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DENSITY-DEPENDENT INCOMPRESSIBLE FLUIDS WITH NON-NEWTONIAN VISCOSITY

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Abstract. We study the system of PDEs describing unsteady flows of incompressible fluids with variable density and non-constant viscosity. Indeed, one considers a stress tensor being a nonlinear function of the symmetric velocity gradient, verifying the properties of p-coercivity and $(p-1)$-growth, for a given parameter $p > 1$. The existence of Dirichlet weak solutions was obtained in [2], in the cases $p \geqslant 12/5$ if $d = 3$ or $p \geqslant 2$ if $d = 2$, d being the dimension of the domain. In this paper, with help of some new estimates (which lead to point-wise convergence of the velocity gradient), we obtain the existence of space-periodic weak solutions for all $p \geqslant 2$. In addition, we obtain regularity properties of weak solutions whenever $p \geqslant 20/9$ (if $d = 3$) or $p \geqslant 2$ (if $d = 2$). Further, some extensions of these results to more general stress tensors or to Dirichlet boundary conditions (with a Newtonian tensor large enough) are obtained.

Keywords: variable density, shear-dependent viscosity, power law, Carreau's laws, weak solution, strong solution, periodic boundary conditions

MSC 2000: 35B10, 35M10, 76A05

1. Introduction

We consider that the domain Ω is a cube (of periodicity) in $\mathbb{R}^d$ $(d = 2$ or $3)$

$$x = (x_i)_{i=1}^d \in \Omega = \prod_{i=1}^d (0, L_i) \quad (0 < L_i < +\infty,\ \forall i = 1, \dots, d)$$

and the time interval is $t \in (0, +\infty)$.

Given $f = (f_i)_{i=1}^d$ (external forces), ϱ^0 and $u^0 = (u_i^0)_{i=1}^d$ (initial density and velocity), the problem is to find ϱ, $u = (u_i)_{i=1}^d$ and π (density, velocity and pressure)

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satisfying the following system of PDEs in $(0,+\infty)\times\Omega$:

$$
\begin{cases} \partial_t \varrho + \partial_i(\varrho u_i) = 0, \quad \partial_i u_i = 0, \\ \partial_t(\varrho u_i) + \partial_j(\varrho u_j u_i) - \partial_j \tau_{ij}(Du) + \partial_i \pi = \varrho f_i, \quad \forall i = 1,\dots,d, \end{cases} \tag{1}
$$

together with the initial conditions

$$
\varrho|_{t=0} = \varrho^0 \qquad \text{and} \qquad (\varrho u)|_{t=0} = \varrho^0 u^0 \quad \text{in } \Omega, \tag{2}
$$

and the space-periodic boundary conditions: for a.e. $t \in (0,+\infty)$,

$$
\varrho(t,\cdot), \quad u_j(t,\cdot), \quad \pi(t,\cdot) \quad \text{and} \quad \partial_k u_j(t,\cdot) \text{ are } L_i\text{-periodic with respect to } x_i, \tag{3}
$$

for all $i, j, k = 1,\dots,d$.

In (1), the summation is over repeated indices, ∂_t, ∂_i denote partial derivatives with respect to t and x_i, Du is the symmetric part of the velocity gradient (with the ij-component equal to $Du_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$) and $\tau = \tau(Du)$ denotes the (symmetric) extra-stress tensor, which is assumed to be a nonlinear function of Du. In such a case, (1) describes a class of non-Newtonian fluids, namely the fluids with shear-dependent viscosity. For simplicity, we consider τ defined by one of the following constitutive laws (see Section 5 for more general assumptions on τ):

$$
\tau(Du) = \{\mu_\infty + \mu_0 |Du|^{p-2}\} Du \quad \text{(power law)}, \tag{4}
$$

$$
\left.\begin{aligned} \tau(Du) &= \{\mu_\infty + \mu_0 (1+|Du|)^{p-2}\} Du, \text{ or} \\ \tau(Du) &= \{\mu_\infty + \mu_0 (1+|Du|^2)^{(p-2)/2}\} Du, \end{aligned}\right\} \quad \text{(Carreau's laws)} \tag{5}
$$

where $\mu_\infty \geqslant 0$, $\mu_0 > 0$ are constants, $|Du| = (Du_{ij} Du_{ij})^{1/2}$ and $p \in (1,+\infty)$. In particular, one can write

$$
\tau(Du) = \mu(|Du|) Du = \mu_\infty Du + \mu_0\, \tau^p(Du).
$$

In the first equality, $\mu(|Du|)$ stands for the generalized viscosity function and, in the latter, we decompose the tensor into two parts: the Newtonian one ($\mu_\infty Du$) and the purely non-Newtonian one ($\mu_0 \tau^p(Du)$). Properties of the tensor τ^p differ for $p \in (2,+\infty)$ and for $p \in (1,2)$, which correspond respectively to *dilatant fluids* and *pseudo-plastic fluids*. For instance, the generalized viscosity function $\mu(|Du|)$ increases with respect to $|Du|$ for dilatant fluids and decreases for pseudo-plastic ones. The case $p = 2$ corresponds to a Newtonian fluid. Constitutive laws (4) and (5) are frequently used in problems related to chemistry, biology, glaciology, geology, … (see [8] and the references cited therein).

The main known results on this subject are the following. In the case $\varrho \equiv$ constant (i.e. for non-Newtonian variants of the Navier-Stokes equations), the existence of Dirichlet weak solutions was obtained by O. A. Ladyzhenskaya in [5] (for Carreau's laws) and J. L. Lions in [6] (for the "p-laplacian" operator, i.e. $\tau^p = |\nabla u|^{p-2}\nabla u$), whenever

$$p \geqslant 11/5 \quad (\text{if } d = 3) \quad \text{or} \quad p \geqslant 2 \quad (\text{if } d = 2).$$

In the proofs, a combination of the monotone operator theory and the compactness method is used. On the other hand, the existence of space-periodic weak solutions was obtained by J. Málek, J. Nečas, M. Rokyta and M. Růžička [9], whenever

$$p > 9/5 \quad (\text{if } d = 3) \quad \text{or} \quad p > 3/2 \quad (\text{if } d = 2)$$

The latter was shown without Newtonian viscosity (i.e. $\mu_\infty = 0$). When $\mu_\infty > 0$, it is not difficult to extend these results to all $p > 1$ ($d = 2$ or 3). Recently, there has been another result on the existence of weak solutions valid for $p > 2(d+1)/(d+2)$, see [3]. On the other hand, J. Málek, J. Nečas and M. Růžička [10] have proved the existence of Dirichlet weak solutions when $d = 3$, for all $p \geqslant 2$, in the cases

$$\mu_\infty > 0 \quad \text{or} \quad (\mu_\infty = 0 \text{ and } (5)).$$

Moreover, if $p \geqslant 9/4$, then the solution is strong and unique.

For density-dependent fluids ($\varrho \neq$ constant), first in the Newtonian case ($p = 2$), the existence of weak solutions was already proved by S. N. Antontsev and A. N. Kazhikhov in [1] (see also [4, 7]). Recently, when a power law is considered, the existence of Dirichlet weak solutions has been obtained in [2], whenever

$$p \geqslant 12/5 \ (\text{if } d = 3) \quad \text{or} \quad p \geqslant 2 \ (\text{if } d = 2)$$

(results that can be easily extended to Carreau's laws). Therefore, there is a "gap" between $p = 2$ and $p = 12/5$ in the $3D$ case.

