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REFLEXIVITY OF INDUCTIVE LIMITS

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Abstract. An inductive locally convex limit of reflexive topological spaces is reflexive iff it is almost regular.

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Throughout the paper $E_1 \subset E_2 \subset \ldots$ is a sequence of locally convex spaces with identity maps $\mathrm{id}_n\colon E_n \to E_{n+1}$, $n \in \mathbb{N}$, continuous and $E = \operatorname{ind} E_n$ its locally convex inductive limit. The strong dual of E_n, resp. E, is denoted by E'_n, resp. E'. For a set $A \subset E$, its closure in E, resp. its polar, is denoted by $\mathrm{cl}_E\, A$, resp. A°. If $A \subset E_n$, then its polar in E'_n is denoted by $A^{\circ n}$.

Definition. An inductive limit $\operatorname{ind} E_n$ is called regular, resp. almost regular, if each set B, bounded in $\operatorname{ind} E_n$ is also bounded in some constituent space E_n, resp. there exists a set A, bounded in some E_n, such that $B \subset \mathrm{cl}_E\, A$.

For $m, n \in \mathbb{N}$, $1 \leqslant m \leqslant n$, we denote by r_n, resp. $r_{m,n}$, the mapping which associates with each $f \in E'$, resp. $f_n \in E'_n$, its restriction to the subspace E_n, resp. E_m. Clearly, $r_m = r_{m,n} \circ r_n$, the projective limit $F = \operatorname{proj}(E'_n, r_n)$ makes sense, and the linear spaces underlying F and E' are the same.

Lemma 1. *Let* $\operatorname{ind} E_n$ *be almost regular. Then the projective topology* $\operatorname{top} F$ *and the strong topology* $\beta(E', E)$ *are the same.*

Proof. Since each map $r_n\colon E' \to E'_n$, $n \in \mathbb{N}$, is continuous and the projective topology of F is the coarsest topology on the linear space underlying E', for which all these maps are continuous, we have $\operatorname{top} F \subset \beta(E', E)$. Hence it is sufficient to show that for any set B, bounded in E, its polar B° is contained in $\operatorname{top} F$.

Since $\operatorname{ind} E_n$ is almost regular, there exists a set A, bounded in some E_n, such that $B \subset \operatorname{cl}_E A$. Then $A^{\circ n} \in \operatorname{top} E'_n$ and $r_n^{-1}(A^{\circ n}) \in \operatorname{top} F$. Further,

$$\begin{aligned} r_n^{-1}(A^{\circ n}) &= \{f \in E';\ |r_n f(x)| \leqslant 1 \text{ for } x \in A\} \\ &= \{f \in E';\ |f(x)| \leqslant 1 \text{ for } x \in \operatorname{cl}_E A\} \\ &\subset \{f \in E';\ |f(x)| \leqslant 1 \text{ for } x \in B\} = B^\circ \end{aligned}$$

Hence, $B^\circ \in \operatorname{top} F$. □

Lemma 2. *Assume* $\operatorname{top} F \supset \beta(E', E)$. *Then* $\operatorname{ind} E_n$ *is almost regular.*

P r o o f. Take a set B bounded in $\operatorname{ind} E_n$. Then $B^\circ \in \beta(E', E) \subset \operatorname{top} F$ and there exists a family $\{B_m;\ 1 \leqslant m \leqslant n\}$, where each set B_m is bounded in E_m, such that $\bigcap\{r_m^{-1}(B_m^{\circ m});\ 1 \leqslant m \leqslant n\} \subset B^\circ$.

Each set B_m, $m \leqslant n$, is also contained and bounded in E_n. Denote by A the balanced convex hull of $\bigcup\{B_m;\ 1 \leqslant m \leqslant n\}$. Then A is bounded in E_n and for each $m \in \mathbb{N}$, $1 \leqslant m \leqslant n$, we have

$$\begin{aligned} A^\circ &= \{f \in E';\ |f(x)| \leqslant 1 \text{ for } x \in A\} \\ &\subset \{f \in E';\ |f(x)| \leqslant 1 \text{ for } x \in B_m\} \subset r_m^{-1} B_m^{\circ m}. \end{aligned}$$

Hence $A^\circ \subset \bigcap\{r_m^{-1} B_m^{\circ m};\ 1 \leqslant m \leqslant n\} \subset B^\circ$. This implies $B^{\circ\circ} \subset A^{\circ\circ}$ and $B \subset B^{\circ\circ} \subset A^{\circ\circ} = \operatorname{cl}_E A$, i.e. $\operatorname{ind} E_n$ is almost regular.

