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A NOTE ON ULTRAMETRIC MATRICES

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Abstract. It is proved in this paper that special generalized ultrametric and special $\mathscr{U}$ matrices are, in a sense, extremal matrices in the boundary of the set of generalized ultrametric and $\mathscr{U}$ matrices, respectively. Moreover, we present a new class of inverse M-matrices which generalizes the class of $\mathscr{U}$ matrices.

Keywords: generalized ultrametric matrix, $\mathscr{U}$ matrix, weighted graph, inverse M-matrix

MSC 2000: 15A09, 15A57, 05C50

1. Introduction

It is a longstanding open problem to characterize the nonnegative matrices whose inverses are M-matrices (see [15]), although the inverse of a nonsingular M-matrix is always a nonnegative matrix. In 1994, Martínez, Michon and San Martín introduced *strictly symmetric ultrametric matrix* $A = (a_{ij})$ whose entries satisfy

$$a_{ij} \geqslant \min\{a_{ik}, a_{kj}\} \quad \text{for all } i, j, k, \tag{1}$$

$$a_{ii} > a_{ij} \quad \text{for all } i \neq j \tag{2}$$

and proved that the inverse of a strictly symmetric ultrametric matrix is a row and column diagonally dominant M-matrix (see [8] and [12]). Later, nonsymmetric ultrametric matrices were independently introduced in [10] and in [13]; i.e., nested block form (for short, NBF) and generalized ultrametric matrices (for short, GUM),

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respectively. After a suitable permutation, every GUM can be put in NBF. In other words, there exists a permutation matrix P such that

$$PAP^t = \begin{pmatrix} A_{11} & b_{12}\mathbf{1}\mathbf{1}^t \\ b_{21}\mathbf{1}^t\mathbf{1} & A_{22} \end{pmatrix} \tag{3}$$

where A_{11} and A_{22} are GUM and $b_{12} \leqslant b_{21}$, $\min\{a_{ij}, a_{ji}\} \geqslant b_{12}$, $\max\{a_{ij}, a_{ji}\} \geqslant b_{21}$ for all i, j, $\mathbf{1}$ is the vector of all one's. Moreover, if A itself and as well as all its principal submatrices which are GUM are of the form (3), then A is called *NBF*. They proved that this class of matrices has similar properties as strictly symmetric ultrametric matrices. In other words, the inverse of a nonsingular GUM is a row and column diagonally dominant M-matrix. Let $A = (a_{ij})$ be an $n \times n$ NBF of the form (3), where A_{11} and A_{22} are $m \times m$ and $(n-m) \times (n-m)$ NBF. An $n \times n$ matrix $B = (b_{ij})$ is called a $\mathscr{U}$ *matrix* (see [11]), if $b_{ij} = a_{ij}$ for $1 \leqslant i \leqslant j \leqslant n$; $b_{ij} = a_{ij}$ for $1 \leqslant j < i \leqslant m$ and $b_{ij} = a_{in}$ for $i > j$ and $m+1 \leqslant i \leqslant n$. Nabben in [11] proved that the inverse of a $\mathscr{U}$ matrix is a column diagonally dominant M-matrix. Many other properties on GUM and other related classes were investigated by many authors (for example, see [3], [4], [14], etc.).

Recently, Fiedler in [6] defined that an $n \times n$ matrix A is called a *special symmetric ultrametric matrix* if A is symmetric nonnegative and satisfies (1) and

$$a_{ii} = \max\{a_{ij}\,;\ j \neq i\} \quad \text{for } i = 1, \ldots, n. \tag{4}$$

Further, he proved that special symmetric ultrametric matrices are, in a sense, extremal matrices in the boundary of the set of strictly symmetric ultrametric matrices. Although they are not inverses of M-matrices, these matrices are in the closure of inverses of weakly row and column diagonally dominant nonsingular M-matrices. In other words, they are the limits of convergent sequences of matrices that are inverses of weakly row and column diagonally dominant M-matrices. Moreover, he gave a simple structure of these matrices using weighted graphs. As for the closure of inverses of M-matrices, the reader may be referred to [2] and [7].

This paper is motivated by the results of Fiedler [6] and Nabben [11]. We introduce special GUM and special $\mathscr{U}$ matrices in Section 2 and 3 respectively, which are, in a sense, extremal matrices in the boundary of the set of GUM and $\mathscr{U}$ matrices. Further, we present a simple construction of these matrices by using doubly edge-weighted paths and mixed edge-weighted paths. The result generalizes the result of Fiedler in [6]. In section 4, we introduce a new class of inverse M-matrices which generalizes the class of $\mathscr{U}$ matrices.

2. Special generalized ultrametric matrices

Definition 2.1. An $n \times n$ matrix $A = (a_{ij})$ is called a special generalized ultrametric matrix (for short, special GUM), if A is a generalized ultrametric matrix and satisfies

$$a_{ii} = \max\{a_{ij}, a_{ji},\ j \neq i\} \quad \text{for } i = 1, \dots, n. \tag{5}$$

Moreover, if A is an NBF and satisfies (5), then A is called *a special NBF*. Clearly, A is a special GUM if and only if there exists a permutation matrix P such that PAP^t is a special NBF. It is easy to see that A is the limit of a convergent sequence of matrices which are inverses of weakly row and column diagonally dominant M-matrices from Theorem B in [6] or in [7]. Moreover, if A is symmetric, it is just a special symmetric ultrametric matrix in [6]. However, a special GUM may be not singular, while each special symmetric ultrametric matrix is always singular.

