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ON THE EMBEDDING OF ORDERED SEMIGROUPS INTO ORDERED GROUP

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Abstract. It was shown in [7] that any right reversible, cancellative ordered semigroup can be embedded into an ordered group and as a consequence, it was shown that a commutative ordered semigroup can be embedded into an ordered group if and only if it is cancellative. In this paper we introduce the concept of L-maher and R-maher semigroups and use a technique similar to that used in [7] to show that any left reversible cancellative ordered L or R-maher semigroup can be embedded into an ordered group.

Keywords: semicommutative semigroups, maher semigroups, ordered semigroups

MSC 2000: 06F05

1. Introuction and premliminaries

The concept of L-semicommutative (R-semicommutative) semigroups was first introduced in [1]. A semigroup $(S,*)$ is called L-semicommutative if and only if $\forall a,b\in S\colon\ a*b*a=a^2*b$ and R-semicommutative if and only if $\forall a,b\in S\colon\ a*b*a=b*a^2$. Clearly any commutative semigroup is both L-semicommutative and R-semicommutative and any cancellative L-semicommutative or R-semicommutative semigroup is commutative.

A semigroup $(S,*)$ is called left (right) reversible if $\forall a,b\in S\colon\ a*S\cap b*S\neq\emptyset$ ($S*a\cap S*b\neq\emptyset$). An R-semicommutative (L-semicommutative) semigroup is left (right) reversible. It is well known that any right reversible cancellative semigroup can be embedded in a group [2], Theorem 1.23. Kehayopulu and Tsingleis [7] proved that a commutative ordered semigroup is embeddable in an ordered group if and only if it is cancellative. The following theorem is an immediate consequence of the main theorem in [7].

Theorem 1. *An L-semicommutative ordered semigroup is embeddable in an ordered group if and only if it is cancellative.*

In order to have the dual result of the above theorem for R-semicommutative semigroups, we introduce the concept of L-maher and R-maher semigroups which are generalizations of L and R-semicommutative semigroups, respectively, and show that both of them are embeddable in ordered groups.

Definition 2. A semigroup $(S, *)$ is called *cancellative* if

$$\forall a, b, c \in S\colon\ c * a = c * b \Longrightarrow a = b \quad \text{and} \quad a * c = b * c \Longrightarrow a = b.$$

A semigroup $(S, *)$ is called *R-maher* if

$$\forall a, b, x, y \in S\colon\ (a * x = b * y \Longrightarrow (a * b * a) * x = (b * a * b) * y)$$

and *L-maher* if

$$\forall a, b, x, y \in S\colon\ (x * a = y * b \Longrightarrow x * (a * b * a) = y * (b * a * b)).$$

Commutative semigroups are both L and R-maher. L-semicommutative semigroups and R-semicommutative semigroups are L and R-maher, respectively. An ordered semigroup $(S, *, \leqslant_S)$ is called *cancellative* if

$$\forall a, b, c \in S\colon\ (c * a \leqslant_S c * b \Rightarrow a \leqslant_S b \quad \text{and} \quad a * c \leqslant_S b * c \Rightarrow a \leqslant_S b).$$

Let $(S, *, \leqslant_S)$, $(T, \diamond, \leqslant_T)$ be ordered semigroups. Let $f\colon S \to T$ be a mapping. Then f is called isotone if $x, y \in S$, $x \leqslant_S y$ implies that $f(x) \leqslant_T f(y)$. f is called reverse isotone if $x, y \in S$, $f(x) \leqslant_T f(y)$ implies that $x \leqslant_S y$. Clearly, any reverse isotone mapping is injective. f is called a homomorphism if it is isotone and

$$\forall x, y \in S\colon\ f(x * y) = f(x) \diamond f(y).$$

f is called an isomorphism if it is an onto reverse isotone homomorphism. S is embeddable in T if $(S, *, \leqslant_S)$ is isomorphic to a subsemigroup of $(T, \diamond, \leqslant_T)$.

2. Constructing an ordered group from an ordered semigroup

Let $(S, *)$ be a left reversible semigroup and $a, b \in S$. We define a relation $E(a, b)$ on S as follows: $(x, y) \in E(a, b) \Longleftrightarrow a * x = b * y$.

