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# OPTIMIZATION AND IDENTIFICATION OF NONLINEAR UNCERTAIN SYSTEMS

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*Abstract.* In this paper we consider the optimal control of both operators and parameters for uncertain systems. For the optimal control and identification problem, we show existence of an optimal solution and present necessary conditions of optimality.

*Keywords*: optimal control, Galerkin method, nonlinear systems, identification problem, necessary condition

*MSC 2000*: 49J20, 49K20, 49K24

## 1. INTRODUCTION

Many physical systems arising from thermodynamics, electrodynamics, and population biology are modelled by differential equations, integrodifferential equations, and nonlinear evolution equations with uncertain parameters or undetermined operators.

In this paper we consider differential equations on Banach spaces as follows:

$$\begin{cases} \dot{x} + A(t,x) = g(t,x), \\ x(0) = x_0, \end{cases}$$

where $A$ is a nonlinear monotone operator from a Banach space $V$ into its dual $V^*$ and $g(t,x)$ is a nonlinear but not necessarily monotone operator. An associated

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control system may be described as

$$\text{(CP)} \qquad \begin{cases} \dot{x} + A(t, x, \lambda) + Bx = g(t, x, \lambda), \\ x(0; \lambda) = x_0(\lambda), \ \ \lambda \in Q_m, \ \ B \in \mathscr{P}_{a,b}, \end{cases}$$

where $Q_m$ is a compact metric space and $\mathscr{P}_{a,b}$ is a suitable subset of $\mathscr{L}(V, V^*)$. Define the cost functional $J(\cdot, \cdot)$ by the form

$$J(\lambda, B) = \int_I f(t, x(\lambda, B)(t), \lambda) \, \mathrm{d}t,$$

where $I = [0, T]$, $T < \infty$ and $x(\lambda, B)$ is a solution function of (CP). The problem is to find $(\lambda^0, B^0) \in Q_m \times \mathscr{P}_{a,b}$ (admissible set) so that

$$\text{(P)} \qquad J(\lambda^0, B^0) \leqslant J(\lambda, B) \ \text{ for all } \ (\lambda, B) \in Q_m \times \mathscr{P}_{a,b}.$$

In recent years optimal control and identification problems have been extensively studied by many authors (see [5], [6], [10], [12], [11] and the references therein) and more generally, functional differential inclusions have been studied by Ahmed and Papageorgious (see [1], [2], [7], [8] and the references therein). These studies were mainly concerned with the question of existence of optimal controls in the uncertain systems.

In this paper, we study the existence of the optimal solution for problem (CP) as well as the optimal pair for the identification problem (P). We also derive necessary conditions of optimality for the identification problem (P).

## 2. Existence and uniqueness of solutions

Let $H$ be a separable Hilbert space and let $V$ be a subspace of $H$ having the structure of a reflexive Banach space, with the embedding $V \hookrightarrow H$ being compact. Identifying $H$ with its dual, we have $V \hookrightarrow H \hookrightarrow V^*$, where $V^*$ is the topological dual of $V$. The system model considered here is based on this evolution triple.

Let $\langle x, y \rangle$ denote the pairing of an element $x \in V$ and an element $y \in V^*$. If $x, y \in H$, then $\langle x, y \rangle = (x, y)$. The norm in any Banach space $X$ will be denoted by $\| \cdot \|_X$.

Let $\{e_1, e_2, \ldots\}$ be a basis of $V$ and set

$$H_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}.$$

In the $n$-dimensional space $H_n$ we introduce the scalar product of Hilbert space $H$. Note that $H_n \subseteq V \subseteq H$.

Let $0 < t \leqslant T < \infty$, $I_t \equiv [0,t]$, $I \equiv [0,T]$, and let $p, q \geqslant 1$ be such that

$$1/p + 1/q = 1 \quad \text{and} \quad 2 \leqslant p < \infty.$$

For simple notation, we write $L_r^t(X) \equiv L_r(I_t, X)$, $L_r(X) \equiv L_r(I, X)$ for $r \geqslant 1$ and a set $X$. For $p$, $q$ satisfying the preceding conditions, it follows from the reflexivity of $V$ that both $L_p^t(V)$ and $L_q^t(V^*)$ are reflexive Banach spaces (see Theorem 1.1.17 of [3]). The pairing of $L_p^t(V)$ and $L_q^t(V^*)$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_t$. In particular, we use $\langle\!\langle \cdot, \cdot \rangle\!\rangle \equiv \langle\!\langle \cdot, \cdot \rangle\!\rangle_T$. Clearly, for $u, v \in L_2(H)$, $\langle\!\langle u, v \rangle\!\rangle = ((u, v))$, where $((\cdot, \cdot))$ is the scalar product in Hilbert space $L_2(H)$. Let $\dot{x} = \frac{\partial}{\partial t} x$. Define

$$W_{p,q} = \{x\colon\ x \in L_p(V),\ \ \dot{x} \in L_q(V^*)\},\ \ \|x\|_{W_{p,q}}^2 = \|x\|_{L_p(V)}^2 + \|\dot{x}\|_{L_q(V^*)}^2.$$