The main purpose of this paper is "filling" this gap, extending these last results to the cases

$$p > 1 \ (\text{if } \mu_\infty > 0) \quad \text{or} \quad p \geqslant 2 \ (\text{if } \mu_\infty = 0 \text{ and } (5)),$$

when space-periodic boundary conditions are considered. Moreover, we will prove additional regularity properties of the weak solutions in the cases

$$\left.\begin{aligned} p \geqslant 20/9 \quad (d = 3) \\ p > 1 \quad (d = 2) \end{aligned}\right\} \quad (\text{if } \mu_\infty > 0)$$

or

$$\left.\begin{aligned} p \geqslant 20/9 \quad (d=3) \\ p \geqslant 2 \quad (d=2) \end{aligned}\right\} \quad (\text{if } \mu_\infty = 0 \text{ and } (5)).$$

For this purpose, the key is to obtain some new estimates for the velocity in an intermediate space between H^1 and H^2, which lead to point-wise convergence of the velocity gradient. The derivation of these estimates is based on the extension of the technique presented in [8, 9] in the case of constant density, to non-constant density flows.

2. Definition of weak solutions and some consequences

First of all, let us introduce some usual notation. For $A \subset \mathbb{R}^k$ let $X(A)$ be the space of scalar functions defined in A. Then $X(A)^d$ (or $X(A)^{d\times d}$) denotes the space of vector-valued (tensor-valued) functions whose components belong to $X(A)$. On the other hand, let $X_{\rm loc}(A)$ represent the space of functions which belong to $X(K)$, for all compact sets $K \subset A$. As usual, $\mathscr{D}(A)$ will denote the space of smooth functions with compact support in A. Furthermore, let $q > 1$ (real) and $k \geqslant 1$ (integer), then L^q ($W^{k,q}$) is used for the standard Lebesgue (Sobolev) spaces. Finally, by $L^q(0,T;X(A))$ we denote the Bochner spaces.

Now, we will introduce the functional spaces related to the weak solutions. In order to consider the periodic conditions, we define

$$\begin{gathered} \mathscr{D}_{\rm per} = \{\psi \in C^\infty(\mathbb{R}^d)\colon\ \psi \text{ is } L_i\text{-periodic with respect to } x_i,\ \forall i = 1,\dots,d\}, \\ L^q_{\rm per} \text{ (or } W^{k,q}_{\rm per}) = \text{the closure of } \mathscr{D}_{\rm per} \text{ in } L^q_{\rm loc}(\mathbb{R}^d) \text{ (in } W^{k,q}_{\rm loc}(\mathbb{R}^d)). \end{gathered}$$

One has the following identities:

$$L^q_{\rm per} = \{g \in L^q_{\rm loc}(\mathbb{R}^d)\colon\ g \text{ is } L_i\text{-periodic with respect to } x_i\}.$$

(idem for $W^{k,q}_{\rm per}$, replacing $L^q_{\rm loc}$ by $W^{k,q}_{\rm loc}$).

Finally, in order to consider the incompressibility equation (free divergence), we define

$$\begin{gathered} \mathscr{V}_{\rm per} = \{\varphi = (\varphi_i)_{i=1}^d \in (\mathscr{D}_{\rm per})^d\colon\ \partial_i\varphi_i = 0 \text{ in } \mathbb{R}^d\}, \\ H_{\rm per} \text{ (or } V^q_{\rm per}) = \text{ the closure of } \mathscr{V}_{\rm per} \text{ in } L^2_{\rm loc}(\mathbb{R}^d)^d \text{ (in } W^{1,q}_{\rm loc}(\mathbb{R}^d)^d). \end{gathered}$$

In the Hilbert case ($q = 2$), we will denote $W^{k,2}_{\rm per} \equiv H^k_{\rm per}$ and $V^2_{\rm per} \equiv V_{\rm per}$.

Remark. In the definition of $L^q_{\rm per}$ and $W^{k,q}_{\rm per}$, a restriction to the cube Ω may be considered, extending the functions to all $\mathbb{R}^d$ by periodicity. In this sense, if $\partial\Omega$ denotes the boundary of Ω, separated in parts

$$\Gamma_i = \partial\Omega \cap \{x_i = 0\}, \quad \Gamma_{i+d} = \partial\Omega \cap \{x_i = L_i\}, \quad \forall i = 1, \ldots, d,$$

one also has

$$\begin{aligned} W^{k,q}_{\rm per} &= \{g \in W^{k,q}(\Omega)\colon\ (\partial^\alpha g)|_{\Gamma_i} = (\partial^\alpha g)|_{\Gamma_{i+d}}\ \forall |\alpha| \leqslant k-1\}, \\ H_{\rm per} &= \{v \in L^2(\Omega)^d\colon\ \partial_i v_i = 0 \text{ in } \Omega, (v\cdot n)|_{\Gamma_i} = -(v\cdot n)|_{\Gamma_{i+d}}\}, \\ V^q_{\rm per} &= \{v \in W^{1,q}(\Omega)^d\colon\ \partial_i v_i = 0 \text{ in } \Omega, v|_{\Gamma_i} = v|_{\Gamma_{i+d}}\}. \end{aligned}$$

Here, n is the exterior normal vector to $\partial\Omega$. Notice that, if $v \in H_{\rm per}$, in particular $v \in L^2(\Omega)^d$ and $\partial_i v_i \in L^2(\Omega)$, then $v \cdot n$ belongs to $H^{-1/2}(\partial\Omega)$.

In the sequel, all spatial norms will be taken in Ω and the space of functions related to the norm will be indicated by a subscript (for instance, $\|u\|_{L^2}$ denotes the $L^2(\Omega)$-norm of u).

Definition 1. Under the following hypotheses for the data:

(H1) $\varrho^0 \in L^\infty_{\rm per}$, $\varrho^0 \geqslant \alpha > 0$ in $\Omega, u^0 \in H_{\rm per}$ and $f \in L^2_{\rm loc}([0,\infty); (L^2_{\rm per})^d)$,

a couple (ϱ, u) is said to be a *weak solution* in $(0, +\infty)$ of the problem (1)–(3) if

$$\varrho \in L^\infty(0,\infty; L^\infty_{\rm per}), \quad 0 < \alpha \leqslant \varrho(t,x) \leqslant \|\varrho^0\|_{L^\infty} \quad \text{a.e. } (t,x), \tag{6}$$

$$u \in L^p_{\rm loc}([0,\infty); V^p_{\rm per}) \cap L^\infty_{\rm loc}([0,\infty); H_{\rm per}), \tag{7}$$

$$\int_0^\infty\!\!\int_\Omega \{\varrho\partial_t\psi + \varrho u_i\partial_i\psi\}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho^0\psi(0)\,\mathrm{d}x = 0 \quad \forall \psi \in \mathscr{D}([0,\infty); \mathscr{D}_{\rm per}), \tag{8}$$

$$\int_0^\infty\!\!\int_\Omega \{\varrho u_i\partial_t\varphi_i + [\varrho u_j u_i - \tau_{ij}(Du)]D\varphi_{ij} + \varrho f_i\varphi_i\}\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{9}$$

$\forall \varphi \in \mathscr{D}((0,\infty); \mathscr{V}_{\rm per})$, and, $\forall v \in V^\sigma_{\rm per}$ (with $\sigma = \max\{p, 2\}$),

$$\int_\Omega \varrho u_i v_i\,\mathrm{d}x \in C[0,\infty) \qquad \text{with } \Big(\int_\Omega \varrho u_i v_i\,\mathrm{d}x\Big)(0) = \int_\Omega \varrho^0 u^0_i v_i\,\mathrm{d}x. \tag{10}$$

Remarks.