Let $T = (V, E)$ be a path on the vertex set $V = \{v_1, \dots, v_n\}$ and the edge set $E = \{E_1, \dots, E_{n-1}\}$, where $E_i = (v_i, v_{i+1})$ for $i = 1, \dots, n-1$. If two nonnegative numbers $\alpha_i \leqslant \beta_i$ are assigned to each edge E_i, for $i = 1, \dots, n-1$ and satisfy the following condition "for any $i < j$, there exists an $i \leqslant p < j$ such that $\alpha_p = \min\{\alpha_k;\ E_k$ is an edge in the path from v_i to $v_j\}$ and $\beta_p = \min\{\beta_k;\ E_k$ is an edge in the path from v_i to $v_j\}$", then T is called a *double edge-weighted path* with two vectors $\vec{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$ and $\vec{\beta} = (\beta_1, \dots, \beta_{n-1})$ (for short, double edge-weighted path). Hence we can define an $n \times n$ nonnegative matrix $C(T) = (c_{ij})$ associated with a double edge-weighted path T as follows:

For $i < j$, $c_{ij} = \min\{\alpha_k;\ E_k$ is an edge in the path from v_i to $v_j\}$; for $i > j$, $c_{ij} = \min\{\beta_k;\ E_k$ is an edge in the path from V_i to $V_j\}$ and $c_{ii} = \max\{\beta_k;\ E_k$ is incident with $v_i\}$.

The main result in this section is that the class of all special generalized ultrametric matrices just coincides with the class of all matrices $C(T)$ with doubly edge-weighted paths, up to permutation.

Lemma 2.2. *Let $A = (a_{ij})$ be an $n \times n$ special* NBF. *Then there exists a double edge-weighted path T with $\vec{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$ and $\vec{\beta} = (\beta_1, \dots, \beta_{n-1})$ such that $A = C(T)$.*

Proof. We prove the assertion by the induction on n. The assertion is trivial for $n = 2$. Assume that the assertion holds for less than n. Since $A = (a_{ij})$ is special NBF, A has the following form

$$A = \begin{pmatrix} A_{11} & b_{12}\mathbf{1}\mathbf{1}^t \\ b_{21}\mathbf{1}^t\mathbf{1} & A_{22} \end{pmatrix}, \tag{6}$$

where A_{11} and A_{22} are $m\times m$ and $(n-m)\times(n-m)$ NBF, respectively; $b_{21}\geqslant b_{12}$, $\min\{a_{ij},a_{ji}\}\geqslant b_{12}$ and $\max\{a_{ij},a_{ji}\}\geqslant b_{21}$ for all i,j. We first suppose that $1<m<n-1$. By the induction hypothesis, there exist double edge-weighted paths T_1 on the vertex set $V_1=\{v_1,\dots,v_m\}$ with two edge-weighted vectors $(\alpha_1,\dots,\alpha_{m-1})$ and $(\beta_1,\dots,\beta_{m-1})$ and T_2 on the vertex set $V_2=\{v_{m+1},\dots,v_n\}$ with two edge-weighted vectors $(\alpha_{m+1},\dots,\alpha_{n-1})$ and $(\beta_{m+1},\dots,\beta_{n-1})$ such that $A_{11}=C(T_1)$ and $A_{22}=C(T_2)$, respectively. Now let T be a path on the vertex set $V=\{v_1,\dots,v_n\}$ obtained from $T_1\cup T_2$ by adding one edge (v_m,v_{m+1}) to $T_1\cup T_2$ with two weight $\alpha_m=b_{12}\leqslant\beta_m=b_{21}$. For $i\leqslant m<j$, by the induction hypothesis, we have $\alpha_m=b_{12}=\min\{a_{kl},a_{lk},k\neq l\}\leqslant\min\{a_{i,i+1},\dots,a_{j-1,j}\}=\min\{\alpha_k;\ E_k$ is an edge in the path from v_i to $v_j\}\leqslant\alpha_m$. Hence $\alpha_m=\min\{\alpha_k;\ E_k$ is edge in the path from v_i to $v_j\}$. Similarly, $\beta_m=b_{21}=\min\{\beta_k,\ E_k$ is an edge in the path from v_i to $v_j\}$. Hence T is a double edge-weighted path on n vertices with $\vec{\alpha}=(\alpha_1,\dots,\alpha_{n-1})$ and $\vec{\beta}=(\beta_1,\dots,\beta_{n-1})$. Moreover, it is easy to see that

$$C(T)=\begin{pmatrix} C(T_1) & \alpha_m\mathbf{1}\mathbf{1}^t\\ \beta_m\mathbf{1}^t\mathbf{1} & C(T_2)\end{pmatrix}.$$

If $m=1$ or $n-1$, we define $C(T_1)=(b_{21})$ or $C(T_2)=(b_{21})$. Therefore, there exists a double edge-weighted path T such that $A=C(T)$. □

Lemma 2.3. *Let $C(T)$ be matrix associated with a doubly edge-weighted path T on n vertices. Then $C(T)$ is a special* NBF.