Then $\{W_{p,q}, \|\cdot\|_{W_{p,q}}\}$ is a Banach space and the embedding $W_{p,q} \hookrightarrow C(I, H)$ is continuous. If $V \hookrightarrow H$ is compact, then $W_{p,q} \hookrightarrow L_p(H)$ is compact (see Proposition 23.23 of [13]). Let $\mathscr{L}(X, Z)$ denote the space of all bounded linear operators from $X$ to $Z$ and $A^*$ the dual of the operator $A$. Let

$$\mathscr{P}_{a,b} = \{B \in \mathscr{L}(V, V^*)\colon\ \|B\|_{\mathscr{L}(V,V^*)} \leqslant b \ \text{ and } \ \langle B\xi, \xi\rangle + a\|\xi\|_H \geqslant 0, \ \text{ for all } \ \xi \in V\}.$$

Consider the space of operators $\mathscr{L}(V, V^*)$ and suppose that is equipped the strong (weak) operator topology which we denote by $\tau_{\mathrm{so}}$ ($\tau_{\mathrm{wo}}$). Given this topology, $\mathscr{L}_s(V, V^*) \equiv (\mathscr{L}(V, V^*), \tau_{\mathrm{so}})$ is a locally convex linear topological vector space which is sequentially complete. Similarly, $\mathscr{L}_w(V, V^*) \equiv (\mathscr{L}(V, V^*), \tau_{\mathrm{wo}})$ with the weak operator topology $\tau_{\mathrm{wo}}$ is also a sequentially complete and locally convex topological space. We shall suppose that $Q_m$ is algebraically contained in a linear topological vector space and that $Q_m$ is a convex and we will denote $Q_m$ a compact metric space with a metric $\tau_m$. We introduce the following assumptions:

H($A$) $A\colon\ I \times V \times Q_m \mapsto V^*$ is an operator.

(1) $t \mapsto A(t, x, \lambda)$ is measurable.

(2) $x \mapsto A(t, x, \lambda)$ is uniformly monotone and hemicontinuous; i.e., there exists a constant $c > 0$ such that

$$\begin{aligned} \langle A(t, x_1, \lambda) - A(t, x_2, \lambda), x_1 - x_2\rangle &\geqslant c\|x_1 - x_2\|_V^p, \\ &\forall\, x_1, x_2 \in V,\ \ t \in I,\ \ \lambda \in Q_m; \\ A(t, x + sy, \lambda) &\xrightarrow{w} A(t, x, \lambda) \in V^*, \\ &\forall\, x, y \in V,\ \ \lambda \in Q_m \ \text{ as } \ s \to 0. \end{aligned}$$

(3) There exist positive constants $c_1$ and $c_2$ such that

$$\langle A(t, x, \lambda), x\rangle \geqslant c_1\|x\|_V^p - c_2,\ \ \forall\, x \in V,\ \ t \in I,\ \ \lambda \in Q_m.$$

(4) There exist a positive constant $c_3$ and a function $c_4(t) \in L_q(I, \mathbb{R}_+)$ such that

$$\|A(t,x,\lambda)\|_{V^*} \leqslant c_4(t) + c_3\|x\|_V^{p-1}, \quad \forall\, x \in V, \ \ t \in I, \ \ \lambda \in Q_m.$$

(5) $\|A(t+\tau,x,\lambda) - A(t,x,\lambda)\|_{V^*} \leqslant O(\tau)(1 + \|x\|_V^{p-1})$, $\forall\, x \in V$, $\lambda \in Q_m$ and $O(\tau)$ is independent of $\lambda$ and $x$.

(6) $\lambda \mapsto A(t,x,\lambda)$ is continuous.

H($g$) $g\colon\ I \times H \times Q_m \mapsto V^*$ is a map.

(1) $t \mapsto g(t,\cdot,\cdot)$ is measurable.

(2) $x \mapsto g(\cdot,x,\cdot)$ is continuous.

(3) There exist $\alpha \geqslant 0$ and $h \in L_q(I, \mathbb{R}_+)$ such that

$$\|g(t,x,\lambda)\|_{V^*} \leqslant h(t) + \alpha\|x\|_H^{2/q}, \quad \forall\, x \in V, \ \ t \in I, \ \ \lambda \in Q_m.$$

(4) $\langle g(t,x,\lambda), x\rangle \leqslant 0$ a.e. $x \in H$.

(5) $g$ is locally Lipschitz continuous with respect to $x$ and for any $b > 0$ there exists $L(b)$ such that $x_1, x_2 \in H$, $\|x_1\|_H, \|x_2\|_H \leqslant b$, $\|g(t,x_1,\lambda) - g(t,x_2,\lambda)\|_{V^*} \leqslant L(b)\|x_1 - x_2\|_H$, $\forall\, t \in I$, $\lambda \in Q_m$.

(6) $\lambda \mapsto g(\cdot,\cdot,\lambda)$ is continuous.

H($\lambda$) $\lambda \mapsto x_0(\lambda)$ is continuous from $Q_m$ into $H$.

H($f$) $f\colon\ I \times H \times Q_m \mapsto \mathbb{R}_+$ is an integrable function.

(1) $(t,x,\lambda) \mapsto f(t,x,\lambda)$ is measurable.

(2) $x \mapsto f(\cdot,x,\cdot)$ is continuous; i.e., if $\lambda_n \to \lambda$ in $Q_m$, then $f(t,\cdot,\lambda_n) \to f(t,\cdot,\lambda)$ a.e.