Clearly, a semigroup $(S, *)$ is R-maher if $(x, y) \in E(a, b) \Longrightarrow (x, y) \in E(a * b * a, b * a * b)$.

The following lemma will be used repeatedly in this paper.

Lemma 3. *Let* $(S,*)$ *be a left reversible semigroup. Then*

1) $\forall a,b\in S\colon\ E(a,b)\neq\emptyset$,
2) $\forall a,b\in S\colon\ E(a,b)=(E(b,a))^{-1}$,
3) *if* $(S,*)$ *is cancellative and* $a,b,c,d\in S$, *then*

$$E(a,b)\cap E(c,d)\neq\emptyset \Longleftrightarrow E(a,b)=E(c,d),$$

4) *if* $(S,*)$ *is cancellative and* $a,b\in S$, *then* $E(a,a)=E(b,b)$.

P r o o f. Parts 1 and 2 follow from the definitions of left reversibility and $E(a,b)$.

3) $(\Rightarrow)$ Assume that $E(a,b)\cap E(c,d)\neq\emptyset$. Let $(x,y)\in E(a,b)\cap E(c,d)$, $(u,v)\in E(c,d)$, $(z,t)\in E(u,x)$. Then

$$a*x=b*y,\quad c*x=d*y,\quad c*u=d*v,\quad u*z=x*t.$$

It then follows that

$$\begin{aligned} d*(v*z)&=(d*v)*z=(c*u)*z=c*(u*z)\\ &=c*(x*t)=(c*x)*t=(d*y)*t=d*(y*t). \end{aligned}$$

Since $(S,*)$ is cancellative, $v*z=y*t$. Also,

$$\begin{aligned} (a*u)*z&=a*(u*z)=a*(x*t)=(a*x)*t\\ &=(b*y)*t=b*(y*t)=b*(v*z)=(b*v)*z. \end{aligned}$$

Since $(S,*)$ is cancellative, it follows that $a*u=b*v$ and thus $(u,v)\in E(a,b)$, which concludes that $E(c,d)\subseteq E(a,b)$. Similarly, $E(a,b)\subseteq E(c,d)$ and therefore $E(a,b)=E(c,d)$.

$(\Leftarrow)$ Immediate.

4) follows immediately from part 3, since $(a,a)\in E(a,a)\cap E(b,b)$. $\square$

In what follows $(S,*)$ is a left reversible cancellative semigroup. We define a relation ξ on $S\times S$ as follows:

$$((a,b),(c,d))\in\xi \Longleftrightarrow E(a,c)=E(b,d).$$

The following lemma shows that ξ is an equivalence relation on S.

Lemma 4. *Let $(S, *)$ be a left reversible, cancellative semigroup. Then ξ is an equivalence relation on $S \times S$.*

P r o o f. From Lemma 3(4) we immediately get that the relation ξ is reflexive.

To show that ξ is symmetric, let $((a, b), (c, d)) \in \xi$. Then $E(a, c) = E(b, d)$ and Lemma 3(2) implies that

$$E(c, a) = (E(a, c))^{-1} = (E(b, d))^{-1} = E(d, b),$$

which shows that $((c, d), (a, b)) \in \xi$.

To show that ξ is transitive, let $((a, b), (c, d)) \in \xi$ and $((c, d), (e, f)) \in \xi$. Then $E(a, c) = E(b, d), E(c, e) = E(d, f)$. Let $(x, y) \in E(a, c), (u, v) \in E(d, f)$ and $(t, z) \in E(y, u)$. Then

$$(1) \qquad a * x = c * y, \; b * x = d * y, \; d * u = f * v, \; c * u = e * v, \; y * t = u * z.$$

By a straightforward calculations we get

$$b * (x * t) = f * (v * z), \quad a * (x * t) = e * (v * z),$$

which means that $(x * t, v * z) \in E(b, f) \cap E(a, e)$. Thus Lemma 3(3) implies that $E(a, e) = E(b, f)$ and therefore $((a, b), (e, f)) \in \xi$. □

Let $(a, b) \in S \times S$. We denote by $[(a, b)]_\xi$ the equivalence class of (a, b) with respect to the equivalence relation ξ.