Proof. We prove the assertion by the induction on n. It is trivial for $n=2$. Assume that the assertion holds for less than n. We assume that there is a double edge-weighted path T on a vertex set $\{v_1,\dots,v_n\}$ with two edge-weighted vectors $\vec{\alpha}$ and $\vec{\beta}$. Then by the definition of a double edge-weighted path, there exists a $1\leqslant m\leqslant n-1$ such that $\alpha_m=\min\{\alpha_k,\ E_k$ is an edge in the path from v_1 to $v_n\}$ and $\beta_m=\min\{\beta_k,\ E_k$ is an edge in the path from v_1 to $v_n\}$. By the definition of $C(T)=(c_{ij})$, we have $c_{ij}=\alpha_m$ for $i\leqslant m<j$, and $c_{ij}=\beta_m$ for $i>m\geqslant j$. Hence $C(T)$ has the following form

$$C(T)=\begin{pmatrix} C(T_1) & \alpha_m\mathbf{1}\mathbf{1}^t\\ \beta_m\mathbf{1}^t\mathbf{1} & C(T_2)\end{pmatrix},$$

where $C(T_1)$ and $C(T_2)$ are matrices associated with double edge-weighed paths T_1 on vertices $\{v_1,\dots,v_m\}$ with two edge-weighted vectors $(\alpha_1,\dots,\alpha_{m-1})$, $(\beta_1,\dots,\beta_{m-1})$ and T_2 on vertices $\{v_{m+1},\dots,v_n\}$ with two edge-weighted vectors $(\alpha_{m+1},\dots,\alpha_{n-1})$, $(\beta_{m+1},\dots,\beta_{n-1})$, respectively. Moreover, $\min\{c_{ij},c_{ji}\}\geqslant\alpha_m$, and $\max\{c_{ij},c_{ji}\}\geqslant\beta_m$. By the induction hypothesis, $C(T_i)$ is a special NBF for $i=1,2$. Moreover,

$c_{ii} = \max\{c_{i1}, \ldots, c_{im}, c_{1i}, \ldots, c_{mi}\} = \max\{c_{i1}, \ldots c_{in}, c_{1i}, \ldots, c_{ni}\}$ for $i = 1, \ldots, m$ and $c_{ii} = \max\{c_{i,m+1}, \ldots, c_{in}, c_{m+1,i}, \ldots, c_{ni}\} = \max\{c_{i1}, \ldots, c_{in}, c_{1i}, \ldots, c_{ni}\}$ for $i = m+1, \ldots, n$. Hence $C(T)$ is a special NBF. □

Theorem 2.4. *Let A be an $n \times n$ nonnegative matrix. Then the following statements are equivalent:*

(i) *A is a special* GUM.

(ii) *There exist a double edge-weighted path T and permutation matrix P such that $PAP^t = C(T)$.*

Proof. (i) $\Longrightarrow$ (ii). By Lemma 4.1 [10], there exists a permutation matrix P such that PAP^t is a special NBF. Hence (ii) follows from Lemma 2.2. The converse directly follows from Lemma 2.3. □

Corollary 2.5. *A is a special* NBF *if and only if there exists a double edge-weighted path such that $A = C(T)$.*

Remark 2.6. For some special GUM, there exists a double edge-weighted path T such that $A = C(T)$. For example,

$$A = \begin{pmatrix} 9 & 9 & 7 & 7 \\ 3 & 9 & 7 & 7 \\ 2 & 2 & 8 & 6 \\ 2 & 2 & 8 & 8 \end{pmatrix}$$

is a special GUM matrix, but there does not exist a double edge-weighted path such that $A = C(T)$. However, if A is a symmetric special ultrametric matrix, there always exists a double edge-weighted path T such that $A = C(T)$ by Theorem 2.2 in [6]. In fact, since the permutation P corresponds to renumbering of the vertices, then Theorem 2.2 in [6] immediately follows from Theorem 2.4. Hence Theorem 2.4 generalizes the result of Fiedler, since in this case, $\vec{\alpha} = \vec{\beta}$. In the next Theorem, we shall investigate the singularity of a special NBF given by a double edge-weighted path.

Theorem 2.7. *Let $A = (a_{ij})$ be an $n \times n$ matrix associated with a double edge-weighted path T and two vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ and $\vec{\beta} = (\beta_1, \ldots, \beta_{n-1})$. Then A is singular if and only if $\alpha_1 = \beta_1 = 0$; or $\alpha_{n-1} = \beta_{n-1} = 0$; or $\alpha_{p-1} = \alpha_p = \beta_{p-1} = \beta_p = 0$; or $\min\{\alpha_p, \ldots, \alpha_{q-1}\} = \alpha_p = \alpha_{q-1} = \min\{\beta_p, \ldots, \beta_{q-1}\} = \beta_{q-1} = \beta_p \geqslant \max\{\beta_{p-1}, \beta_q\}$ for some $1 < p < q \leqslant n$.*

Proof. Sufficiency: If $\alpha_1 = \beta_1 = 0$, or $\alpha_{n-1} = \beta_{n-1} = 0$, or $\alpha_{p-1} = \alpha_p = \beta_{p-1} = \beta_p = 0$, then all entries of the first, or last, or p-th rows of A are zero.