1. By virtue of the regularity of (6)–(7), every term in (8)–(9) makes sense.

2. From (8), one has (cf. [7])

$$\partial_t \varrho + \partial_i(\varrho u_i) = 0 \quad \text{in } \mathscr{D}'((0,\infty)\times\mathbb{R}^d),$$

(11) $$\varrho \in C([0,\infty); L^q(\Omega)) \ \forall q < +\infty \quad \text{and} \quad \varrho_{|t=0} = \varrho^0 \ \text{a.e. in } \Omega,$$

(12) $$\|\varrho(t,\cdot)\|_{L^q} = \|\varrho^0\|_{L^q} \quad \forall t \in [0,\infty),\ \forall q \in [1,\infty).$$

3. Taking in (9) $\varphi = \varphi_1(t)\varphi_2(x)$ with $\varphi_1 \in \mathscr{D}(0,\infty)$ and $\varphi_2 \in \mathscr{V}_{\text{per}}$, one has

$$\langle \partial_t(\varrho u_i) + \partial_j(\varrho u_j u_i) - \partial_j \tau_{ij}(Du) - \varrho f_i, (\varphi_2)_i \rangle = 0 \quad \text{in } \mathscr{D}'(0,\infty)$$

for all $i = 1,\dots,d$, where $\langle\cdot,\cdot\rangle$ denotes the duality in Ω in the distributional sense. Consequently, from De Rham's lemma, we have that there exists a distribution $\pi \in \mathscr{D}'((0,\infty)\times\Omega)$ such that (ϱ, u, π) satisfies $(1)_2$ in $\mathscr{D}'((0,\infty)\times\Omega)$, cf. [2].

4. Incompressibility equation $\partial_i u_i = 0$ and periodic conditions on u have been imposed in the "essential" form; $u(t,\cdot) \in V^p_{\text{per}}$ for a.e. $t \in (0,+\infty)$. However, the initial conditions $(\varrho u)|_{t=0} = \varrho^0 u^0$ are verified in the sense of (10) (see also (13) below), because ϱu is only continuous in the weak topology of V^σ_{per}.

3. Main results

Theorem 1 (Existence of a weak solution. The case $\mu_\infty > 0$)**.** *If we assume $\mu_\infty > 0$ and hypothesis* (H1)*, then there exists a weak solution (ϱ, u) in $(0,+\infty)$ of the problem* (1)–(3) *for all $p > 1$. Furthermore, $u \in L^\sigma_{\text{loc}}([0,\infty); V^\sigma_{\text{per}})$ (with $\sigma = \max\{p,2\}$) and*

(13) $$\|\varrho^{1/2}u\|^2_{L^2} \in C[0,\infty) \qquad \textit{with} \quad \|\varrho^{1/2}u\|^2_{L^2}(0) = \|(\varrho^0)^{1/2}u^0\|^2_{L^2}.$$

Theorem 2 (Regularity of weak solutions. The case $\mu_\infty > 0$)**.** *Under hypotheses of Theorem* 1*, if moreover one assumes*

(H2) $$u^0 \in V^\sigma_{\text{per}},\ p \geqslant 20/9 \ \ (\textit{if } d = 3) \quad \textit{or} \quad p > 1 \ \ (\textit{if } d = 2),$$

then any weak solution (furnished by Theorem 1*) verifies*

(14) $$u \in L^2_{\text{loc}}([0,\infty); H^2(\Omega)^d) \cap L^\infty_{\text{loc}}([0,\infty); V^\sigma_{\text{per}}),$$
$$\partial_t u \in L^2_{\text{loc}}([0,\infty); H_{\text{per}})$$

(recall that $\sigma = \max\{p,2\}$).

Theorem 3 (The case $\mu_\infty = 0$ and Carreau's laws)**.** *If we assume* $\mu_\infty = 0$, (H1) *and the tensor* τ *is defined as in* (5), *we have the statements of Theorems* 1 *and* 2 *whenever* $p \geqslant 2$, *i.e. for all* $p \geqslant 2$ *there exists a weak solution* (ϱ, u) *in* $(0, +\infty)$ *of* (1)–(3), *which has the additional regularity* (14) *if*

$$\text{(H2}'\text{)} \qquad u^0 \in V^p_{\text{per}},\ p \geqslant 20/9 \ \ (\textit{if } d = 3) \quad \textit{or} \quad p \geqslant 2 \ \ (\textit{if } d = 2).$$

Remarks.

1. When the Newtonian viscosity vanishes ($\mu_\infty = 0$), we get no result for pseudo-plastic fluids (i.e. for $p < 2$).
2. In particular, (14) implies that $u \in C([0, \infty); H_{\text{per}})$. In addition, from (11), (10) and the bound form below of ϱ^0 given in (H1), the following initial conditions for the velocity can be proved ([2]):

$$u|_{t=0} = u^0 \quad \text{a.e. in } \Omega.$$

3. Under the regularity of (14), one can show that the pressure $\pi \in L^2_{\text{loc}}([0, \infty); L^{\sigma'}_{\text{loc}})$ (denoting by σ' the conjugate exponent of σ; $1/\sigma + 1/\sigma' = 1$) and (ϱ, u, π) verifies $(1)_2$ in the $L^2_{\text{loc}}([0, \infty); W^{-1,\sigma'}_{\text{loc}})$ sense. Moreover, we can directly deduce that $\partial_j \tau^p_{ij}(Du) \in L^2_{\text{loc}}([0, \infty); L^{6/(p+1)}(\Omega))$, in the following cases:

$$\left.\begin{array}{ll} 2 \leqslant p \leqslant 5 & (d = 3) \\ 2 \leqslant p < \infty & (d = 2) \end{array}\right\} \quad \text{(for the power law)}$$

or

$$\left.\begin{array}{ll} p \leqslant 5 & (d = 3) \\ p < \infty & (d = 2) \end{array}\right\} \quad \text{(for Carreau's laws).}$$

Then, in these cases, $\partial_i \pi \in L^2_{\text{loc}}([0, \infty); L^{6/(p+1)}(\Omega))$ and the system of PDEs (equivalent to $(1)_2$),

$$(15) \qquad \varrho\{\partial_t u_i + u_j \partial_j u_i\} - \partial_j \tau_{ij}(Du) + \partial_i \pi = \varrho f_i, \quad \forall i = 1, \ldots, d$$

is satisfied point-wise a.e. in $(0, \infty) \times \Omega$. On the other hand, the continuity equation $(1)_1$ is only satisfied in a variational sense, because a better regularity result for the density ϱ is not known. It will be said that (ϱ, u, π) (with the above properties) is a *semi-strong solution* in $(0, +\infty)$ of the problem (1)–(3).