Lemma 5. *Let $(S, *)$ be a left reversible cancellative semigroup and let $a, b, c, d \in S$. Then*

1) $[(a, a)]_\xi = [(b, b)]_\xi$,
2) $((a, b), (c, d)) \in \xi \iff ((b, a), (d, c)) \in \xi$.

P r o o f. 1) Since $E(a, b) = E(a, b)$, it follows by the definition of ξ that $((a, a), (b, b)) \in \xi$ and thus $[(a, a)]_\xi = [(b, b)]_\xi$.

2) The definition of ξ implies that

$$\begin{aligned} &((a, b), (c, d)) \in \xi \iff E(a, c) = E(b, d) \iff \\ &E(b, d) = E(a, c) \iff ((b, a), (d, c)) \in \xi. \end{aligned}$$

□

Let $\odot$ be defined on $(S \times S)/\xi$ as follows:

$$[(a, b)]_\xi \odot [(c, d)]_\xi = [(c * y, b * x)]_\xi \quad \text{for some } (x, y) \in E(a, d).$$

The following lemma shows that $\odot$ is an associative binary operation on $(S \times S)/\xi$.

Lemma 6. *Let $(S,*)$ be a left reversible cancellative semigroup. Then $\odot$ is an associative binary operation on $(S\times S)/\xi$.*

P r o o f. We first show that $\odot$ is well defined. So let $[(a,b)]_\xi = [(a',b')]_\xi$ and $[(c,d)]_\xi = [(c',d')]_\xi$ in $(S\times S)/\xi$. Then

$$[(a,b)]_\xi \odot [(c,d)]_\xi = [(c*y, b*x)]_\xi \quad \text{for some } (x,y)\in E(a,d), \tag{2}$$

$$[(a',b')]_\xi \odot [(c',d')]_\xi = [(c'*y', b'*x')]_\xi \quad \text{for some } (x',y')\in E(a',d'). \tag{3}$$

Since $((a,b),(a',b')),((c,d),(c',d')) \in \xi$, we have $E(a,a') = E(b,b')$, $E(c,c') = E(d,d')$. Also, $a*x = d*y$, $a'*x' = d'*y'$. Let $(\alpha,\beta)\in E(b*x, b'*x')$. Then

$$\begin{gathered} b*x*\alpha = b'*x'*\beta \Longrightarrow (x*\alpha, x'*\beta)\in E(b,b') = E(a,a') \\ \Longrightarrow (a*x)*\alpha = (a'*x')*\beta \Longrightarrow (d*y)*\alpha = (d'*y')*\beta \Longrightarrow \\ (y*\alpha, y'*\beta)\in E(d,d') = E(c,c') \Longrightarrow (c*y)*\alpha = (c'*y')*\beta. \end{gathered}$$

Thus $(\alpha,\beta)\in E(c*y, c'*y')$ and $E(b*x,b'*x')\cap E(c*y,c'*y')\neq\emptyset$; hence $E(b*x,b'*x') = E(c*y,c'*y')$. This means that $((b*x,c*y),(b'*x',c'*y')\in\xi$. Then (2) and (3) and Lemma 5(2) imply that $[(a,b)]_\xi\odot[(c,d)]_\xi = [(a',b')]_\xi\odot[(c',d')]_\xi$. Therefore, $\odot$ is well defined.

It remains to show that $\odot$ is associative. So, let $[(a,b)]_\xi, [(c,d)]_\xi, [(e,f)]_\xi \in (S\times S)/\xi$. Then

$$\begin{aligned} &([(a,b)]_\xi \odot [(c,d)]_\xi)\odot[(e,f)]_\xi = [(c*y,b*x)]_\xi \odot [(e,f)]_\xi \\ &= [(e*z,(b*x)*t)]_\xi \quad \text{for some } (x,y)\in E(a,d),(t,z)\in E(c*y,f), \end{aligned} \tag{4}$$

$$\begin{aligned} &[(a,b)]_\xi \odot ([(c,d)]_\xi \odot [(e,f)]_\xi) = [(a,b)]_\xi \odot [(e*y',d*x')]_\xi \\ &= [((e*y')*z',(b*t')]_\xi \quad \text{for some } (x',y')\in E(c,f),(t',z')\in E(a,d*x'). \end{aligned} \tag{5}$$