Hence A is singular. Now we may assume that $\min\{\alpha_p,\ldots,\alpha_{q-1}\} = \alpha_p = \alpha_{q-1} = \min\{\beta_p,\ldots,\beta_{q-1}\} = \beta_{q-1} = \beta_p \geqslant \max\{\beta_{p-1},\beta_q\}$ for some $1 < p < q \leqslant n$. We shall show that the p-th and q-th rows of A are the same. In fact, for $j < p$, $a_{pj} = \min\{\beta_k;\ E_k$ is an edge in the path from vertex v_p to vertex $v_j\} = \min\{\beta_k;\ E_k$ is an edge in the path from vertex v_q to vertex $v_j\} = a_{qj}$, since $\min\{\beta_p,\ldots,\beta_{q-1}\} = \beta_p \geqslant \beta_{p-1}$. For $j > q$, $a_{pj} = \min\{\alpha_k;\ E_k$ is an edge in the path from vertex v_p to vertex $v_j\} = \min\{\alpha_k;\ E_k$ is an edge in the path from vertex v_q to vertex $v_j\} = a_{qj}$, since $\min\{\alpha_p,\ldots,\alpha_{q-1}\} = \alpha_p \geqslant \alpha_q$. For $p < j < q$, $a_{pj} = \min\{\alpha_k;\ E_k$ is an edge in the path from vertex v_p to vertex $v_j\} = \alpha_p = \beta_p = \beta_{q-1} = \min\{\beta_k;\ E_k$ is an edge in the path from vertex v_q to vertex $v_j\} = a_{qj}$, since $\alpha_p = \min\{\alpha_p,\ldots,\alpha_{q-1}\}$ and $\beta_{q-1} = \min\{\beta_p,\ldots,\beta_{q-1}\}$. Moreover, $a_{pp} = \max\{\beta_k;\ E_k$ is incident with $v_p\} = \beta_p = \max\{\beta_k;\ E_k$ is an edge in the path from v_p to vertex $v_q\} = a_{qp}$ and $a_{qq} = \max\{\beta_k;\ E_k$ is incident with $v_q\} = \beta_{q-1} = \min\{\alpha_k;\ E_k$ is an edge in the path from v_p to vertex $v_q\} = a_{pq}$. Hence A is singular.

Necessity. Assume that A is singular. If all entries of p-th row of A are zero, then by the definition of $A = C(T)$, if $p = 1$, then $\alpha_1 = \beta_1 = 0$; or if $p = n$, then $\alpha_{n-1} = \beta_{n-1} = 0$; or if $1 < p < n$, then $\alpha_{p-1} = \alpha_p = \beta_{p-1} = \beta_p = 0$. Now we assume that A does not contain a row of zeros. By Theorem 4.4 in [10], there exist two rows of A, say p-th and q-th rows for $p < q$, which are the same. So $a_{pj} = a_{qj}$ for $j = 1,\ldots,n$. Hence, for $p < j < q$,

$$\begin{aligned} a_{pp} &= \max\{\beta_{p-1},\beta_p\} \geqslant \beta_p \geqslant \alpha_p \geqslant \min\{\alpha_p,\ldots,\alpha_{j-1}\} = a_{pj} \\ &\geqslant \min\{\alpha_p,\ldots,\alpha_{q-1}\} = a_{pq} = a_{qq} = \max\{\beta_{q-1},\beta_q\} \geqslant \beta_{q-1} \\ &\geqslant \min\{\beta_j,\ldots,\beta_{q-1}\} = a_{qj} \geqslant \min\{\beta_p,\ldots,\beta_{q-1}\} = a_{qp} = a_{pp}. \end{aligned}$$

Therefore, $\min\{\alpha_p,\ldots,\alpha_{q-1}\} = \alpha_p = \beta_p = \min\{\beta_p,\ldots,\beta_{q-1}\} = \beta_{q-1} = \alpha_{q-1} \geqslant \max\{\beta_{p-1},\beta_q\}$. □

Corollary 2.8. *Let A be the $n\times n$ matrix associated with a double edge-weighted path T and two vectors $\vec\alpha = (\alpha_1,\ldots,\alpha_{n-1})$ and $\vec\beta = (\beta_1,\ldots,\beta_{n-1})$. Let S denote the set of such indices $k \in \{1,\ldots,n\}$ for which $S = \{k\colon\ \alpha_k = \beta_k \geqslant \max\{\beta_{k-1},\beta_{k+1}\}\ \}$. Then the nullity $\nu(A)$ and the rank of A satisfy the inequalities $\nu(A) \geqslant |S|$ and $\operatorname{rank}(A) \leqslant n - |S|$ respectively.*

P r o o f. If $k \in S$, then the k-th and $(k+1)$-th rows of A are the same. In fact, for $j < k$, $a_{kj} = \min\{\beta_i;\ E_i$ is an edge in the path from v_k to vertex $v_j\} = \min\{\beta_i;\ E_i$ is an edge in the path from v_{k+1} to vertex $v_j\} = a_{k+1,j}$, since $\beta_k \geqslant \beta_{k-1}$. For $j > k+1$, $a_{kj} = \min\{\alpha_i;\ E_i$ is an edge in the path from v_k to vertex $v_j\} = \min\{\alpha_i;\ E_i$ is an edge in the path from v_{k+1} to vertex $v_j\} = a_{k+1,j}$, since $\alpha_k \geqslant$