In the following, let $\{F_n, n \in \mathbb{N}\}$ be a family of locally convex spaces and $\{p_{m,n};\ 1 \leqslant m \leqslant n\}$ a family of linear continuous mappings $p_{m,n}\colon F_n \to F_m$ such that for any $k, m, n \in \mathbb{N}$, $k \leqslant m \leqslant n$, we have $p_{k,n} = p_{k,m} \circ p_{m,n}$. Moreover, let L be a linear space and $p_n\colon L \to F_n$, $n \in \mathbb{N}$, be a linear injective mapping such that $p_m = p_{m,n} \circ p_n$ for any $m \leqslant n$. Then the projective limit $\operatorname{proj}(F_n, p_n)$ exists. Denote it by F. Finally, let F'_n, $n \in \mathbb{N}$, resp. F', be the strong dual of F_n, resp. of F. Then, each mapping $i_n\colon f \mapsto f \circ p_n\colon F'_n \to F'$, $n \in \mathbb{N}$, is linear and the inductive limit $G = \operatorname{ind}(F'_n i_n)$ makes sense. □

Lemma 3. *The linear spaces underlying* F' *and* G *are the same.*

P r o o f. The vector space underlying G is the linear hull of the union $\bigcup\{i_n F'_n;\ n \in \mathbb{N}\}$ and $i_n F'_n \subset F'$, $n \in \mathbb{N}$. Hence $G \subset F'$. Take $f \in F'$. Then $U = f^{-1}(-1, 1) \in \operatorname{top} F$ and there exists a family $\{U_m \in \operatorname{top} F_m;\ 1 \leqslant m \leqslant n\}$ such that $\bigcap\{p_m^{-1} U_m;\ 1 \leqslant m \leqslant n\} \subset U$. Further, $V_m = p_{m,n}^{-1} U_m \in \operatorname{top} F_n$ for $1 \leqslant m \leqslant n$. Put $V = \bigcap\{V_m;\ 1 \leqslant m \leqslant n\}$ and denote by M the linear hull of V, equipped with the topology of F_n. Then the linear mapping $f \circ p_n^{-1}\colon M \to \mathbb{R}$ is majorized by the

Minkowski functional $\varphi\colon F_n \to \mathbb{R}$ of V. Since φ is a continuous seminorm on F_n, the mapping $f \circ p_n^{-1}\colon M \to \mathbb{R}$ has a continuous extension $g\colon F_n \to \mathbb{R}$. Then for $x \in F$, we have $f(x) = (f \circ p_n^{-1} \circ p_n)(x) = (f \circ p_n^{-1})(p_n x) = g(p_n x) = (g \circ p_n)(x) = (i_n g)(x)$ and $f = i_n g \in i_n F_n' \subset G$. $\square$

Lemma 4. *Each mapping* $i_n\colon F_n' \to F'$, $n \in \mathbb{N}$, *is continuous.*

Proof. Take $U \in \operatorname{top} F'$. Then there exists a set B, bounded in F such that its polar $B^\circ \subset U$. The set $p_n B$ is bounded in F_n. Hence for its polar $(p_n B)^{\circ n} \subset F_n'$ we have $(p_n B)^{\circ n} = \{f \in F_n';\ |f(x)| \leqslant 1 \text{ for } x \in p_n B\} \in \operatorname{top} F_n'$.

For $f \in (p_n B)^{\circ n}$ and $x \in B$, we have $|(i_n f)(x)| = |(f \circ p_n)(x)| = |f(p_n(x))| \leqslant 1$. This implies $i_n f \in B^\circ$ and $i_n(p_n B)^{\circ n} \subset B^\circ \subset U$. $\square$

Lemma 5. $\operatorname{top} G = \operatorname{top} F'$.

Poof. Since the topology of the inductive limit G is the finest one for which all mappings $i_n\colon F_n' \to F'$, $n \in \mathbb{N}$, are continuous, we have $\operatorname{top} G \supset \operatorname{top} F'$. To prove the other inclusion, take a closed, balanced, and convex neighborhood $U \in \operatorname{top} G$.