Let $(\alpha,\beta)\in E(e*z,(e*y')*z')$. Then

$$\begin{aligned} (e*z)*\alpha = ((e*y')*z')*\beta &\Longrightarrow z*\alpha = (y'*z')*\beta \\ &\Longrightarrow (f*z)*\alpha = (f*y')*(z'*\beta) \\ &\Longrightarrow c*y*t*\alpha = c*x'*z'*\beta \\ &\Longrightarrow y*t*\alpha = x'*z'*\beta \\ &\Longrightarrow d*y*t*\alpha = d*x'*z'*\beta \\ &\Longrightarrow a*x*t*\alpha = a*t'*\beta \\ &\Longrightarrow x*t*\alpha = t'*\beta \\ &\Longrightarrow (b*x*t)*\alpha = (b*t')*\beta. \end{aligned}$$

This concludes that $(\alpha,\beta)\in E(b*x*t,b*t')$ and $E(e*z,(e*y')*z')\cap E((b*x)*t,b*t')\neq\emptyset$. Hence $E(e*z,(e*y')*z')=E((b*x)*t,b*t')$ and thus $((e*z,(b*x)*t),((e*y')*z',b*t'))\in\xi$. The combination of the last statement with (4) and (5) implies that $\odot$ is associative. $\square$

Theorem 7. *Let $(S,*)$ be a left reversible cancellative semigroup. Then $((S\times S)/\xi,\odot)$ is a group.*

Proof. By Lemma 6, $\odot$ is an associative binary operation on $(S\times S)/\xi$. It remains to show that $(S\times S)/\xi$ contains an identity element with respect to the binary operation $\odot$ and that for each element in $(S\times S)/\xi$ there exists an inverse element in $(S\times S)/\xi$ with respect to $\odot$.

For any $a\in S$, $[(a,a)]_\xi$ is the identity of $(S\times S)/\xi$ with respect to $\odot$. Indeed, if $[(c,d)]_\xi\in(S\times S)/\xi$ then

$$[(a,a)]_\xi\odot[(c,d)]_\xi=[(c*y,a*x)]_\xi\quad\text{for some } (x,y)\in E(a,d).$$

Take $(\alpha,\beta)\in E(a*x,d)$, then

$$\begin{aligned}(a*x)*\alpha=d*\beta\Rightarrow(d*y)*\alpha=d*\beta\Rightarrow y*\alpha=\beta\Rightarrow c*(y*\alpha)=c*\beta\\ \Rightarrow(\alpha,\beta)\in E(c*y,c)\Rightarrow E(a*x,d)\cap E(c*y,c)\neq\emptyset,\end{aligned}$$

hence $E(a*x,d)=E(c*y,c)$. Thus $((c*y,a*x),(c,d))\in\xi$ and $[(c*y,a*x)]_\xi=[(c,d)]_\xi$. Similarly $[(c,d)]_\xi\odot[(a,a)]_\xi=[(c,d)]_\xi$.

To show that each element of $(S\times S)/\xi$ has an inverse in $(S\times S)/\xi$, let $[(a,b)]_\xi\in(S\times S)/\xi$. Then $[(b,a)]_\xi\in(S\times S)/\xi$ and

$$[(a,b)]_\xi\odot[(b,a)]_\xi=[(b*y,b*x)]_\xi\quad\text{for some } (x,y)\in E(a,a).$$

Since $a*x=a*y$ and $(S,*)$ is cancellative, we have $x=y$ and

$$[(a,b)]_\xi\odot[(b,a)]_\xi=[(b*x,b*x)]_\xi=[(a,a)]_\xi\quad\text{(see Lemma 3(4))}.$$

Similarly, if $(x,y)\in E(b,b)$ then $x=y$ and

$$[(b,a)]_\xi\odot[(a,b)]_\xi=[(a*y,a*x)]_\xi=[(a*x,a*y)]_\xi=[(a,a)]_\xi\quad\text{(see Lemma 3(4))}.$$

This yields $[(a,b)]_\xi^{-1}=[(b,a)]_\xi$. $\square$

Since every R-semicommutative semigroup is left reversible, we obtain

Corollary 8. *Let $(S,*)$ be an R-semicommutative cancellative semigroup. Then $((S\times S)/\xi,\odot)$ is a group.*

Let $(S,*,\leqslant_S)$ be a left reversible cancellative ordered semigroup. As in [7], we define a relation "$\preceq$" on $(S\times S)/\xi$ as follows:

$$([(a,b)]_\xi,[(c,d)]_\xi)\in\preceq\iff\exists(x,y)\in[(a,b)]_\xi,(z,t)\in[(c,d)]_\xi;\\ (\forall(k,h)\in E(y,t)\colon\ x*k\leqslant_S z*h).$$

The following lemma and its proof is a modification of equivalent statements proved in [7].