4. In Theorem 2, if we consider (H2) but only with $u^0 \in H_{\text{per}}$, then using a standard "cut-off method" (in a neighborhood of $t = 0$), see [9] for instance, we

can show the following regularity results (similar to (14) but excluding the time $t = 0$):

$$u \in L^2_{\mathrm{loc}}((0,\infty); H^2(\Omega)^d) \cap L^\infty_{\mathrm{loc}}((0,\infty); V^\sigma_{\mathrm{per}}),$$
$$\partial_t u \in L^2_{\mathrm{loc}}((0,\infty); H_{\mathrm{per}}).$$

4. Outline of the proofs

For simplicity, we outline the proofs of Theorems 1 and 2 simultaneously, i.e. for $\mu_\infty > 0$, and only some possible modifications to circumvent the case $\mu_\infty = 0$ (Theorem 3) will be pointed out. We will start the proofs with $u^0 \in V^\sigma_{\mathrm{per}}$ considering the case $u^0 \in H_{\mathrm{per}}$ at the end.

The general method of these proofs is: construction of a sequence of approximated solutions (using a discretization in a space of Galerkin's type) and a limit process using compactness in order to control the nonlinear terms.

During all this section, we will denote by $C, C_1, \dots$ different positive constants.

4.1 Approximated solutions and a priori estimates.

According to the method used in [2], we begin with a basis of V^σ_{per}, furnished by regular functions (at least C^1), $\{w_1, \dots, w_m, \dots\}$, and let V_m be the subspace spanned by $w_1, \dots, w_m$.

It will be said that (ϱ_m, u_m) is an m-th approximated solution to (1)–(3) if $\varrho_m \in C^1([0,\infty) \times \mathbb{R}^d)$, ϱ_m is L_i-periodic with respect to x_i, $u_m \in C^1([0,\infty); V_m)$ and (ϱ_m, μ_m) verifies

$$\partial_t \varrho_m + u_{m_i} \partial_i \varrho_m = 0 \ \text{ in } (0,\infty) \times \mathbb{R}^d, \quad \varrho_m|_{t=0} = \varrho^0_m \ \text{ in } \mathbb{R}^d, \tag{16}$$

$$\begin{cases} \displaystyle\int_\Omega \{\varrho_m(\partial_t u_{m_i} + u_{m_j} \partial_j u_{m_i}) v_i + \tau_{ij}(Du_m) Dv_{ij}\} \,\mathrm{d}x = \int_\Omega \varrho_m f_i v_i \,\mathrm{d}x \\ \forall v \in V_m \ \forall t \in (0,\infty), \quad u_m|_{t=0} = u^0_m \text{ in } \Omega. \end{cases} \tag{17}$$

Here we choose $\varrho^0_m \in C^1_{\mathrm{per}}$ and $u^0_m \in V_m$ such that

$$\varrho^0_m \to \varrho^0 \ \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^d) \ \forall q < \infty \quad \text{ with } \ \alpha \leqslant \varrho^0_m \leqslant \|\varrho^0\|_{L^\infty}, \ u^0_m \to u^0 \ \text{ in } V^\sigma_{\mathrm{per}}.$$

For instance, we can define $\varrho^0_m = \varrho^0 \star \xi_{1/m}$ where (ξ_ε) is a standard mollified sequence, and $u^0_m = P_m u^0$ with P_m the projection operator from V^σ_{per} onto V_m.

Notice that in the above definition of the m-th approximated solution, the PDEs system (15) has been discretized in space by a Galerkin method in (17), but the continuity equation $(1)_1$ remains to be of dimension infinity in (16); it can be called a Galerkin semi-discretization method.

The existence of an m-th approximated solution (ϱ_m, u_m) can be obtained arguing as in [2]: first, one rewrites the problem as a fixed point equation, then, one deduces appropriate estimates and, finally, Schauder's Theorem can be applied. Moreover, one deduces the following a priori estimates (uniformly with respect to m):

$$(\varrho_m) \text{ is bounded in } L^\infty(0,\infty; L^\infty(\mathbb{R}^d)),\ \alpha \leqslant \varrho_m(t,x) \leqslant \|\varrho^0\|_{L^\infty}, \tag{18}$$

$$(u_m) \text{ is bounded in } L^\infty(0,T; H_{\text{per}}),\ \forall T > 0, \tag{19}$$

$$(Du_m) \text{ is bounded in } L^\sigma(0,T; L^\sigma(\Omega)^{d\times d}),\ \forall T > 0 \tag{20}$$

(recall that $\sigma = \max\{p, 2\}$). Estimation (18) can be deduced making a construction of ϱ_m as a solution of (16) (a transport equation in the whole $\mathbb{R}^d$) by means of the characteristics method. On the other hand, taking $v = u_m(t) \in V_m, \forall t \in (0,\infty)$ as a test function in (17), one has the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2} u_m\|_{L^2}^2 + \mu_\infty \|Du_m\|_{L^2}^2 + \mu_0 \int_\Omega \tau_{ij}^p(Du_m)(Du_m)_{ij}\,\mathrm{d}x = \int_\Omega \varrho_m u_{m_i} f_i\,\mathrm{d}x.$$

Using the p-coercivity property of the non-Newtonian tensor τ^p (cf. [9]):

$$\tau_{ij}^p(D)D_{ij} \geqslant C \begin{cases} |D|^p & \text{(power law)} \\ \big(|D|^p - 1\big) & \text{(Carreau's laws)} \quad \forall D \in \mathbb{R}_S^{d\times d} \end{cases} \tag{P1}$$

where $\mathbb{R}_S^{d\times d}$ denotes the space of the symmetric real $d \times d$ matrices, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho_m^{1/2} u_m\|_{L^2}^2 + \mu_\infty \|Du_m\|_{L^2}^2 + \mu_0 C \|Du_m\|_{L^p}^p \\ &\qquad \leqslant \frac{1}{2}\|\varrho_m^{1/2} u_m\|_{L^2}^2 + \frac{1}{2}\|\varrho_m^{1/2} f\|_{L^2}^2 + \mu_0 C|\Omega|, \end{aligned}$$

thus, thanks to Gronwall's lemma, (19) and (20) hold.

Remark. More precisely, the p-coercivity property is verified as follows: $\tau_{ij}^p(D) \times D_{ij} = |D|^p$ when the power law (4) is considered, and for Carreau's laws (5) (see [9])

$$\tau_{ij}^p(D)D_{ij} \geqslant \begin{cases} \max\{|D|^2, |D|^p\} & \text{if } p \geqslant 2, \\ C\big(|D|^p - 1\big) & \text{if } 1 < p < 2. \end{cases}$$

Let $1 < q < +\infty$ be a given exponent. If we apply Korn's inequality

$$\|\nabla v\|_{L^q} \leqslant K \|Dv\|_{L^q} \quad \forall v \in W^{1,q}$$

and the generalized Poincaré inequality

$$\|v\|_{W^{1,q}} \leqslant C_q\biggl(\|\nabla v\|_{L^q} + \biggl|\int_\Omega v\,\mathrm{d}x\biggr|\biggr) \quad \forall v \in W^{1,q},$$

one has the inequality

$$\|v\|_{W^{1,q}} \leqslant C\biggl\{\|Dv\|_{L^q} + \biggl|\int_\Omega v\,\mathrm{d}x\biggr|\biggr\} \quad \forall v \in W^{1,q}.$$

In particular,

$$\|v\|_{W^{1,q}} \leqslant C\bigl\{\|Dv\|_{L^q} + \|v\|_{L^2}\bigr\} \quad \forall v \in W^{1,q} \cap L^2. \tag{21}$$