For each $n \in \mathbb{N}$, we have $i_n^{-1} U \in \operatorname{top} F_n'$ hence there exists a balanced convex set $B_n \subset F_n$, bounded in F_n, such that $B_n^{\circ n} = \{f \in F_n';\ |f(x)| \leqslant 1 \text{ for } x \in B_n\} \subset i_n^{-1} U$. The set $B = \bigcap\{p_n^{-1} B_n;\ n \in \mathbb{N}\}$ is balanced, convex, and bounded in $F = \operatorname{proj}(F_n, p_n)$. The polar B° is the F'-closure of the convex hull of the union $\bigcup\{(p_n^{-1} B_n)^\circ;\ n \in \mathbb{N}\}$. Further, $(p_n^{-1} B_n)^\circ = \{f \in F';\ |f(x)| \leqslant 1 \text{ for } x \in p_n^{-1} B_n\} = \{f \in F';\ |(f \circ p_n^{-1})(y)| \leqslant 1, y \in B_n\} = i_n B_n^{\circ n} \subset U$.

Hence we have $B^\circ \subset U$, where $B^\circ \in \operatorname{top} F'$. $\square$

Theorem. *Let* $E_1 \subset E_2 \subset \ldots$ *be a sequence of reflexive locally convex spaces with identity maps* $\mathrm{id}_n\colon E_n \to E_{n+1}$, $n \in \mathbb{N}$, *continuous. Then its locally convex inductive limit* $\operatorname{ind} E_n$ *is reflexive iff it is almost regular.*

Proof. It follows from Lemmas 1–5 that almost regularity of $\operatorname{ind} E_n$ implies its reflexivity.

Assume $\operatorname{ind} E_n$ to be reflexive and that the spaces F_n, resp. mappings p_n, $n \in \mathbb{N}$, from Lemmas 3–5 are the same as the duals E_n', resp. mappings r_n, from Lemmas 1, 2.

Take a bounded set $B \subset E = \operatorname{ind} E_n$. We have to construct a set A, bounded in some E_n, such that $B \subset \operatorname{cl}_E A$. By Lemma 5, we have $E = F'$ and the set B is also bounded in F'. Hence, $B^\circ = \{f \in F;\ |f(x)| \leqslant 1 \text{ for } x \in B\} \in \operatorname{top} F$ and there exists a closed balanced convex $U \in \operatorname{top} E_n'$ such that $r_n^{-1} \subset B^\circ$.

The balanced convex set $A = \{x \in E_n;\ |f(x)| \leqslant 1 \text{ for } x \in U\}$ is weakly bounded in E_n. Hence it is also bounded in the topology of E_n. Since $A \subset E_n$, we have

$$\begin{aligned} A^\circ &= \{f \in E';\ |f(x)| \leqslant 1 \text{ for } x \in A\} \\ &\subset \{r_n^{-1} g \in E';\ g \in E'_n, |g(x)| \leqslant 1 \text{ for } x \in A\} = r_n^{-1} A^{\circ n}. \end{aligned}$$

Take $f \in A^{\circ n} \subset E'_n$, $f \neq 0$. There exists $\alpha > 0$ and $g \in U$ such that $f = \alpha g$. Let $\beta = \sup\{\lambda > 0, \lambda g \in U\}$. Then $g \neq 0$ implies $\beta \neq +\infty$ and we can put $h = \beta g$. Since the set U is closed convex, and balanced, we have $h \in U$.

Let $\lambda = \alpha\beta^{-1}$ and $\varepsilon \in (0,1)$. The choice of β implies existence of $x_\varepsilon \in A$ for which $|h(x_\varepsilon)| > 1 - \varepsilon$. Then $1 \geqslant |f(x_\varepsilon)| = |\lambda h(x_\varepsilon)| > \lambda(1-\varepsilon)$. Thus $\lambda \leqslant 1 = \inf\{(1-\varepsilon)^{-1};\ \varepsilon \in (0,1)\}$ and $f \in U$. So far, we have $A^\circ \subset r_n^{-1} A^{\circ n} \subset r_n^{-1} U \subset B^\circ$. This implies $B \subset B^{\circ\circ} \subset A^{\circ\circ} = \operatorname{cl}_E A$. $\square$

References

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