Lemma 9. *Let $(S,*,\leqslant_S)$ be a left reversible cancellative ordered semigroup. Then*

1) $([(a,b)]_\xi,[(c,d)]_\xi)\in\preceq\iff\forall(x,y)\in[(a,b)]_\xi,\ \forall(z,t)\in[(c,d)]_\xi,$ $(\forall(k,h)\in E(y,t)\colon\ x*k\leqslant_S z*h)$;
2) $([(a,b)]_\xi,[(c,d)]_\xi)\in\preceq\iff\exists(x,y)\in[(a,b)]_\xi,\exists(z,t)\in[(c,d)]_\xi,$ $\exists(k,h)\in E(y,t)\colon\ (x*k\leqslant_S z*h)$.

Proof. $(\Longleftarrow)$ immediate.

$(\Longrightarrow)$ Let $([(a,b)]_\xi,[(c,d)]_\xi)\in\preceq$, $(x,y)\in[(a,b)]_\xi$, $(z,t)\in[(c,d)]_\xi$ and $(k,h)\in E(y,t)$. Since $([(a,b)]_\xi,[(c,d)]_\xi)\in\preceq$ there exists, by definition of $\preceq$, $(x_1,y_1)\in([(a,b)]_\xi$, $(z_1,t_1)\in[(c,d)]_\xi)$ such that

$$\forall(u,v)\in E(y_1,t_1)\colon\ x_1*u\leqslant_S z_1*v. \tag{6}$$

Since $(k,h)\in E(y,t)$, we have $y*k=t*h$. Since $((x,y),(x_1,y_1))\in\xi$, $E(x,x_1)=E(y,y_1)$. Similarly, $E(z,z_1)=E(t,t_1)$. Let $(s,w)\in E(x_1,x),(e,f)\in E(z_1,z)$. Then $x*w=x_1*s$, $y*w=y_1*s$, $z*f=z_1*e$, $t*f=t_1*e$. Let $(\lambda,\varrho)\in E(w,k)$, $(\alpha,\beta)\in E(h*\varrho,f)$. Then $w*\lambda=k*\varrho$ and $h*\varrho*\alpha=f*\beta$. Hence,

$$(y_1*s)*\lambda*\alpha=(y*w)*\lambda*\alpha=y*(k*\varrho)*\alpha=(t*h)*\varrho*\alpha\\ =t*(h*\varrho*\alpha)=t*(f*\beta)=(t*f)*\beta=(t_1*e)*\beta.$$

This shows that $(s*\lambda*\alpha,e*\beta)\in E(y_1,t_1)$ and by (6) we have

$$x_1*(s*\lambda*\alpha)\leqslant_S z_1*(e*\beta)=(z*f)*\beta=z*(h*\varrho*\alpha).$$

On the other hand,

$$(x_1*s)*\lambda*\alpha=(x*w)*\lambda*\alpha=x*(w*\lambda)*\alpha=x*(k*\varrho)*\alpha.$$

Therefore, $x * k * \varrho * \alpha \leqslant_S z * h * \varrho * \alpha$. Since $(S, *, \leqslant_S)$ is cancellative, we have $x * k \leqslant_S z * h$. Since $(x, y), (z, t)$ and (k, h) were arbitrarily chosen from $[(a, b)]_\xi$, $[(c, d)]_\xi$ and $E(y, t)$, respectively, we have

$$\forall (x, y) \in [(a, b)]_\xi, \forall (z, t) \in [(c, d)]_\xi\colon\ (\forall (k, h) \in E(y, t)\colon\ x * k \leqslant_S z * h).$$

2) $(\Longrightarrow)$ immediate.