$\beta_{k+1} \geqslant \alpha_{k+1}$. Moreover, $a_{kk} = a_{k,k+1} = a_{k+1,k} = a_{k+1,k+1} = \beta_k$. Hence the vector $\vec{\varphi_k} = (0, \ldots, 0, 1, -1, 0, \ldots, 0)^t$ belongs to the null-space of A, where the k-th and $(k+1)$-th components of $\vec{\varphi_k}$ are 1 and -1, respectively. Since all these vectors φ_k are linearly independent, $\nu(A) \geqslant |S|$ and $\operatorname{rank}(A) \leqslant n - |S|$. □

3. Special $\mathscr{U}$ matrices

Let U be an $n \times n$ $\mathscr{U}$ matrix. Then $U = (u_{ij})$ has the following form

$$U = \begin{pmatrix} U_{11} & \tau \mathbf{1}\mathbf{1}^t \\ b\mathbf{1}^t & U_{22} \end{pmatrix}, \tag{7}$$

where U_{11} is an $m \times m$ matrix in NBF, $\tau = \min\{u_{ij}, i, j = 1, \ldots, n\}$ and b is the last column of U_{22}.

Definition 3.1. An $n \times n$ matrix $U = (u_{ij})$ is called special $\mathscr{U}$ matrix if U is a $\mathscr{U}$ matrix in the form (7) satisfying $u_{ii} = \max\{u_{ij}, u_{ji};\ j = 1, \ldots, m \text{ and } j \neq i\}$ for $i = 1, \ldots, m$; $u_{ii} = \max\{u_{i,i+1}, \ldots, u_{in}, u_{1i}, \ldots, u_{i-1,i}\}$ for $i = m+1, \ldots, n$.

Clearly, each special $\mathscr{U}$ matrix U is always singular, since the last two rows of U are the same. Furthermore, it follows from Theorem B in [6] that each special $\mathscr{U}$ matrix is the limit of a convergent sequence of matrices which are inverses of column diagonally dominant M-matrices.

Let T_1 be a double edge-weighted path on the vertex set $V_1 = \{v_1, \ldots, v_m\}$ with the two vectors $(\alpha_1, \ldots, \alpha_{m-1})$ and $(\beta_1, \ldots, \beta_{m-1})$ and T_2 be an edge weighted path on the vertex set $V_2 = \{v_{m+1}, \ldots, v_n\}$ with the edge-weighted vector $(\alpha_{m+1}, \ldots, \alpha_{n-1})$. Let $T = T_1 \cup T_2$ be the path obtained by adding an edge (v_m, v_{m+1}) with weight α_m satisfying $\alpha_m = \min\{\alpha_i,\ i = 1, \ldots, n-1\}$. Then T is called a *mixed edge-weighted path* with the two vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ and $\vec{\beta} = (\beta_1, \ldots, \beta_{m-1})$.

Now we may define an $n \times n$ nonnegative matrix $B(T) = (b_{ij})$ associated with a mixed edge-weighted path T on n vertices and the two vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ and $\vec{\beta} = (\beta_1, \ldots, \beta_{m-1})$ as follows:

For $i < j$, $b_{ij} = \min\{\alpha_k, E_k$ is an edge in the path from vertex v_i to vertex $v_j\}$; for $j < i \leqslant m$ $b_{ij} = \min\{\beta_k, E_k$ is an edge in the path from vertex v_i to vertex $v_j\}$; for $j < i$ and $m < i < n$, $b_{ij} = \min\{\alpha_k, E_k$ is an edge in the path from vertex v_i to vertex $v_n\}$ and $b_{nj} = \alpha_{n-1}$ for $j = 1, \ldots, n-1$. Moreover, $b_{ii} = \max\{\beta_k, E_k$ is incident with vertex $v_i\}$ for $i = 1, \ldots, m$ and $b_{ii} = \max\{\alpha_k, E_k$ is incident with vertex $v_i\}$ for $i = m+1, \ldots, n$.

In this section, we prove that the set of special $\mathscr{U}$ matrices just coincides with the set of nonnegative matrices associated with mixed edge-weighted paths.

Lemma 3.2. *Let U be an $n \times n$ special $\mathscr{U}$ matrix in the form* (7). *Then there exists a mixed edge-weighted path T such that $U = B(T)$.*