Under the above assumptions we consider the following initial value problem:

$$\begin{cases} \dot{x}(t) + A(t,x,\lambda) + Bx(t) = g(t,x(t),\lambda), \\ x(0) = x_0(\lambda), \ \ \lambda \in Q_m, \ \ B \in \mathscr{P}_{a,b}. \end{cases} \tag{2.1}$$

For given $x_0(\lambda) \in H$, we seek a function $x \in W_{p,q}$ such that (2.1) is satisfied in a weak sense. For $x \in L_p(V)$, $\lambda \in Q_m$, we set

$$A(x,\lambda)(t) = A(t,x(t),\lambda), \quad G(x,\lambda)(t) = g(t,x(t),\lambda), \ \ t \in I.$$

Note that $A\colon\ L_p(V) \times Q_m \mapsto L_q(V^*)$ is bounded, uniformly monotone, hemicontinuous and coercive and also the operator $G\colon\ L_p(V) \times Q_m \mapsto L_q(V^*)$ is bounded.

The purpose of this section is to prove the existence and uniqueness of solution for equation (2.1) based on Galerkin approximation.

Let $\lambda \in Q_m$ be an arbitrary fixed parameter. We get the following lemma.

**Lemma 2.1.** *If $x^n \to x^0$ weakly in $W_{p,q}$, then $G(x^n,\lambda) \to G(x^0,\lambda)$ in $L_q(V^*)$.*

P r o o f. Since the embedding $V \hookrightarrow H$ is compact, the embedding $W_{p,q} \hookrightarrow L_p(H)$ is compact as well. Since $x^n \xrightarrow{w} x^0 \in W_{p,q}$, there exists a constant $b > 0$ such that $\|x^0\|_{C(I,H)} \leqslant b$, $\|x^n\|_{C(I,H)} \leqslant b$. By virtue of assumption H($g$) and the embedding $L_p(H) \hookrightarrow L_q(H) \hookrightarrow L_q(V^*)$, we have

$$\begin{aligned}\|G(x^n,\lambda) - G(x^0,\lambda)\|_{L_q(V^*)} &= \Big(\int_I \|g(t,x^n(t),\lambda) - g(t,x^0(t),\lambda)\|_{V^*}^q \,\mathrm{d}t\Big)^{1/q}\\ &\leqslant L(b)\Big(\int_I \|x^n(t) - x^0(t)\|_H^q \,\mathrm{d}t\Big)^{1/q}\\ &\leqslant L^*\Big(\int_I \|x^n(t) - x^0(t)\|_H^p \,\mathrm{d}t\Big)^{1/p},\end{aligned}$$

where $L^*$ is a constant depending on $p$, $q$, $b$ and the Lebesgue measure of $I$. Hence the conclusion follows. □

**Remark.** It is convenient to write system (2.1) as an operator equation in

$$(2.2)\quad W_{p,q}^0(\lambda) \equiv \{x \in W_{p,q};\ x(0) = x_0(\lambda)\}\colon \begin{cases} \dot{x} + A(x,\lambda) + Bx = G(x,\lambda),\\ x \in W_{p,q}^0(\lambda),\ B \in \mathscr{P}_{a,b},\ \lambda \in Q_m.\end{cases}$$

**Lemma 2.2.** *There exists $b > 0$ such that*

$$\|x\|_{C(I,H)} \leqslant b, \quad \|x\|_{L_p(V)} \leqslant b, \quad \|\dot{x}\|_{L_q(V^*)} \leqslant b$$

*for any solution $x$ (if one exists) of equation* (2.1).

P r o o f. If $x$ is any solution of (2.1), then for each $t \in I$,

$$\langle\!\langle \dot{x}, x\rangle\!\rangle_t + \langle\!\langle A(x,\lambda), x\rangle\!\rangle_t + \langle\!\langle Bx, x\rangle\!\rangle_t = \langle\!\langle G(x,\lambda), x\rangle\!\rangle_t.$$

Using the assumptions and the Cauchy inequality, for any $\varepsilon > 0$ we have

$$(2.3)\quad \begin{aligned}&\frac{1}{2}\Big(\|x(t)\|_H^2 - \|x(0)\|_H^2) + \int_0^t (c_1\|x(\sigma)\|_V^p - c_2\Big)\,\mathrm{d}\sigma\\ &\qquad\leqslant a\int_0^t \|x(\sigma)\|_H^2\,\mathrm{d}\sigma + \int_0^t \|g(\sigma,x(\sigma),\lambda)\|_{V^*}\|x(\sigma)\|_V\,\mathrm{d}\sigma.\end{aligned}$$

From (2.3), we have

$$\begin{aligned}\|x(t)\|_H^2 + 2c_1\int_0^t \|x(\sigma)\|_V^p\,\mathrm{d}\sigma &\leqslant 2c_2T + \|x(0)\|_H^2 + 2a\int_0^t \|x(\sigma)\|_H^2\,\mathrm{d}\sigma\\ &\quad + 2\int_0^t (h(\sigma) + \alpha\|x(\sigma)\|_H^{2/q})\|x(\sigma)\|_V\,\mathrm{d}\sigma.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\|x(t)\|_H^2 + 2c_1 \int_0^t \|x(\sigma)\|_V^p \,\mathrm{d}\sigma \leqslant 2c_2T + \|x(0)\|_H^2 + 2a \int_0^t \|x(\sigma)\|_H^2 \,\mathrm{d}\sigma \\ + (2/q\varepsilon^q) \int_0^t (h(\sigma) + \alpha \|x(\sigma)\|_H^{2/q})^q \,\mathrm{d}\sigma \\ + (2\varepsilon^p/p) \int_0^t \|x(\sigma)\|_V^p \,\mathrm{d}\sigma.\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small and $h \in L_q(I, \mathbb{R}^+)$, one can easily verify that there exist positive constants $c_5$, $c_6$, $c_7$ such that

$$\|x(t)\|_H^2 + c_5 \|x\|_{L_p^t(V)}^p \leqslant c_6 + c_7 \int_0^t \|x(\sigma)\|_H^2 \,\mathrm{d}\sigma. \tag{2.4}$$