Then, using (19), (20) and (21), one has

$$(u_m) \quad \text{is bounded in} \quad L^\sigma(0,T;V^\sigma_{\mathrm{per}}), \quad \forall T > 0. \tag{22}$$

On the other hand, from (20) and the $(p-1)$-growth property of τ^p:

$$|\tau^p(D)| \leqslant C\begin{cases} |D|^{p-1} & \text{(power law)}, \\ (1+|D|)^{p-1} & \text{(Carreau's laws)} \quad \forall D \in \mathbb{R}^{d\times d}_S, \end{cases} \tag{P2}$$

one has

$$\{\tau^p(Du_m)\} \ \text{is bounded in} \ L^{\sigma/(p-1)}(0,T;L^{\sigma/(p-1)}(\Omega)^{d\times d}), \quad \forall T > 0. \tag{23}$$

4.2 Compactness of (ϱ_m) and (u_m).

By virtue of estimates (18)–(23), every term of the approximated equations (16)–(17) is bounded. But, in order to control the passage to the limit in the nonlinear terms, some compactness properties will be necessary. Arguing as in [2], one has that for all $T > 0$,

$$(u_m) \text{ is relatively compact in } L^\sigma(0,T;L^s(\Omega)^d), \quad \forall s < \sigma_\star$$

where $\sigma_\star$ is the exponent related to σ by Sobolev's embeddings. Indeed, this compactness of u_m is a consequence of the following estimate of the time derivative type (see [2]): $\forall h:\ 0 < h < T,\ \exists C > 0$ such that

$$\int_0^{T-h} \|u_m(t+h) - u_m(t)\|^2_{L^2}\,\mathrm{d}t \leqslant Ch^{1/\sigma'} \tag{24}$$

where σ' is the conjugate exponent of σ.

On the other hand, taking into account (12) (norms $L^q(\Omega)$ of ϱ_m and ϱ are conserved for all $q < \infty$ and $t \in [0,\infty)$), one has that for all $T > 0$,

$$\varrho_m \to \varrho \ \text{ in } \ L^q((0,T)\times\Omega).$$

Hence, the periodicity of ϱ is deduced.

Remark (Measure-valued solution). All the properties showed until now imply, using compactness and monotony arguments, the existence of a measure-valued solution, that in particular verifies (13) and (10), see [2] for the case of Dirichlet boundary conditions. In fact, this result holds for the more general hypotheses: $\varrho^0 \geqslant 0$ in Ω and $f \in L^1(0,T;L^2(\Omega)^d)$.

4.3 Supplementary estimates of (u_m)**.**

We shall find a weak solution passing to the limit as $m \to \infty$ in (16)–(17). Here, the key point is to prove that, as $m \to +\infty$,

$$\int_0^\infty\int_\Omega \tau^p_{ij}(Du_m)D\varphi_{ij}\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_0^\infty\int_\Omega \tau^p_{ij}(Du)D\varphi_{ij}\,\mathrm{d}x\,\mathrm{d}t. \tag{25}$$

Thanks to the continuity of τ^p, (25) holds if we prove point-wise convergence of Du_m to Du (a.e. in $(0,\infty)\times\Omega$). We are going to obtain it by showing that (u_m) is relatively compact in a space of the $L^2(0,T;H^1)$ type. For this, we will need some supplementary estimates of u_m.

Taking $v = \partial_t u_m(t) \in V_m$ as a test function in (17) and using appropriately the upper and lower bounds of the density, we easily deduce

$$\frac{\alpha}{2}\|\partial_t u_m\|^2_{L^2} + \frac{\mathrm{d}}{\mathrm{d}t}J(u_m) \leqslant \frac{C}{\alpha}\big(\|f\|^2_{L^2} + I(u_m,\nabla u_m)\big), \tag{26}$$

where we have used the notation

$$I(u,\nabla u) = \int_\Omega |u|^2|\nabla u|^2\,\mathrm{d}x,$$
$$J(u) = \frac{\mu_\infty}{2}\|Du\|^2_{L^2} + \mu_0\int_\Omega U_p(Du)\,\mathrm{d}x,$$

U_p being a potential function of τ^p, i.e.

$$U_p : \mathbb{R}^{d\times d}_S \to \mathbb{R}_+, \quad U_p \in C^2 \qquad \text{and} \qquad \frac{\partial U_p}{\partial D_{ij}}(D) = \tau^p_{ij}(D) \quad \forall D \in \mathbb{R}^{d\times d}_S, \forall ij.$$

In fact, one can determine U_p as

$$U_p(D) = \frac{1}{2}\int_0^{|D|^2} s^{(p-2)/2}\,\mathrm{d}s = \frac{1}{p}|D|^p \quad \text{(power law)},$$

$$\left.\begin{aligned} U_p(D) &= \frac{1}{2}\int_0^{|D|^2}(1+s^{1/2})^{p-2}\,\mathrm{d}s, \quad \text{or} \\ U_p(D) &= \frac{1}{2}\int_0^{|D|^2}(1+s)^{(p-2)/2}\,\mathrm{d}s \end{aligned}\right\} \quad \text{(Carreau's laws).}$$

In addition, $\int_\Omega U_p(Du)\,\mathrm{d}x$ is "almost" equivalent to $\|Du\|^p_{L^p}$. Indeed, one has (cf. [9])

$$\text{(P3)}\qquad \begin{aligned} C_1\|Du\|^p_{L^p} &\leqslant \int_\Omega U_p(Du) \leqslant C_2\|Du\|^p_{L^p} \quad \text{(power law)},\\ C_1\big(\|Du\|^p_{L^p}-|\Omega|\big) &\leqslant \int_\Omega U_p(Du) \leqslant C_2\big(\|Du\|^p_{L^p}+|\Omega|\big) \quad \text{(Carreau's laws)}. \end{aligned}$$

Remark. More precisely, in the case of the power law (4) one has

$$\int_\Omega U_p(Du) = \frac{1}{p}\|Du\|^p_{L^p}.$$

On the other hand, the behavior of Carreau's laws can be again splitted into the cases $p \geqslant 2$ and $1 < p < 2$. When $p \geqslant 2$, one also has

$$\int_\Omega U_p(Du)\,\mathrm{d}x \geqslant \max\left\{\frac{1}{2}\|Du\|^2_{L^2}, \frac{1}{p}\|Du\|^p_{L^p}\right\}.$$

In order to complete the proofs of Theorems 1 and 2, we are going to use another differential inequality together with (26). For this, we consider a special basis (w_m) formed by eigenfunctions of the Stokes operator A ($Aw_m = \lambda_m w_m$). Then $Au_m(t) \in V_m$ can be taken as a test function in (17), which yields

$$\text{(27)}\qquad \begin{aligned} \mu_\infty\|\Delta u_m\|^2_{L^2} &+ \mu_0\int_\Omega \tau^p_{ij}(Du_m)D(Au_m)_{ij}\,\mathrm{d}x\\ &= \int_\Omega \varrho_m\{f-\partial_t u_m - u_m\cdot\nabla u_m\}\cdot Au_m\,\mathrm{d}x. \end{aligned}$$

