w}, \quad k_1 = 1, \ldots, \xi_w - 1, \; w = 1, \ldots, M.
$$

Thus, the function r^{c_w} and by virtue of (2.14), each of the functions

$$\left.\frac{\partial^{k_1} r^m}{\partial m^{k_1}}\right|_{m=c_w} = r^{c_w}\left(lnr\right)^{k_1}, \quad k_1 = 1, \dots, \xi_w - 1,\ w = 1, \dots, M$$

satisfy equation (1.8). Since the given equation is linear, by the superposition principle the sum

$$\sum_{w=1}^{M} \sum_{k_1=0}^{\xi_w - 1} A_{wk_1} r^{c_w} \left(lnr\right)^{k_1} \tag{2.16}$$

also satisfies (1.8). Similarly, the factors of (2.13)

$$\left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\tau_s} = \left[m - (\alpha_s + \mathrm{i}\beta_s)\right]^{\tau_s} \left[m - (\alpha_s - \mathrm{i}\beta_s)\right]^{\tau_s}, \ s = 1, \dots, N$$

and the expressions

$$\frac{\mathrm{d}^{k_2}}{\mathrm{d}m^{k_2}} \left[m - (\alpha_s \pm \mathrm{i}\beta_s)\right]^{\tau_s}, \quad k_2 = 1, \dots, \tau_s - 1,\ s = 1, \dots, N$$

are zero for $m = \alpha_s \pm \mathrm{i}\beta_s$. Hence by (2.14) and (2.15), for $k_2 = 0, 1, \dots, \tau_s - 1$, $s = 1, \dots, N$ each of the functions

$$\left.\frac{\partial^{k_2} r^m}{\partial m^{k_2}}\right|_{m=\alpha_s \pm \mathrm{i}\beta_s} = r^{\alpha_s \pm \mathrm{i}\beta_s}\left(lnr\right)^{k_2} = r^{\alpha_s}\left(lnr\right)^{k_2}\left[\cos\left(\beta_s lnr\right) \pm \mathrm{i}\sin\left(\beta_s lnr\right)\right]$$

and their superposition

$$\sum_{s=1}^{N} \sum_{k_2=0}^{\tau_s - 1} r^{\alpha_s}\left(lnr\right)^{k_2}\left[B_{sk_2}\cos\left(\beta_s lnr\right) + C_{sk_2}\sin\left(\beta_s lnr\right)\right] \tag{2.17}$$

satisfy (1.8). Therefore, the sum of (2.16) and (2.17) gives (2.12). Thus the theorem is proved. □

Remark. If $k_1 = 0$ and c_w, $w = 1, \dots, M$ are even natural numbers in (2.16), then we obtain polynomial solutions which are analytic everywhere including the singularity hyperplanes $x_i = x_i^0$.

3. Solution of type $u = u(r)$

In this section we will show that all solutions for equation (1.8) which depend only on r can be expressed by formula (2.12).

Lemma 2. *Let q be an arbitrary positive integer. Then for the function $u = u(r)$*

$$L_j^q u = \mathrm{e}^{-2qt}\left\{\prod_{k=0}^{q-1}\varphi_j\left(D-2k\right)\right\}u \tag{3.1}$$

holds where $D = \mathrm{d}/\mathrm{d}t, r = \mathrm{e}^t$ and φ_j are given by (2.2).

P r o o f. We will prove this lemma by induction on q. Noticing the definition of r given by (1.10), if we apply operator L_j to $u = u(r)$ we find

$$\begin{aligned} L_j u = {}& \lambda_j r^2 \frac{\mathrm{d}^4 u}{\mathrm{d}r^4} + (5n\lambda_j - 6\lambda_j + \mu_j)\, r\frac{\mathrm{d}^3 u}{\mathrm{d}r^3} \\ & + (15\lambda_j - 15n\lambda_j + 3n\mu_j - 3\mu_j + 1)\,\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} \\ & + (15n\lambda_j - 15\lambda_j - 3n\mu_j + 3\mu_j + n - 1 + \varrho_j)\, r^{-1}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{\gamma_j u}{r^2}. \end{aligned} \tag{3.2}$$

It is easy to see that L_j becomes an Euler type operator. We let $r = \mathrm{e}^t$. Then we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} &= \mathrm{e}^{-t} D, \\ \frac{\mathrm{d}^2}{\mathrm{d}r^2} &= \mathrm{e}^{-2t}\left(D^2 - D\right), \\ \frac{\mathrm{d}^3}{\mathrm{d}r^3} &= \mathrm{e}^{-3t}\left(D^3 - 3D^2 + 2D\right), \\ \frac{\mathrm{d}^4}{\mathrm{d}r^4} &= \mathrm{e}^{-4t}\left(D^4 - 6D^3 + 11D^2 - 6D\right). \end{aligned}$$