It follows from Gronwall's lemma that the above inequality implies

$$\|x(t)\|_H \leqslant c_8 \quad \forall\, t \in I,$$

for some constant $c_8$ depending on $c_6$ and $c_7$. Again, by virtue of assumptions (3)–(4) of H($A$), (3) of H($g$), definition of $\mathscr{P}_{a,b}$ and inequality (2.4), it is easy to verify that there exist positive constants $c_9$, $c_{10}$ such that

$$\|x\|_{L_p(V)} \leqslant c_9, \quad \|\dot{x}\|_{L_q(V^*)} \leqslant c_{10}.$$

Choosing $b = \max\{c_8, c_9, c_{10}\}$, the assertion follows. □

**Lemma 2.3.** *The solution of* (2.1), *if one exists, is unique.*

P r o o f. Let $x_1, x_2 \in W_{p,q}^0(\lambda)$ be two solutions of (2.1). Using integration by parts and the monotonicity of the operator $A$ and the definition of $\mathscr{P}_{a,b}$, we obtain

$$\begin{aligned}\frac{1}{2}\|x_1(t) - x_2(t)\|_H^2 + c\|x_1 - x_2\|_{L_p^t(V)}^p \leqslant a \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma \\ + \int_0^t \langle g(\sigma, x_1(\sigma), \lambda) - g(\sigma, x_2(\sigma), \lambda), x_1(\sigma) - x_2(\sigma)\rangle_{V^*,V} \,\mathrm{d}\sigma.\end{aligned}$$

By virtue of assumption H($g$), Lemma 2.2, and the Cauchy inequality, for any $\varepsilon > 0$ we have

$$\begin{aligned}\frac{1}{2}\|x_1(t) - x_2(t)\|_H^2 + c\|x_1 - x_2\|_{L_p^t(V)}^p &\leqslant |a| \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma \\ &\quad + \int_0^t \|g(\sigma, x_1(\sigma), \lambda) - g(\sigma, x_2(\sigma), \lambda)\|_{V^*} \|x_1(\sigma) - x_2(\sigma)\|_V \,\mathrm{d}\sigma \\ &\leqslant |a| \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma \\ &\quad + L(b) \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H \|x_1(\sigma) - x_2(\sigma)\|_V \,\mathrm{d}\sigma \\ &\leqslant |a| \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma + (L(b)/2\varepsilon) \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma \\ &\quad + (L(b)\varepsilon/2) \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_V^2 \,\mathrm{d}\sigma.\end{aligned}$$

Using the compact embedding $L_p^t(V) \hookrightarrow L_2^t(V)$, we obtain

$$\begin{aligned}\|x_1(t) - x_2(t)\|_H^2 + 2c\|x_1 - x_2\|_{L_p^t(V)}^p \leqslant (2|a| + L(b)/\varepsilon) \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma \\ + L_1\varepsilon \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_V^p \,\mathrm{d}\sigma,\end{aligned}$$

where $L_1$ is a constant depending on $b$ and the embedding constant. Consequently, for sufficiently small $\varepsilon > 0$, there exists a constant $c' > 0$ such that

$$\|x_1(t) - x_2(t)\|_H^2 + c'\|x_1 - x_2\|_{L_p^t(V)}^p \leqslant (2|a| + L(b)/\varepsilon) \int_0^t \|x_1(\sigma) - x_2(\sigma)\|_H^2 \,\mathrm{d}\sigma.$$

Using Gronwall's lemma, uniqueness follows from the above inequality. □

**Theorem 2.1.** *Under assumptions* H($A$) *and* H($g$), *problem* (2.1) *has a unique solution.*

P r o o f. Let a sequence $\{x_0^n\}$ be an approximation of the given initial state $x_0(\lambda) \in H$, i.e., $x_0^n(\lambda) \in H_n$, $x_0^n(\lambda) \to x_0(\lambda)$ in $H$ as $n \to \infty$. We consider the sequence $x^n(t) = \sum\limits_{k=1}^{n} C_{k,n}(t) e_k$ and seek a function $x^n$ such that

$$(2.5)\qquad \begin{cases} \langle \dot{x}^n(t), e_j\rangle + \langle A(t, x^n(t), \lambda), e_j\rangle + \langle Bx^n(t), e_j\rangle \\ \quad = \langle g(t, x^n(t), \lambda), e_j\rangle, \quad j = 1, 2, \ldots, n; \\ x^n(0) = x_0^n(\lambda); \\ x^n \in L_p(I, H_n), \quad \dot{x}^n \in L_q(I, H_n). \end{cases}$$