Estimating the right hand side of (27), we have

$$\text{(28)}\qquad \begin{aligned} \frac{\mu_\infty}{2}\|\Delta u_m\|^2_{L^2} &+ \mu_0\int_\Omega \tau^p_{ij}(Du_m)D(Au_m)_{ij}\\ &\leqslant C\Big(\|f\|^2_{L^2}+\|\partial_t u_m\|^2_{L^2}+I(u_m,\nabla u_m)\,\mathrm{d}x\Big). \end{aligned}$$

First, since we are considering periodic boundary conditions, $Au_m = -\Delta u_m$ (in fact $-\Delta w_m = \lambda_m w_m$). Then, the second term of (28) can be bounded from below as follows, cf. [9]:

$$\begin{aligned} \int_\Omega \tau^p_{ij}(Du_m)D(Au_m)_{ij}\,\mathrm{d}x &= -\int_\Omega \frac{\partial U_p}{\partial D_{ij}}(Du_m)\Delta(Du_m)_{ij}\,\mathrm{d}x\\ = \int_\Omega \frac{\partial^2 U_p}{\partial D_{ij}\partial D_{kl}}(Du_m)\partial_n(Du_m)_{kl}\partial_n(Du_m)_{ij}\,\mathrm{d}x &\geqslant CI_p(Du_m), \end{aligned}$$

where

$$
(29) \qquad I_p(Du) = \begin{cases} \displaystyle\int_\Omega |Du|^{p-2} |\nabla(Du)|^2 \, \mathrm{d}x & \text{(power law)}, \\ \displaystyle\int_\Omega (1+|Du|)^{p-2} |\nabla(Du)|^2 \, \mathrm{d}x & \text{(Carreau's laws)}. \end{cases}
$$

Secondly, performing an "adequate balance" between (26) and (28), we can eliminate $\|\partial_t u_m\|_{L^2}^2$ from the right hand side of (28), hence (for simplicity, the different positive constants will be omitted)

$$
(30) \qquad \|\partial_t u_m\|_{L^2}^2 + \|\Delta u_m\|_{L^2}^2 + I_p(Du_m) + \frac{\mathrm{d}}{\mathrm{d}t} J(u_m) \leqslant \|f\|_{L^2}^2 + I(u_m, \nabla u_m).
$$

Finally, using that

$$
(31) \qquad \|u\|_{H^2}^2 \leqslant C\{\|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2\} \quad \forall u \in H^2 \cap H^1_{\text{per}},
$$

(which is a consequence of the H^2 regularity of the periodic Poisson problem), if one adds $\|u_m\|_{L^2}^2$ to both parts of (30), one concludes (again omitting the constants):

$$
(32) \qquad \begin{aligned} &\|\partial_t u_m\|_{L^2}^2 + \|u_m\|_{H^2}^2 + I_p(Du_m) + \frac{\mathrm{d}}{\mathrm{d}t} J(u_m) \\ &\quad \leqslant \|f\|_{L^2}^2 + I(u_m, \nabla u_m) + \|u_m\|_{L^2}^2. \end{aligned}
$$

In the case $\mu_\infty = 0$, (32) also holds if we consider $p \geqslant 2$ and Carreau's laws. Indeed, using $(29)_2$ and Korn's inequality, one has

$$
I_p(Du) \geqslant \int_\Omega |\nabla(Du)|^2 \, \mathrm{d}x \geqslant C \int_\Omega |\partial^2 u|^2 \, \mathrm{d}x,
$$

where $\partial^2 u$ denotes the tensor of all second derivatives of u. Therefore, using (31), we arrive at

$$
I_p(Du) + \|u\|_{L^2}^2 \geqslant C\|u\|_{H^2}^2 \quad \forall u \in H^2 \cap H^1_{\text{per}}.
$$

Then, we can control the right hand side terms of (27) and deduce (32).

4.4 Proof of Theorem 1 for $u^0 \in V_{\text{per}}$ and p large enough ($p \geqslant 12/5$ if $d = 3$ or $p \geqslant 2$ if $d = 2$).

In these cases, we only use inequality (26). Indeed, applying the Hölder and Young inequalities (with exponents $p/2$ and $p/(p-2)$), we bound $I(u_m, \nabla u_m)$ as follows:

$$
\begin{aligned} I(u_m, \nabla u_m) &\leqslant \|u_m\|_{L^{2p/(p-2)}}^2 \|\nabla u_m\|_{L^p}^2 \\ &\leqslant \frac{p-2}{p} \|u_m\|_{L^{2p/(p-2)}}^2 + \frac{2}{p} \|u_m\|_{L^{2p/(p-2)}}^2 \|\nabla u_m\|_{L^p}^p. \end{aligned}
$$

Then, using (P3) and Korn's inequality, one obtains

$$I(u_m, \nabla u_m) \leqslant \frac{p-2}{p}\|u_m\|^2_{L^{2p/(p-2)}} + C\|u_m\|^2_{L^{2p/(p-2)}}\left\{\frac{1}{C_1}\int_\Omega U_p(Du)\,\mathrm{d}x + |\Omega|\right\}. \tag{33}$$

Since $2p/(p-2) \leqslant p_\star$ if and only if $p \geqslant 12/5$ (if $d=3$) or $p \geqslant 2$ (if $d=2$), in these cases (22) implies that

$$\|u_m\|^2_{L^{2p/(p-2)}} \text{ is bounded in } L^1(0,T).$$

On the other hand,

$$J(u_m^0) \leqslant \frac{\mu_\infty}{2}\|Du_m^0\|^2_{L^2} + \mu_0 C_2\{\|Du_m^0\|^p_{L^p} + |\Omega|\} \leqslant C\{\|u^0\|^p_{V^p} + |\Omega|\}. \tag{34}$$

Thus, thanks to (33) and (34), we can argue by the Gronwall lemma in (26), getting

$$(u_m) \text{ is bounded in } L^\infty(0,T;V^p_{\mathrm{per}}) \text{ and } (\partial_t u_m) \text{ is bounded in } L^2(0,T;H_{\mathrm{per}}).$$

Under these conditions, by a monotony argument (using, in particular, the Minty trick), one can show the existence of a weak solution of (1)–(3) (cf. [2] for the Dirichlet case).

4.5 Existence of semi-strong solution. Proof of Theorem 2.

In the cases of Theorem 2, we bound $I(u_m, \nabla u_m)$ as follows:

When $p \geqslant 12/5$ (if $d=3$) and $p \geqslant 2$ (if $d=2$), we have already got (33).