Thus, substituting into (3.2), we obtain

$$\begin{aligned} L_j u = {}& \mathrm{e}^{-2t}\big[\lambda_j D^4 + (5n\lambda_j - 12\lambda0_j + \mu_j)\, D^3 \\ & + (44\lambda_j - 30n\lambda_j + 3n\mu_j - 6\mu_j + 1)\, D^2 \\ & + (40n\lambda_j - 48\lambda_j - 6n\mu_j + 8\mu_j + n - 2 + \varrho_j)\, D + \gamma_j\big]u \\ = {}& \mathrm{e}^{-2t}\varphi_j(D)u. \end{aligned} \tag{3.3}$$

Hence, (3.1) is true for $q = 1$. Now we suppose that (3.1) is true for $q - 1$, that is,

$$L_j^{q-1} u = \mathrm{e}^{-2(q-1)t}\left\{\prod_{k=0}^{q-2}\varphi_j\left(D-2k\right)\right\}u. \tag{3.4}$$

Applying the operator L_j on both sides of (3.4), we find

$$L_j^q u = L_j\left(\mathrm{e}^{-2(q-1)t}\left\{\prod_{k=0}^{q-2}\varphi_j\left(D-2k\right)\right\}u\right).$$

We know from (3.3) that $L_j = \mathrm{e}^{-2t}\varphi_j(D)$, therefore the right-hand side of the above equality can be written as

$$L_j^q u = \mathrm{e}^{-2t}\varphi_j(D)\left(\mathrm{e}^{-2(q-1)t}\left\{\prod_{k=0}^{q-2}\varphi_j\left(D-2k\right)\right\}u\right). \tag{3.5}$$

From the theory of ordinary differential equations it is known that, for any two polynomials of the operator D with constant coefficients G and H and for any constant a, the following relation holds [1]:

$$G\left(D\right)\left\{\mathrm{e}^{-at}H\left(D\right)u\right\} = \mathrm{e}^{-at}G\left(D-a\right)H\left(D\right)u. \tag{3.6}$$

Using this property, we can write (3.5) as

$$\begin{aligned} L_j^q u &= \mathrm{e}^{-2t}\mathrm{e}^{-2(q-1)t}\varphi_j\left(D-2\left(q-1\right)\right)\prod_{k=0}^{q-2}\varphi_j\left(D-2k\right)u \\ &= \mathrm{e}^{-2qt}\left\{\prod_{k=0}^{q-1}\varphi_j\left(D-2k\right)\right\}u. \end{aligned}$$

Thus, the proof is complete. We remark that the product of the operators $\prod\limits_j \varphi_j$ is commutative.

Lemma 3. *Let p and $q_1, \ldots, q_p$ be arbitrary positive integers. Then*

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right)u = \mathrm{e}^{-2Q(p)t}\prod_{j=1}^{p}\prod_{k=0}^{q_j-1}\varphi_j\left(D-2\left[Q(p)-Q(j)\right]-2k\right)u. \tag{3.7}$$

Using (3.1), this is easily proved in a manner similar to the proof of Lemma 2.

Now, we will establish the following theorem.

Theorem 2. *All solutions for equation* (1.8) *of the type* $u = u(r)$ *can be expressed by formula* (2.12).

P r o o f. Equating (3.7) to zero, we find the following ordinary differential equation with constant coefficients and of order $4Q(p) = 4\,(q_1 + \ldots + q_p)$:

$$\prod_{j=1}^{p} \prod_{k=0}^{q_j - 1} \varphi_j \left(D - 2\left[Q(p) - Q(j)\right] - 2k\right) u = 0. \tag{3.8}$$

The characteristic equation of (3.8) is

$$\prod_{j=1}^{p} \prod_{k=0}^{q_j - 1} \varphi_j \left(m - 2\left[Q(p) - Q(j)\right] - 2k\right) = 0.$$

This was obtained in Lemma 1. Therefore, from Theorem 1 we know that this equation has the factors

$$\prod_{w=1}^{M} (m - c_w)^{\xi_w} \qquad \text{and} \qquad \prod_{s=1}^{N} \left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\tau_s}.$$

Hence the solution of (3.8) is given by

$$\begin{aligned} u(t) = {} & \sum_{w=1}^{M} \sum_{k_1=0}^{\xi_w - 1} A_{wk_1} t^{k_1} \mathrm{e}^{c_w t} \\ & + \sum_{s=1}^{N} \sum_{k_2=0}^{\tau_s - 1} \mathrm{e}^{\alpha_s t} t^{k_2} \left[B_{sk_2} \cos\left(\beta_s t\right) + C_{sk_2} \sin\left(\beta_s t\right)\right]. \end{aligned} \tag{3.9}$$

Since $\mathrm{e}^t = r$, we set $t = lnr$ in (3.9) arriving at formula (2.12). Thus the theorem is proved. □

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