$(\Longleftarrow)$ Let $(x, y) \in [(a, b)]_\xi, (z, t) \in [(c, d)]_\xi$ and let $(k, h) \in E(y, t)$ be such that $x * k \leqslant_S z * h$. Let (u, v) be arbitrarily chosen in $E(y, t)$ and $(\lambda, \varrho) \in E(k, u)$. Then

$$\begin{aligned}(t * h) * \lambda &= (y * k) * \lambda = y * (k * \lambda) = y * (u * \varrho)\\ &= (y * u) * \varrho = (t * v) * \varrho \Rightarrow h * \lambda = v * \varrho,\\ x * (u * \varrho) &= x * (k * \lambda) = (x * k) * \lambda \leqslant_S (z * h) * \lambda\\ &= z * (h * \lambda) = z * (v * \varrho) \Rightarrow x * u \leqslant_S z * v.\end{aligned}$$

□

Theorem 10. *Let $(S, *, \leqslant_S)$ be a left reversible cancellative ordered semigroup. Then $((S \times S)/\xi, \odot, \preceq)$ is an ordered group.*

P r o o f. Theorem 8 shows that $((S \times S)/\xi, \odot)$ is a group. It remains to show that it is ordered by $\preceq$. So let $[(a, b)]_\xi \in (S \times S)/\xi$. Then $(a, b) \in [(a, b)]_\xi$ and for any $(x, y) \in E(b, b)$ we have $x = y$ and thus $a * x \leqslant_S a * y$. It then follows, by the definition of $\preceq$, that $([(a, b)]_\xi, [(a, b)]_\xi) \in \preceq$. Therefore $\preceq$ is reflexive.

Let $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq$, $([(c, d)]_\xi, [(a, b)]_\xi) \in \preceq$. Then, by Lemma 9(1),

$$\forall (x, y) \in [(a, b)]_\xi, \forall (z, t) \in [(c, d)]_\xi,\ \forall (k, h) \in E(y, t)\colon\ x * k \leqslant_S z * h.$$

Let $(k, h) \in E(b, d)$. Since $(a, b) \in [(a, b)]_\xi$ and $(c, d) \in [(c, d)]_\xi$, we have $a * k \leqslant_S c * h$. Also, $(h, k) \in E(d, b)$ and thus $c * h \leqslant_S a * k$, which implies that $a * k = c * h$. Hence, we have $(k, h) \in E(b, d) \cap E(a, c)$ and $E(a, c) \cap E(b, d) \neq \emptyset$. It means that $E(a, c) = E(b, d)$ and $[(a, b)]_\xi = ([c, d)]_\xi$. Therefore $\preceq$ is anti-symmetric.

Let $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq, ([(c, d)]_\xi, [(e, f)]_\xi) \in \preceq, (k, h) \in E(b, d)$ and $(u, v) \in E(d, f)$. Since $(a, b) \in [(a, b)]_\xi, (c, d) \in [(c, d)]_\xi$, it follows by Lemma 9(2) that $a * k \leqslant_S c * h$. Similarly, since $(c, d) \in [(c, d)]_\xi, (e, f) \in [(e, f)]_\xi$, it follows by Lemma 9(2) that $c * u \leqslant_S e * v$. Let $(\lambda, \varrho) \in E(h, u)$. Then

$$\begin{aligned}(b * k) * \lambda &= (d * h) * \lambda = d * (h * \lambda) = d * (u * \varrho) = (d * u) * \varrho = (f * v) * \varrho\\ &\Rightarrow (k * \lambda, v * \varrho) \in E(b, f).\end{aligned}$$

Also, since $a * k \leqslant_S c * h$ and $c * u \leqslant_S e * v$, we have

$$(a * k) * \lambda \leqslant_S (c * h) * \lambda = c * (h * \lambda) = c * (u * \varrho) = (c * u) * \varrho \leqslant_S (e * v) * \varrho.$$

Thus, we have $(a, b) \in [(a, b)]_\xi, (e, f) \in [(e, f)]_\xi$ and $(k * \lambda, v * \varrho) \in E(b, f)$ such that $a * (k * \lambda) \leqslant_S e * (v * \varrho)$. Hence by Lemma 9(2) $([a, b]_\xi, [(e, f)]_\xi) \in \preceq$. Therefore $\preceq$ is transitive.