P r o o f. We prove the assertion by induction on n. It is trivial for $n = 2$ and assume that the assertion holds for less than n. If $1 < m < n-1$, we assume that $U = (u_{ij})$ is a special $\mathscr{U}$ matrix in the form (7), where $\tau = \min\{u_{ij},\ i, j = 1, \ldots, n\}$, U_{11} is special NBF. Hence by Lemma 2.2, there exists a double edge-weighted path T_1 on vertex set $V_1 = \{v_1, \ldots, v_m\}$ with $(\alpha_1, \ldots, \alpha_{m-1})$ and $(\beta_1, \ldots, \beta_{m-1})$ such that $U_{11} = C(T_1) = (c_{ij})$. On the other hand, clearly, U_{22} has the following form

$$U_{22} = \begin{pmatrix} U_{33} & \tau_1 \mathbf{1}\mathbf{1}^t \\ b_2 \mathbf{1}^t & U_{44} \end{pmatrix},$$

where U_{33}^t is $(p-m) \times (p-m)$ special NBF with $m+1 \leqslant p < n$, $\tau_1 \geqslant \tau$ and b_2 is the last column of U_{44}. Hence

$$W_{22} = (w_{ij}) = \begin{pmatrix} U_{33}^t & \tau_1 \mathbf{1}\mathbf{1}^t \\ b_2 \mathbf{1}^t & U_{44} \end{pmatrix}$$

is special $\mathscr{U}$ matrix. By the induction hypothesis, there exists a mixed edge-weighted path $T_2 = T_{21} \cup T_{22}$ on vertices $T_{21} = \{v_{m+1}, \ldots, v_p\}$ with the two vectors $(\gamma_{m+1}, \ldots, \gamma_{p-1}) \leqslant (\delta_{m+1}, \ldots, \delta_{p-1})$ and on vertices $T_{22} = \{v_{p+1}, \ldots, v_n\}$ with $(\gamma_{p+1}, \ldots, \gamma_{n-1})$. Moreover, the edge (v_p, v_{p+1}) is assigned with $\gamma_p = \min\{\gamma_i,\ i = m+1, \ldots, n-1\} = \tau_1$. Hence we may define a mixed edge-weighted path T on the vertex set $V = \{v_1, \ldots, v_n\}$ with the two vectors $(\alpha_1, \ldots, \alpha_{n-1})$ and $(\beta_1, \ldots, \beta_{m-1})$, where $\alpha_m = \tau$, $\alpha_i = \delta_i$ for $i = m+1, \ldots, p-1$ and $\alpha_i = \gamma_i$ for $i = p, \ldots, n-1$. If $m = 1$ or $m = n-1$, we have $U_{11} = (\tau)$ or $U_{22} = (\tau)$, respectively. Then we may show that the matrix $B(T) = (b_{ij})$ associated with a mixed edge-weighted path T on the vertex set V and the two vectors $(\alpha_1, \ldots, \alpha_{n-1})$ and $(\beta_1, \ldots, \beta_{m-1})$ is just U. In fact, if $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m$, then $b_{ij} = c_{ij} = u_{ij}$. If $1 \leqslant i \leqslant m$, $m+1 \leqslant j \leqslant n$, then $b_{ij} = \min\{\alpha_k;\ E_k$ is an edge in the path T from vertex v_i to vertex $v_j\} = \tau = u_{ij}$. If $m+1 \leqslant i < j \leqslant p$; $b_{ij} = \min\{\alpha_k,\ E_k$ is an edge in the path from vertex v_i to vertex $v_j\} = \min\{\delta_k,\ E_k$ is edge in the path from vertex v_i to vertex $v_j\} = u_{ij}$. If $m+1 \leqslant i \leqslant p$ and $p+1 \leqslant j \leqslant n$, then $b_{ij} = \min\{\alpha_k,\ E_k$ is edge in the path from vertex v_i to vertex $v_j\} = \gamma_p = \tau_1 = u_{ij}$. If $p+1 \leqslant i < j \leqslant n$, then $b_{ij} = \min\{\alpha_k,\ E_k$ is an edge in the path from vertex v_i to vertex $v_j\} = \min\{\gamma_k,\ E_k$ is an edge in the path from vertex v_i to vertex $v_j\} = w_{ij} = u_{ij}$. If $i > j$ and $i \geqslant m+1$, then $b_{ij} = \min\{\alpha_k,\ E_k$ is edge in the path from vertex v_i to vertex $v_n\} = b_{in} = u_{in} = u_{ij}$. Moreover, for $1 \leqslant i \leqslant m$, $b_{ii} = \max\{\beta_k,\ E_k$ is incident with $v_i\} = c_{ii} = u_{ii}$. For $m+1 \leqslant i \leqslant p$, $b_{ii} = \max\{\alpha_k,\ E_k$ is incident with $v_i\} = \max\{\delta_k,\ E_k$ is incident with $v_i\} = w_{ii} = u_{ii}$, since $\delta_k \geqslant \gamma_k \geqslant \gamma_p$ for $m+1 \leqslant k \leqslant p-1$. For $p+1 \leqslant i \leqslant n$, $b_{ii} = \max\{\alpha_k,\ E_k$ is incident with $v_i\} = \max\{\gamma_k,\ E_k$ is incident with $v_i\} = u_{ii}$. $\square$

Lemma 3.3. *Let U be an $n \times n$ nonnegative matrix associated with a mixed edge-weighted path T. Then U is a special $\mathcal{U}$ matrix.*