When $1 < p < 2$ (if $d=2$), applying the Gagliardo-Niremberg's inequality $\|u\|_{L^4} \leqslant C\|u\|^{1/2}_{L^2}\|u\|^{1/2}_{H^1}$ one has

$$\begin{aligned} I(u_m, \nabla u_m) &\leqslant \|u_m\|^2_{L^4}\|\nabla u_m\|^2_{L^4} \\ &\leqslant C\|u_m\|_{L^2}\|u_m\|_{H^1}\|\nabla u_m\|_{L^2}\|\nabla u_m\|_{H^1} \\ &\leqslant \varepsilon\|u_m\|^2_{H^2} + C_\varepsilon\|u_m\|^2_{L^2}\|u_m\|^2_{H^1}\|\nabla u_m\|^2_{L^2}. \end{aligned}$$

Using Korn's inequality, the second term on the right hand side can be bounded by

$$C_\varepsilon\|u_m\|^2_{L^2}\|u_m\|^2_{H^1}J(u_m).$$

When $20/9 \leqslant p < 12/5$ (if $d=3$), using (21) and the inequality (see [9]),

$$\|\nabla u\|_{L^{3p}} \leqslant C I_p(Du)^{1/p},$$

one obtains

$$\begin{aligned} I(u_m, \nabla u_m) &\leqslant \|u_m\|_{L^{p\star}}^2 \|\nabla u_m\|_{L^p}^{(5p-8)/2} \|\nabla u_m\|_{L^{3p}}^{(12-5p)/2} \\ &\leqslant C\|u_m\|_{V_p}^{(5p-4)/2} I_p(Du_m)^{(12-5p)/2p} \\ &\leqslant \varepsilon I_p(Du_m) + C_\varepsilon \|u_m\|_{V_p}^{(5p-4)p/(7p-12)}. \end{aligned}$$

Since $p < (5p-4)p/(7p-12) \leqslant 2p$, if we use (P3) and (21), the second term on the right hand side can be bounded by

$$C_\varepsilon \|u_m\|_{V_p}^{\{(5p-4)p/(7p-12)\}-p} C\left\{ \frac{1}{C_1} \int_\Omega U_p(Du_m)\,\mathrm{d}x + |\Omega| + \|u_m\|_{L^2}^p \right\}.$$

Finally, by (22), $\|u_m\|_{V_p}^{\{(5p-4)p/(7p-12)\}-p}$ is bounded in $L^1_{\text{loc}}(0,\infty)$.

Consequently, inserting in (32) the above estimates for $I(u_m, \nabla u_m)$ and taking into account (34), we obtain the following estimates by means of Gronwall's lemma: $\forall T > 0$,

$$(u_m) \text{ is bounded in } L^2(0,T; H^2(\Omega)^d) \cap L^\infty(0,T; V_{\text{per}}^\sigma),$$

$$(\partial_t u_m) \text{ is bounded in } L^2(0,T; H_{\text{per}}).$$

Thanks to a classical compactness result of "Aubin-Lions" type, cf. [6], using the triplet of spaces $(H^2 \cap V_{\text{per}}) \hookrightarrow\hookrightarrow V_{\text{per}} \hookrightarrow H_{\text{per}}$ (we denote by $\hookrightarrow\hookrightarrow$ a compact embbeding and by $\hookrightarrow$ a continuous one), these estimates imply that

$$(u_m) \text{ is relatively compact in } L^2(0,T; V_{\text{per}}), \quad \forall T > 0.$$

Hence, it is possible to complete the proof of Theorem 2 by a standard limit process.

4.6 Existence of a weak solution with $u^0 \in H_{\text{per}}$ (Proof of Theorem 1).

Now, we consider $u_m^0 \in V_m$ with $u_m^0 \to u^0$ in H_{per} (instead of in V_{per}^σ as before). Then, the bound for $I(u_m, \nabla u_m)$ can not be used as in the previous case. Now, using interpolation inequalities, we have

$$I(u_m, \nabla u_m) \leqslant \|u_m\|_{L^6}^2 \|\nabla u_m\|_{L^2} \|\nabla u_m\|_{L^6} \leqslant \varepsilon \|u_m\|_{H^2}^2 + C_\varepsilon \|u_m\|_{H^1}^4 \|\nabla u_m\|_{L^2}^2.$$

The second term on the right hand side can be bounded using (21) for $q = 2$ as follows:

$$C_\varepsilon \|u_m\|_{H^1}^2 \{\|Du_m\|_{L^2}^2 + \|u_m\|_{L^2}^2\} \|Du_m\|_{L^2}^2.$$

Now, we use (21) and the fact that

$$
(35)\qquad \begin{cases} \|Du_m\|_{L^2}^2 \leqslant \dfrac{2}{\mu_\infty} J(u_m) \quad (\text{when } \mu_\infty > 0) \\ \|Du_m\|_{L^2}^2 \leqslant \dfrac{2}{p}\|Du_m\|_{L^p}^p + \dfrac{p-2}{p}|\Omega| \\ \qquad\qquad \leqslant C(\Omega)\{J(u_m)+1\} \ (\text{when } \mu_\infty = 0 \text{ and } p \geqslant 2). \end{cases}
$$

Inserting in (32) the above bound for $I(u_m, \nabla u_m)$ one has (omitting constants)

$$
\begin{aligned}
&\|\partial_t u_m\|_{L^2}^2 + \|u_m\|_{H^2}^2 + I_p(Du_m) + \frac{\mathrm{d}}{\mathrm{d}t} J(u_m) \\
&\leqslant \|f\|_{L^2}^2 + \|u_m\|_{H^1}^2 \{\|u_m\|_{L^2}^2 + 1\}(1 + J(u_m))^2.
\end{aligned}
$$

Dividing by $(1+J(u_m))^2$, one has (omitting constants)

$$
\frac{\|\partial_t u_m\|_{L^2}^2 + \|u_m\|_{H^2}^2}{(1+J(u_m))^2} - \frac{\mathrm{d}}{\mathrm{d}t}\{(1+J(u_m))^{-1}\} \leqslant \|f\|_{L^2}^2 + \|u_m\|_{H^1}^2\{\|u_m\|_{L^2}^2 + 1\}
$$

and the right hand side is bounded in $L^1_{\mathrm{loc}}([0,\infty))$. Therefore, integrating in $(0,T)$,

$$
\int_0^T \frac{\|\partial_t u_m\|_{L^2}^2 + \|u_m\|_{H^2}^2}{(1+J(u_m))^2}\,\mathrm{d}t + \frac{1}{1+J(u_m^0)} \leqslant \frac{1}{1+J(u_m(T))} + C.
$$

In particular, although $J(u_m^0)$ could be not bounded, one has

$$
(36)\qquad \int_0^T \frac{\|u_m\|_{H^2}^2}{(1+J(u_m))^2}\,\mathrm{d}t \leqslant C.
$$

Using (36), we can get the following estimate of u_m (of the "$L^{2/3}(0,T;H^2)$" type):

$$
\begin{aligned}
\int_0^T \|u_m\|_{H^2}^{2/3}\,\mathrm{d}t &= \int_0^T \left(\frac{\|u_m\|_{H^2}^2}{(1+J(u_m))^2}\right)^{1/3} (1+J(u_m))^{2/3}\,\mathrm{d}t \\
&\leqslant C^{1/3}\left(\int_0^T (1+J(u_m))\,\mathrm{d}t\right)^{2/3} \leqslant C.
\end{aligned}
$$

Finally, with help of the interpolation estimates between the H^k spaces

$$
\|u\|_{H^{1+\theta}} \leqslant C\|u\|_{H^1}^{1-\theta}\|u\|_{H^2}^{\theta} \qquad \text{for } 0 \leqslant \theta \leqslant 1,
$$

one concludes

$$
(u_m) \text{ is bounded in } L^{2/(1+2\theta)}(0,T;H^{1+\theta}), \quad \forall T > 0.
$$

Since $2/(1+2\theta) \geqslant 1$ whenever $0 \leqslant \theta \leqslant 1/2$, the above estimate together with (24) and an adequate compactness result (considering the triplet of spaces $(H^{1+\theta} \cap V_{\text{per}}) \hookrightarrow\hookrightarrow V_{\text{per}} \hookrightarrow H_{\text{per}}$), cf. [11], imply that $\forall T > 0$,

$$\text{(37)} \qquad (u_m) \text{ is relatively compact in } L^{2/(1+2\theta)}(0,T;V_{\text{per}}) \qquad \text{for } 0 \leqslant \theta \leqslant 1/2.$$

In particular, one can deduce point-wise convergence a.e. in $(0,\infty) \times \Omega$ from Du_m to Du, hence (25) holds. Finally, by a standard limit process, one can complete the proof of Theorem 1.