It follows from the existence theorem of Carathéodory for the ordinary differential equation in $\mathbb{R}^n$ (see [13]) and Lemma 2.2 that, for each $n \in \mathbb{N}$, the finite dimensional system (2.5) has a unique solution $x^n$. It can be seen from Lemma 2.3 that $\{x^n\}$ is contained in a bounded subset of $W^0_{p,q}(\lambda)$. Hence, by assumption H$(A)$, $\{A(x^n,\lambda)\}$ is bounded in $L_q(V^*)$. Since $B \in \mathscr{P}_{a,b}$, $\{Bx^n\}$ is bounded in $L_q(V^*)$ and also since $L_p(V)$ and $L_q(V^*)$ are reflexive Banach spaces, there exists a subsequence, again denoted by $\{x^n\}$, an element $x \in L_p(V)$ with its distributional derivative $\dot{x} \in L_q(V^*)$ and $W \in L_q(V^*)$ such that

$$\begin{aligned} x^n &\overset{w}{\to} x^0 \in L_p(V), \\ \dot{x}^n &\overset{w}{\to} \dot{x}^0 \in L_q(V^*), \\ A(x^n,\lambda) &\overset{w}{\to} W \in L_q(V^*), \\ Bx^n &\overset{w}{\to} Bx^0 \in L_q(V^*) \quad \text{as } n \to \infty. \end{aligned}$$

Combining the assumptions with Lemma 2.1, we have

$$\begin{aligned} G(x^n,\lambda) &\to G(x^0,\lambda) \in L_q(V^*), \\ x^n(0) &\to x_0(\lambda) \in H, \\ x^n(T) &\overset{w}{\to} z \in H \quad \text{as } n \to \infty. \end{aligned}$$

Let $\psi \in C^\infty(I,\mathbb{R})$ and $v \in H_n$. Using equation (2.5) and integration by parts, one can obtain

$$\begin{aligned} (x^n(T),\psi(T)v) - (x^n(0),\psi(0)v) &= \int_0^T \langle x^n(t), \dot{\psi}(t)v\rangle \,\mathrm{d}t \\ + \int_0^T \langle g(t,x^n(t),\lambda) &- A(t,x^n(t),\lambda) - Bx^n(t), \psi(t)v\rangle \,\mathrm{d}t. \end{aligned}$$

Letting $n \to \infty$, we have

$$(z,\psi(T)v) - (x_0(\lambda),\psi(0)v) = \langle\!\langle G(x^0,\lambda) - W - Bx^0, \psi v\rangle\!\rangle + \langle\!\langle \dot{\psi}v, x^0\rangle\!\rangle.$$

Using this, one can easily verify that the limit elements $x^0$, $W$, $z$, $Bx^0$ satisfy

$$\begin{cases} \dot{x}^0 + W + Bx^0 = G(x^0,\lambda), \\ x(0) = x_0(\lambda), \;\; x^0(T) = z, \;\; x^0 \in W_{p,q}. \end{cases}$$

Again using equation (2.5) and integration by parts, we have

$$\frac{1}{2}(\|x^n(T)\|_H^2 - \|x^n(0)\|_H^2) = \langle\!\langle G(x^n,\lambda) - A(x^n,\lambda) - Bx^n, x^n\rangle\!\rangle.$$

By virtue of the fact that $\liminf_{n\to\infty} \|x^n(T)\|_H \geqslant \|x^0(T)\|_H$, we obtain

$$\begin{aligned}\langle\!\langle W, x^0\rangle\!\rangle &\leqslant \liminf_{n\to\infty} \langle\!\langle A(x^n,\lambda), x^n\rangle\!\rangle\\ &\leqslant \limsup_{n\to\infty} \langle\!\langle A(x^n,\lambda), x^n\rangle\!\rangle\\ &\leqslant \langle\!\langle G(x^0,\lambda), x^0\rangle\!\rangle - \langle\!\langle Bx^0, x^0\rangle\!\rangle + \tfrac{1}{2}(\|x^0(0)\|_H^2 - \|x^0(T)\|_H^2)\\ &= \langle\!\langle W, x^0\rangle\!\rangle.\end{aligned}$$

Since $A$ is monotone and hemicontinuous, $W = A(x^0,\lambda)$ (see [13]). Thus the limit element $x^0$ satisfies equation (2.2) and hence is a solution of (2.1). The uniqueness follows from Lemma 2.3. This completes the proof. □

## 3. Existence of optimality for both operators and parameters

In this section, we consider the identification problem (P) of nonlinear system (2.1). Find a pair $(\lambda^0, B^0) \in Q_m \times \mathscr{P}_{a,b}$, such that $J(\lambda^0, B^0) \leqslant J(\lambda, B)$ for all $(\lambda, B) \in Q_m \times \mathscr{P}_{a,b}$, where

$$J(\lambda, B) = \int_0^T f(t, x(\lambda, B)(t), \lambda)\,\mathrm{d}t. \tag{3.1}$$

In the following, we assume that the initial datum $x_0(\lambda) \equiv x_0$ is fixed.

**Lemma 3.1.** *Consider the identification problem* (P). *Suppose that assumptions* H($A$), H($g$), H($\lambda$), *and* H($f$) *hold. Then the mapping* $(\lambda, B) \to x(\lambda, B)$ *is continous from* $Q_m \times \mathscr{L}_s(V, V^*)$ *to* $C(I, H)$ *and the functional* $(\lambda, B) \to J(\lambda, B)$ *is lower semicontinuous on* $Q_m \times \mathscr{L}_s(V, V^*)$.