P r o o f. We prove the assertion by induction on n. Clearly, the assertion holds for $n = 1$ and $n = 2$. Assume U is associated with a mixed edge-weighted path $T = T_1 \cup T_2$ on vertex set $V = \{v_1, \ldots, v_n\}$ with the two vectors $(\alpha_1, \ldots, \alpha_{n-1})$ and $(\beta_1, \ldots, \beta_{m-1})$. Moreover, $\alpha_m = \min\{\alpha_i,\ i = 1, \ldots, n-1\}$. Clearly, U has the following form

$$U = B(T) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} := (b_{ij}),$$

where B_{11} is the $m \times m$ matrix associated with a double edge-weighted path T_1 on the vertex set $V_1 = \{v_1, \ldots, v_m\}$ with two vectors $(\alpha_1, \ldots, \alpha_{m-1})$ and $(\beta_1, \ldots, \beta_{m-1})$; $B_{12} = \alpha_m \mathbf{1}\mathbf{1}^t$; $B_{21} = b\mathbf{1}^t$ and b is the last column of B_{22}. Let $C_{22} = (c_{ij})$ be the $(n-m) \times (n-m)$ matrix associated with the double edge-weighted path T_2 and the two vectors $(\alpha_{m+1}, \ldots, \alpha_{n-1})$ and $(\alpha_{m+1}, \ldots, \alpha_{n-1})$. Then by Theorem 2.4, C_{22} is a special NBF. Further, the matrix

$$C = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^t & C_{22} \end{pmatrix} := (c_{ij})$$

is a NBF. Moreover, for $m + 1 \leqslant i < j < n$, we have $b_{ij} = \min\{\alpha_k;\ E_k$ is an edge in the path from vertex v_i to vertex $v_j\} = c_{ij}$. For $m + 1 \leqslant j < i \leqslant n$, we have $b_{ij} = \min\{\alpha_k;\ E_k$ is an edge in the path from vertex v_i to vertex $v_n\} = b_{in}$. Therefore by the definition of $\mathcal{U}$ matrix, $B(T)$ is a $\mathcal{U}$ matrix. Now we show that $B(T)$ is a special $\mathcal{U}$ matrix. Since B_{11} is an $m \times m$ matrix associated with a double edge-weighted path T_1 and the vectors $(\alpha_1, \ldots, \alpha_{m-1})$ and $(\beta_1, \ldots, \beta_{m-1})$, B_{11} is special NBF by Theorem 2.4. Hence $b_{ii} = \max\{\beta_k, E_k$ is incident with $v_i\} = \max\{b_{ij}, b_{ji};$ $j \neq i,\ j = 1, \ldots, m\}$ for $i = 1, \ldots, m$; $b_{ii} = \max\{\alpha_i, E_k$ is incident with $V_i\} = \max\{\alpha_{i-1}, \alpha_i\} = \max\{b_{i,i+1}, \ldots, b_{in}, b_{1i}, \ldots, b_{i-1,i}\}$ for $i = m+1, \ldots, n-1$, since $b_{i,i+1} \geqslant b_{i,i+2} \geqslant \ldots \geqslant b_{in}$ and $b_{1i} \leqslant b_{2i} \leqslant \ldots \leqslant b_{i-1,i}$. Moreover, $b_{nn} = \max\{\alpha_k;\ E_k$ is incident with $v_n\} = \alpha_{n-1} = \max\{b_{1n}, \ldots, b_{n-1,n}\}$, since $b_{1n} \leqslant b_{2n} \leqslant \ldots \leqslant b_{n-1,n}$. Hence $B(T)$ is a special $\mathcal{U}$ matrix. □

We immediately obtain the main result in this section.

Theorem 3.4. *A nonnegative matrix U is a special $\mathcal{U}$ matrix if and only if there exists a mixed edge-weighted path T such that $U = B(T)$.*

4. A new class of inverse M-matrices

In this section, we shall define a new class of inverse M-matrices which generalizes the class of $\mathscr{U}$ matrices. Let T_1 be a double weighted path on the vertex set $V_1 = \{v_1, \ldots, v_m\}$ and two vectors $(\alpha_1, \ldots, \alpha_{m-1}) \leqslant (\beta_1, \ldots \beta_{m-1})$. Let T_2 be a double weighted path on the vertex set $V_2 = \{v_{m+1}, \ldots, v_n\}$ and two vectors $(\alpha_{m+1}, \ldots, \alpha_{n-1})$ and $(\beta_{m+1}, \ldots, \beta_{n-1})$ satisfying $\beta_i \leqslant 1$ for $i = m+1, \ldots, n-1$. Then let $T = T_1 \cup T_2$ be a path on the vertex set $V = \{v_1, \ldots, v_n\}$ obtained by adding an edge (v_m, v_{m+1}) which is assigned two positive numbers α_m and β_m satisfying $\alpha_m = \min\{\alpha_i,\ i = 1, \ldots, n-1\}$ and $\beta_m = \min\{\beta_m, \ldots, \beta_{n-1}\}$. Hence we call such a weighted path T with $(\alpha_1, \ldots, \alpha_{n-1})$ and $(\beta_1, \ldots, \beta_{n-1})$ *quasi-double edge-weighted path*.