We have shown that the relation $\preceq$ is a partial order on the set $(S \times S)/\xi$.

Let $[(a, b)]_\xi \preceq [(c, d)]_\xi, [(e, f)]_\xi \in (S \times S)/\xi$ and $(k, h) \in E(b, d)$. Then $a * k \leqslant_S c * h$ and

$$[(a, b)]_\xi \odot [(e, f)]_\xi = [(e * v, b * u)]_\xi \quad \text{for some } (u, v) \in E(a, f);$$

$$[(c, d)]_\xi \odot [(e, f)]_\xi = [(e * v', d * u')]_\xi \quad \text{for some } (u', v') \in E(c, f).$$

Remember that $[(e * v, b * u)]_\xi \preceq [(e * v', d * u')]_\xi$ if and only if there exist $(l, m) \in [(e * v, b * u)]_\xi, (n, r) \in [(e * v', d * u')]_\xi$ and $(p, q) \in E(m, r)$ such that $l * p \leqslant_S n * q$. This can be achieved as follows:

Let $(x, y) \in E(k, u), (z, t) \in E(h * x, u')$. Then

$$\begin{aligned} b * (u * y) * z &= b * (k * x) * z = (b * k) * (x * z) = (d * h) * (x * z) \\ &= d * (h * x * z) = d * (u' * t) \Rightarrow (y * z, t) \in E(b * u, d * u'). \end{aligned}$$

Also,

$$\begin{aligned} (f * v) * (y * z) &= (a * u) * (y * z) = a * (u * y) * z = a * (k * x) * z \\ &= (a * k) * (x * z) \leqslant_S (c * h) * (x * z) = c * (h * x * z) \\ &= c * (u' * t) = (c * u') * t = (f * v') * t \Rightarrow v * y * z \leqslant_S v' * t \\ &\Rightarrow e * v * y * z \leqslant_S e * v' * t. \end{aligned}$$

Take $(l, m) = (e * v, b * u) \in [(e * v, b * u)]_\xi$, $(n, r) = (e * v', d * u') \in [(e * v', d * u')]_\xi$ and $(p, q) = (y * z, t) \in E(b * u, d * u')$. Then $l * p \leqslant_S n * q$ and, by Lemma 9(2), we have

$$[(e * v, b * u)]_\xi \preceq [(e * v', d * u')]_\xi.$$

Hence, $[(a, b)]_\xi \odot [(e, f)]_\xi \preceq [(c, d)]_\xi \odot [(e, f)]_\xi$.

Similarly, $[(e, f)]_\xi \odot [(a, b)]_\xi \preceq [(e, f)]_\xi \odot [(c, d)]_\xi$. □

3. Embedding theorems

The following theorem is an immediate consequence of the main theorem of [7].

Theorem 11. *Any right reversible ordered L-maher semigroup is embeddable in an ordered group if and only if it is cancellative.*

The dual of the above theorem, for the left reversible semigroups, is still true as stated and proved in the next theorem.

Theorem 12. *Let $(S, *, \leqslant_S)$ be a left reversible ordered R-maher semigroup. Then $(S, *, \leqslant_S)$ is embeddable in an ordered group if and only if it is cancellative.*

Proof. ($\Longleftarrow$) Assume that $(S, *, \leqslant_S)$ is cancellative. Define $f\colon S \to (S \times S)/\varrho$ as follows: $\forall x \in S$: $f(x) = [(x^2, x)]_\xi$. To show that f is a homomorphism, let $x, y \in S$. Then

$$f(x) \odot f(y) = [(x^2, x)]_\xi \odot [(y^2, y)]_\xi = [(y^2 * t, x * z)]_\xi \quad \text{for some } (z, t) \in E(x^2, y).$$

Let $u \in S$. Then $(u, y * u) \in E(x * y, x)$ and $(u, y * u) \in E((x * y)^2, (x * y * x))$.