5. Some extensions and open problems

5.1 More general non-Newtonian tensors.

The same sort of results can be obtained for a non-Newtonian tensor τ^p defined by a potential function U_p, i.e.

$$\text{(38)} \qquad U_p : \mathbb{R}_S^{d\times d} \to \mathbb{R}_+,\ U_p \in C^2 \quad \text{and} \quad \frac{\partial U_p}{\partial D_{ij}}(D) = \tau_{ij}^p(D) \quad \forall D \in \mathbb{R}_S^{d\times d},$$

which posesses the following properties:

$$\text{(39)} \qquad U_p(0) = \frac{\partial U_p}{\partial D_{ij}}(0) = 0,$$

$$\text{(40)} \qquad \left|\frac{\partial^2 U_p}{\partial D_{ij}\partial D_{kl}}(D)\right| \leqslant C_1 \begin{cases} |D|^{p-2} \\ (1+|D|)^{p-2} \end{cases} \qquad \forall D \in \mathbb{R}_S^{d\times d},$$

$$\text{(41)} \qquad \frac{\partial^2 U_p}{\partial D_{ij}\partial D_{kl}}(D)E_{ij}E_{kl} \geqslant C_2 \begin{cases} |D|^{p-2}|E|^2 \\ (1+|D|)^{p-2}|E|^2 \end{cases} \qquad \forall D, E \in \mathbb{R}_S^{d\times d}.$$

The above assertion is due to the fact that (38)–(41) lead to (P1), (P2) and (P3), cf. [9]. Indeed, (38)–(40) imply (P2) and the upper bound of (P3), whereas (38), (39) and (41) imply (P1) and the lower bound of (P3). We understand (40) and (41) in the sense that only one of the two conditions is considered: either $(40)_1$ and $(41)_1$, which play the role of a power law, or $(40)_2$ and $(41)_2$ in the role of Carreau's laws.

On the other hand, a perturbation σ^p of τ^p can be admitted (and then, the non-Newtonian tensor is $\tau^p + \sigma^p$), where τ^p is determined by (38)–(41) and σ^p is determined by a potential function V_p ($\sigma^p_{ij} \equiv \partial V_p / \partial D_{ij}$) such that

$$V_p(0) = \frac{\partial V_p}{\partial D_{ij}}(0) = 0 \quad \forall ij,$$

$$\left| \frac{\partial^2 V_p}{\partial D_{ij} \partial D_{kl}}(D) \right| \leqslant C_1' \begin{cases} |D|^{p-2} \\ (1+|D|)^{p-2} \end{cases} \quad \forall D \in \mathbb{R}^{d\times d}_S,$$

$$\frac{\partial^2 V_p}{\partial D_{ij} \partial D_{kl}}(D) E_{ij} E_{kl} \geqslant 0 \quad \forall D, E \in \mathbb{R}^{d\times d}_S.$$

Hence, it is easy to check that, in these cases, one will have the same results.

5.2 Dirichlet boundary conditions.

According to [2], when one considers Dirichlet boundary conditions, the existence of weak solutions in the cases $p < 12/5$ (if $d = 3$) and $p < 2$ (if $d = 2$) are interesting open problems. The argument made in this paper for the periodic case now fails because of $Au_m \neq -\Delta u_m$ (since Δu_m do not vanish on $\partial\Omega$). In the case $\varrho \equiv$ *constant* and $p \geqslant 2$, this difficulty has been circumvented in [10], multiplying $-\Delta u_m$ by a cut-off function χ_ε and considering the "mixed formulation" (with pressure), since the new test function $-\chi_\varepsilon \Delta u_m$ is not solenoidal.

On the other hand, when $1 < p < 2$ and Carreau's laws are considered (i.e. assuming $(40)_2$), results similar to Theorems 1 and 2 can be proved for the Dirichlet problem, assuming that the Newtonian viscosity "dominates" the non-Newtonian one in the following sense:

$$\mu_\infty > C\mu_0 \quad \text{with } C > 0 \text{ large enough.}$$

The main idea of this proof is to estimate the second term of (28) by

$$-\mu_0 \int_\Omega \tau^p_{ij}(Du_m) D(Au_m)_{ij} \, \mathrm{d}x = \mu_0 \int_\Omega \frac{\partial^2 U_p}{\partial D_{ij} \partial D_{kl}}(Du_m) \partial_j (Du_m)_{kl} (Au_m)_i \, \mathrm{d}x$$
$$\leqslant \mu_0 C_1 \sum_{i,j,k,l=1}^{d} \int_\Omega (1+|Du_m|)^{p-2} |\partial_j (Du_m)_{kl}| |(Au_m)_i| \, \mathrm{d}x \leqslant \mu_0 C_1 d^2 \|\partial^2 u_m\|^2_{L^2}$$

(in the last estimate, we have used that $1 < p < 2$). Then, we could control this bound with the first term of (28), $\mu_\infty \|\Delta u_m\|^2_{L^2}$, letting μ_∞ to be large enough (namely, whenever $\mu_\infty > C\mu_0 C_1 d^2$, with C the constant from inequality (31)).

5.3 Some open problems.

Finally, we point out some related open problems:

1. Viscosity dependent on the density. If we assume $\mu_\infty = \mu_\infty(\varrho)$ and $\mu_0 = \mu_0(\varrho)$, with $\mu_\infty, \mu_0 : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and strictly increasing functions, the existence of a weak solution is deduced in [2] when $p \geqslant 12/5$ (if $d = 3$) and $p \geqslant 2$ (if $d = 2$), using compactness and monotony arguments. The extension of this result to more general cases of this paper, is an interesting open problem. Arguing as in this paper, the main difficulty is to control the term

$$\int_\Omega \{\mu_\infty(\varrho_m)(Du_m)_{ij} + \mu_0(\varrho_m)\tau^p_{ij}(Du_m)\} D(Au_m)_{ij}\,\mathrm{d}x,$$

which appears when Au_m is taken as a test function.

2. The case $\mu_\infty = 0$ and $p < 2$ (for a power law or Carreau's laws). From a physical point of view, this problem is more interesting for Carreau's laws, cf. [8]. Now, the main difficulty is to control the term

$$\int_\Omega \varrho_m \partial_t u_m \cdot \Delta u_m$$

which appears when $-\Delta u_m$ is taken as a test function.

3. Uniqueness, even for semi-strong solutions and p large. The more delicate question is that we do not know whether the additional regularity of u given in (14) implies additional regularity for the density gradient $\nabla\varrho$ (for instance, of the L^q type). With only $\varrho \in L^\infty$, a standard uniqueness argument fails. On the other hand, a uniqueness result for a weak solution, whenever a sufficiently regular solution exists, could be developed, i.e. we obtain weak-regular uniqueness, cf. [7] for the density-dependent Newtonian case.

4. Existence of strong solutions (with regularity of the L^q type for $\nabla\varrho$). At our knowledge, it is an open problem, even in the restrictive cases for small time (local in time solution) or for large time but assuming small data.

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