For a quasi-double edge-weighted path T, we may define an $n \times n$ nonnegative matrix $W(T)$ as follows: $w_{ii} \geqslant \max\{\beta_k,\ E_k$ is incident with vertex $v_i\}$ for $i = 1, \ldots, m-1$; $w_{mm} \geqslant \beta_{m-1}$; $w_{ii} \geqslant \max\{\alpha_k,\ E_k$ is incident with vertex $v_i\}$ for $i = m+1, \ldots, n$. For $i < j$, $w_{ij} = \min\{\alpha_k,\ E_k$ is edge in the path from vertex v_i to vertex $v_j\}$. For $m \geqslant i > j$, $w_{ij} = \min\{\beta_k,\ E_k$ is edge from vertex v_i to vertex $v_j\}$; for $j < i \leqslant n$ and $i \geqslant m+1$, $w_{ij} = w_{in} f_{ij}$, where $f_{ij} = \beta_m$ for $i > m \geqslant j$ and $f_{ij} = \min\{\beta_k,\ E_k$ is edge from vertex v_i to vertex $v_j\}$ for $i > j \geqslant m+1$. The set of all matrices $W(T)$ given by the above definition and up to permutation matrices is denoted by $\mathscr{W}$. From the definition, let $A \in \mathscr{W}$. If $\beta_i = 1$ for $i = m+1, \ldots, n-1$, then there exists a permutation matrix such that $\mathrm{PAP}^t \in \mathscr{U}$. Hence the class of $\mathscr{U}$ is just the proper subclass of $\mathscr{W}$. Now we present the main result of this Section.

Theorem 4.1. *Let $A \in \mathscr{W}$. Then A is nonsingular if and only if A does not contain a row or column of zeros, and no two rows or two columns are the same. If A is nonsingular, then A^{-1} is a column diagonally dominant M-matrix.*

P r o o f. If A does contain a row or column of zeros, or two rows or two columns are the same, then A is singular. We prove the rest of the assertion by induction on n. Assume that the assertion holds for less than n. By the definition of $\mathscr{W}$, there exists a permutation matrix P such that

$$\mathrm{PAP}^t = \begin{pmatrix} A_{11} & \alpha_m \mathbf{1}\mathbf{1}^t \\ \beta_m b \mathbf{1}^t & A_{22} \end{pmatrix}$$

where A_{11} is an $m \times m$ NBF and $A_{22} \in \mathscr{W}$, and b is the last column of A_{22}. Clearly A_{ii} does not contain a row or column of zeros, and no two rows or two columns are the same for $i = 1, 2$. Hence by Theorem 4.4 by [10], A_{11} is nonsingular. Further, A_{22} is nonsingular by the induction hypothesis. Moreover, the Schur complement of A_{11} in A is

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12} = A_{22} - \alpha_m\beta_m(\mathbf{1}^t A_{11}^{-1}\mathbf{1})b\mathbf{1}^t.$$

By Theorem 3.5 in [13], $\beta_m\alpha_m(\mathbf{1}^t A_{11}^{-1}\mathbf{1}) \leqslant \beta_m$. Hence A/A_{11} is a nonnegative matrix and is in $\mathscr{W}$. By the induction hypothesis, $(A/A_{11})^{-1}$ is a column diagonally dominant M-matrix. On the other hand,

$$A/A_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21} = A_{11} - \alpha_m\beta_m\mathbf{1}\mathbf{1}^t;$$

thus A/A_{22} is nonsingular GUM, whose inverse is a column diagonally dominant M-matrix in [10] or [13]. Using the Sherman-Morrison formula, A is nonsingular and

$$A^{-1} = \begin{pmatrix} (A/A_{22})^{-1} & -A_{11}^{-1}\alpha_m\mathbf{1}\mathbf{1}^t(A/A_{11})^{-1} \\ -A_{22}^{-1}\beta_m b\mathbf{1}^t(A/A_{22})^{-1} & (A/A_{11})^{-1} \end{pmatrix}.$$

Since

$$\begin{aligned} -A_{22}^{-1}A_{21}(A/A_{22})^{-1} &= -e_{n-p}\beta_m\mathbf{1}^t(A/A_{22})^{-1} \leqslant 0, \\ -A_{11}^{-1}A_{12}(A/A_{11})^{-1} &= -(\alpha_m A_{11}^{-1}\mathbf{1})(\mathbf{1}^t(A/A_{11})^{-1}) \leqslant 0, \end{aligned}$$

where $e_{n-m} = (0,\ldots,0,1)^t$, A^{-1} is an M-matrix. Moreover, we have

$$\mathbf{1}^t(A/A_{22})^{-1} - \mathbf{1}^t A_{22}^{-1}b\mathbf{1}^t(A/A_{22})^{-1} = (1-\beta_m)\mathbf{1}^t(A/A_{22})^{-1} \geqslant 0$$

and

$$\begin{aligned} &\mathbf{1}^t(A/A_{11})^{-1} - \mathbf{1}^t A_{11}^{-1}\alpha_m\mathbf{1}\mathbf{1}^t(A/A_{11})^{-1} \\ &\qquad = (1-\alpha_m\beta_m\mathbf{1}^t(A/A_{11})^{-1}\mathbf{1})\mathbf{1}^t(A/A_{11})^{-1} \geqslant 0, \end{aligned}$$

since $1-\alpha_m\beta_m\mathbf{1}^t(A/A_{11})^{-1}\mathbf{1}) \geqslant 1-\beta_m \geqslant 0$. Hence A^{-1} is a column diagonally dominant M-matrix. □

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