Hence, $E(x*y, x) \cap E((x*y)^2, x*y*x) \neq \emptyset$ and we have $((x*y*x, x), ((x*y)^2, x*y)) \in \xi$. So

$$f(x * y) = [(x * y)^2, (x * y)]_\xi = [(x * y * x, x)]_\xi.$$

Also, since $x^2 * z = y * t$, we have $y * x^2 * z = y^2 * t$. Hence,

$$f(x) \odot f(y) = [(y^2 * t, x * z)]_\xi = [(y * x^2 * z, x * z)]_\xi.$$

Similarly, if $v \in S$ then $(v, z * v) \in E(x * z, x) \cap E(y * x^2 * z, y * x^2)$. Hence, we have $((y * x^2 * z, x * z), (y * x^2, x)) \in \xi$ and

$$f(x) \odot f(y) = [(y * x^2 * z, x * z)]_\xi = [(y * x^2, x)]_\xi.$$

Since $x^2 * z = y * t$, we have $(x * z, t) \in E(x, y)$ and since $(y * x^2) * (x * z) = (y * x) * (x^2 * z) = (y * x) * (y * t)$, we have $(x * z, t) \in E(y * x^2, y * x * y)$. Thus, we have $(x * z, t) \in E(x, y) \cap E(y * x^2, y * x * y) \neq \emptyset$. Therefore, $((y * x^2, x), (y * x * y, y)) \in \xi$ and

$$f(x) \odot f(y) = [(y * x^2, x)]_\xi = [(y * x * y, y)]_\xi.$$

Since $(S, *)$ is R-maher and $(x*z, t) \in E(x, y)$, we have $(x*z, t) \in E(x*y*x, y*x*y)$. This implies that $((x * y * x, x), (y * x * y, y)) \in \xi$ and

$$f(x) \odot f(y) = [(y^2 * t, x * z)]_\xi = [(y * x * y, y)]_\xi = [(x * y * x, x)]_\xi = f(x * y).$$

It remains to show that f is isotone and reverse isotone. So, let $a, b, x, y \in S$ be such that $a \leqslant_S b$ and $f(x) \preceq f(y)$. We will show that $f(a) \preceq f(b)$ and $x \leqslant_S y$. We have

$$f(a) = [(a^2, a)]_\xi, \qquad f(b) = [(b^2, b)]_\xi.$$

Let $(k,h) \in E(a,b)$. Then

$$a * k = b * h \Longrightarrow a^2 * k = a * b * h \leqslant_S b^2 * h.$$

Thus, $(a^2,a) \in [(a^2,a)]_\xi, (b^2,b) \in [(b^2,b)]_\xi, (k,h) \in E(a,b)$ and $a^2*k \leqslant_S b^2*h$, which implies that $f(a) \preceq f(b)$. Therefore, f is isotone.

Since $f(x) \preceq f(y), [(x^2,x)]_\xi \preceq [(y^2,y)]_\xi$ and since $(x^2,x) \in [(x^2,x)]_\xi, (y^2,y) \in [(y^2,y)]_\xi$, there exists $(k,h) \in E(x,y)$ such that $x^2 * k \leqslant_S y^2 * h$. Hence, $x^2 * k = x * (x * k) = x * (y * h)$. Thus, $x * y * h \leqslant_S y^2 * h$. Since $(S,*,\leqslant_S)$ is cancellative, we have $x \leqslant_S y$ and f is reverse isotone.

$(\Longrightarrow)$ Assume that $(S,*,\leqslant_S)$ is embeddable in an ordered group $(G,\oslash,\leqslant_G)$. Then there exists a reverse isotone (thus isotone) homomorphism $f\colon (S,*,\leqslant_S) \longrightarrow (G,\oslash,\leqslant_G)$. Now assume that $a,b,c \in S$ are such that $a * c \leqslant_S b * c$. Then

$$\begin{aligned} &f(a * c) \leqslant_G f(b * c) \Rightarrow f(a) \oslash f(c) \leqslant_G f(b) \oslash f(c) \\ &\xRightarrow{(G,\oslash) \text{ is a group}} f(a) \leqslant_G f(b) \xRightarrow{f \text{ reverse isotone}} a \leqslant_S b. \end{aligned}$$

Similarly, $c * a \leqslant_S c * b$ implies that $a \leqslant_S b$. □

Since every ordered R-semicommutative semigroup is a left reversible R-maher ordered semigroup, we get

Corollary 13. *An R-semicommutative ordered semigroup is embeddable in an ordered group if and only if it is cancellative.*

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