Proof. Suppose $\lambda^n \to \lambda^0 \in Q_m$ and $B^n \to B^0 \in \mathscr{L}_s(V, V^*)$. Let $\{x^n\}$ and $\{x^0\}$ denote the solutions of the system (2.1) corresponding to $(\lambda^n, B^n)$ and $(\lambda^0, B^0)$, respectively. Defining $y^n = x^n - x^0$, one observes that $y^n$ is the solution of the problem

$$\begin{cases}\dot y^n(t) + (A(t, x^n(t), \lambda^n) - A(t, x^0(t), \lambda^n)) + B^n y^n(t)\\ \quad = -(A(t, x^0(t), \lambda^n) - A(t, x^0(t), \lambda^0)) + (B^0 - B^n)x^0(t)\\ \qquad + g(t, x^n(t), \lambda^n) - g(t, x^0(t), \lambda^0),\\ x^n(0) - x^0(0) = 0.\end{cases} \tag{3.2}$$

Scalar multiplying the first equation of (3.2) on either side by $y^n$ and using Young's inequality, we have

$$
\begin{aligned}
\frac{1}{2}\|y^n(t)\|_H^2 + c\|y^n\|_{L_p^t(V)}^p \leqslant{}& |a|\int_0^t \|y^n(\sigma)\|_H^2 \,\mathrm{d}\sigma + L(b)\int_0^t \|y^n(\sigma)\|_H\|y^n(\sigma)\|_V \,\mathrm{d}\sigma \\
&+ \int_0^t \|(B^0 - B^n)x^0(\sigma)\|_{V^*}\|y^n(\sigma)\|_V \,\mathrm{d}\sigma \\
&+ \int_0^t \|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*}\|y^n(\sigma)\|_V d\sigma \\
\leqslant{}& |a|\int_0^t \|y^n(\sigma)\|_H^2 \,\mathrm{d}\sigma + \frac{L(b)}{2\varepsilon}\int_0^t \|y^n(\sigma)\|_H^2 \,\mathrm{d}\sigma + \frac{L(b)\varepsilon}{2}\int_0^t \|y^n(\sigma)\|_V^2 \,\mathrm{d}\sigma \\
&+ \frac{1}{2\varepsilon}\int_0^t \|(B^0 - B^n)x^0(\sigma)\|_{V^*}^2 \,\mathrm{d}\sigma + \varepsilon\int_0^t \|y^n(\sigma)\|_V^2 \,\mathrm{d}\sigma \\
&+ \frac{1}{2\varepsilon}\int_0^t \|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma \\
\leqslant{}& \Big(|a| + \frac{L(b)}{2\varepsilon}\Big)\int_0^t \|y^n(\sigma)\|_H^2 \,\mathrm{d}\sigma + \frac{\varepsilon(L(b)+2)}{2}\int_0^t \|y^n(\sigma)\|_V^2 \,\mathrm{d}\sigma \\
&+ \frac{1}{2\varepsilon}\int_0^t \|(B^0 - B^n)x^0(\sigma)\|_{V^*}^2 \,\mathrm{d}\sigma \\
&+ \frac{1}{2\varepsilon}\int_0^t \|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma.
\end{aligned}
$$

Choosing $\varepsilon > 0$ sufficiently small and the compact embedding $L_p^t(V) \hookrightarrow L_2^t(V)$ with an embedding constant $b'$, we obtain

$$
\begin{aligned}
\|y^n(t)\|_H^2 + 2c\|y^n\|_{L_P^t(V)}^p \leqslant{}& \Big(2|a| + \frac{L(b)}{\varepsilon}\Big)\int_0^t \|y^n(\sigma)\|_H^2 \,\mathrm{d}\sigma \\
&+ L_2\varepsilon\int_0^t \|y^n(\sigma)\|_V^p \,\mathrm{d}\sigma + \frac{1}{\varepsilon}\int_0^t \|(B^0 - B^n)x^0(\sigma)\|_{V^*}^2 \,\mathrm{d}\sigma \\
&+ \frac{1}{\varepsilon}\int_0^t \|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma,
\end{aligned}
$$

where $L_2 = (L(b)+2)b'$. Taking $\varepsilon = \frac{c}{L_2}$, we have

$$
\begin{aligned}
\|y^n(t)\|_H^2 + c\|y^n\|_{L_p^t(V)}^p \leqslant{}& \Big(2|a| + \frac{L_2 L(b)}{c}\Big)\int_0^t \|y^n(\sigma)\|_H^2 \,\mathrm{d}\sigma \\
&+ \frac{L_2}{c}\Big(\int_0^t \|(B^0 - B^n)x^0(\sigma)\|_{V^*}^2 \,\mathrm{d}\sigma \\
&+ \int_0^t \|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma\Big).
\end{aligned}
$$

Defining

$$\psi^n(t) = \|y^n(t)\|_H^2 + c\int_0^t \|y^n(\sigma)\|_V^p \,\mathrm{d}\sigma,$$

it follows from the above inequality that

$$\psi^n(t) \leqslant \Big(2|a| + \frac{L_2 L(b)}{c}\Big)\int_0^t \psi^n(\sigma)\,\mathrm{d}\sigma + K_n(t)$$

where

$$K_n(t) = \frac{L_2}{c}\bigg(\int_0^t \|(B^0 - B^n)x^0(\sigma)\|_{V^*}^2 \,\mathrm{d}\sigma + \int_0^t \|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma\bigg).$$

Using Gronwall's lemma, one concludes that

$$\psi^n(t) \leqslant \exp\bigg(\Big(2|a| + \frac{L_2 L(b)}{c}\Big)T\bigg)K_n(t) \tag{3.3}$$

for all $t \in I$. Since $\lambda^n \to \lambda^0 \in Q_m$, $B^n \to B^0 \in \mathscr{L}_s(V, V^*)$, and $x^0 \in L_p(I, V)$, it is clear that $\|(B^0 - B^n)x^0\|_{V^*} \to 0$ almost everywhere on $I$ and there exists a finite number $\gamma$ such that

$$\|(B^0 - B^n)x^0(t)\|_{V^*} \leqslant \gamma\|x^0(t)\|_V \quad \text{for all } t \in I.$$

On the other hand, since $\lambda \to g(\cdot, \cdot, \lambda)$ is continuous, we get

$$\|g(\sigma, x^0(\sigma), \lambda^n) - g(\sigma, x^0(\sigma), \lambda^0)\|_{V^*} \to 0$$

almost everywhere on $I$. Thus by the Lebesgue dominated convergence theorem, it follows that $\psi^n(t) \to 0$ as $n \to \infty$ uniformly on $I$. Hence one may conclude from (3.3) that $x^n \to x^0$ in $C(I, H)$ as well as in $L_p(I, V)$, and in particular $x^n(t) \to x^0(t)$ in $H$ for all $t \in I$. This proves the continuity of the map $(\lambda, B) \to x(\lambda, B)$ as desired.

Define

$$J(\lambda^n, B^n) = \int_I f(t, x^n(t), \lambda^n)\,\mathrm{d}t \quad \text{and} \quad J(\lambda^0, B^0) = \int_I f(t, x^0(t), \lambda^0)\,\mathrm{d}t,$$

where $x^n$ and $x^0$ are the solutions of the system (2.1) corresponding to $(\lambda^n, B^n)$ and $(\lambda^0, B^0)$, respectively. Since, by assumption H($f$), for almost all $t \in I$, $x \to f(t, x, \cdot)$ is continuous on $H$, we have

$$f(t, x^0(t), \lambda^0) \leqslant \liminf_n f(t, x^n(t), \lambda^n) \text{ almost everywhere on } I,$$

and consequently, by Fatou's lemma,

$$\int_I f(t, x^0(t), \lambda^0)\,\mathrm{d}t \leqslant \liminf_n \int_I f(t, x^n(t), \lambda^n)\,\mathrm{d}t.$$

Clearly, this is equivalent to

$$J(\lambda^0, B^0) \leqslant \liminf_n J(\lambda^n, B^n).$$

This completes the proof of the lemma. $\square$

**Lemma 3.2.** *The set $\mathscr{P}_{a,b}$ considered as a subset of $\mathscr{L}(V, V^*)$ is sequentially compact in the strong operator topology $\tau_{\rm so}$.*

P r o o f. For proof, see Lemma 1.2 of [3]. $\square$

**Theorem 3.1.** *Suppose that assumptions* H$(A)$, H$(g)$, H$(\lambda)$ *and* H$(f)$ *hold. Then there exists $(\lambda^0, B^0) \in Q_m \times \mathscr{P}_{a,b}$ such that*

$$J(\lambda^0, B^0) \leqslant J(\lambda, B) \quad \textit{for all } (\lambda, B) \in Q_m \times \mathscr{P}_{a,b}.$$

P r o o f. Define $l = \inf\{J(\lambda, B),\ (\lambda, B) \in Q_m \times \mathscr{P}_{a,b}\}$. Since $f(t, x, \lambda) > -\infty$ for $(t, x) \in I \times H$, the infimum is well defined and $l > -\infty$. Let $\{(\lambda^k, B^k)\}$ be a minimizing sequence from $Q_m \times \mathscr{P}_{a,b}$, i.e., $\lim_k J(\lambda^k, B^k) = l$. Then by Lemma 3.2, there exists $\{(\lambda^{k_i}, B^{k_i})\} \subset \{(\lambda^k, B^k)\}$ relabeled as $\{(\lambda^k, B^k)\}$ and a $(\lambda^0, B^0) \in Q_m \times \mathscr{P}_{a,b}$ such that $\lambda^k \to \lambda^0 \in Q_m$, $B^k \to B^0 \in \mathscr{P}_{a,b}(\tau_{\rm so})$. Since $(\lambda, B) \to J(\lambda, B)$ is lower semicontinuous on $Q_m \times \mathscr{P}_{a,b}$ (see Lemma 3.1), we have

$$l \leqslant J(\lambda^0, B^0) \leqslant \liminf_k J(\lambda^k, B^k) \leqslant \lim_k J(\lambda^k, B^k) = l.$$

Hence $J(\lambda^0, B^0) = l$ implies that $J(\cdot, \cdot)$ attains its infimum on $Q_m \times \mathscr{P}_{a,b}$. This completes the proof. $\square$

## 4. Necessary conditions of optimality

We consider necessary conditions of optimality for the identification problem (P). We note that usually the mapping $(\lambda, B) \to x(\lambda, B)$ from $Q_m \times \mathscr{L}(V, V^*)$ to $L_p(I, V)$ is unique. In this section we assume that $p = q = 2$ and $L^p_+ = \{x(\cdot) \in L^p\colon\ x(\cdot) \geqslant 0\}$, $p = 1, 2, \infty$. In order to derive the necessary optimality conditions, we need some additional assumptions

H$(A)_1$ $A\colon\ I \times V \times Q_m \to V^*$ is an operator.

(1) $A$ satisfies condition H$(A)$.

(2) $x \to A(t, x, \lambda)$ is continuously Frechét differentiable and strong uniformly monotone in $t \in I$.

(3) $\|A'_x(t, x, \lambda)\|_{V^*} \leqslant a_1(t) + b_1(t)\|x\|_V$ a.e. $a_1(\cdot) \in L^2_+, b_1(\cdot) \in L^\infty_+$ and $\langle A'_x(t, x, \lambda)h, h\rangle \geqslant \beta\|h\|^2_V$ a.e. $\beta > 0$, $h \in V$.

(4) $\lambda \to A(t, x, \lambda)$ is continuously Frechét differentiable and $\|A'_\lambda(t, x, \lambda)\|_{V^*} \leqslant \delta_1$ a.e. $\delta > 0$.

(5) $A'_x(t, x, \lambda)$ is continuous on $Q_m$.

H$(g)_1$ $g\colon\ I \times H \times Q_m \to V^*$ is a map.

(1) $g$ satisfies condition H$(g)$.

(2) $x \to g(t, x, \lambda)$ is Frechét differentiable, $\langle g'_x(t, x, \lambda)h, h\rangle \leqslant 0$ a.e. and $\|g'_x(t, x, \lambda)\|_{V^*} \leqslant a_2(t) + b_2\|x\|_H$ a.e. $a_2(\cdot) \in L^2_+$, $b_2 > 0$.

(3) $\lambda \to g(t, x, \lambda)$ is Frechét differentiable, $\|g'_\lambda(t, x, \lambda)\| \leqslant \delta_2$ a.e., $\delta_2 > 0$.

(4) $(x, \lambda) \to g'_x(t, x, \lambda)$ is continuous and $(x, \lambda) \to g'_\lambda(t, x, \lambda)$ is continuous.

H$(f)_1$ $f\colon\ I \times H \times Q_m \to \mathbb{R}^+$ is an integrable function.

(1) $f$ satisfies condition H$(f)$.

(2) $x \to f(t, x, \lambda)$ is Frechét differentiable and $(x, \lambda) \to f'_x(t, x, \lambda)$ is continuous.

(3) $\lambda \to f(t, x, \lambda)$ is Frechét differentiable and $(x, \lambda) \to f'_\lambda(t, x, \lambda)$ is continuous.

(4) $\|f'_x(t, x, \lambda)\|_H \leqslant a_3(t) + b_3(t)\|x\|^2_H$ a.e. with $a_3(\cdot) \in L^1_+$, $b_3(\cdot) \in L^\infty_+$ and $\|f'_\lambda(t, x, \lambda)\|_H \leqslant \delta_3$ a.e. $\delta_3 > 0$.

For the proof of necessary conditions of optimality, we shall make use of the Gâteaux differential of $x(\lambda, B)$ with respect to the parameter and operator $(\lambda, B)$. Indeed, we show that the Gâteaux differential of $x$ at $(\lambda^0, B^0)$ in the direction $(\lambda-\lambda^0, B - B^0)$ defined by

$$\hat{x}(\lambda^0, B^0; \lambda - \lambda^0, B - B^0) = w - \lim_{\varepsilon\to 0} \frac{x(\lambda^0 + \varepsilon(\lambda - \lambda^0), B^0 + \varepsilon(B - B^0)) - x(\lambda^0, B^0)}{\varepsilon}$$

exists and that it is the solution of a related differential equation.

In the next lemma we present the Gâteaux differentiability of the mapping $(\lambda, B) \to x(\lambda, B)$ in the weak sense.

**Lemma 4.1.** *Consider system* (2.1) *and suppose that assumptions* $\mathrm{H}(A)_1$, $\mathrm{H}(g)_1$ *and* $\mathrm{H}(f)_1$ *hold. Let* $x(\lambda, B)$ *denote the (weak) solution of the problem* (2.1) *corresponding to* $(\lambda, B) \in Q_m \times \mathscr{P}_{a,b}$. *Then at each point* $(\lambda, B) \in Q_m \times \mathscr{P}_{a,b}$ *the function* $(\lambda, B) \to x(\lambda, B)$ *has a weak Gâteaux differential in the direction* $(\lambda - \lambda^0, B - B^0)$, *denoted* $\hat{x}(\lambda^0, B^0; \lambda - \lambda^0, B - B^0)$, *and it is the solution of the Cauchy problem*

$$(4.1)\qquad \begin{cases} \dot{e} + A'_x(t, x^0, \lambda^0)e + B^0 e - g'_x(t, x^0, \lambda^0)e \\ \quad = -A'_\lambda(t, x^0, \lambda^0; \lambda - \lambda^0) + (B^0 - B)x^0 + g'_\lambda(t, x^0, \lambda^0; \lambda - \lambda^0), \\ e(0) = 0 \end{cases}$$

*satisfying* $\hat{x} \in L_2(I, V) \cap L_\infty(I, H)$, *where* $x^0 = x(\lambda^0, B^0)$ *is the solution of* (2.1) *corresponding to* $\lambda = \lambda^0, B = B^0$. *Here,*

$$A'_x(t, x^0, \lambda^0) = w - \lim_{\varepsilon \to 0} \frac{A(t, x^\varepsilon, \lambda^0) - A(t, x^0, \lambda^0)}{x^\varepsilon - x^0},$$

$$A'_\lambda(t, x^0, \lambda^0; \lambda - \lambda^0) = w - \lim_{\varepsilon \to 0} \frac{A(t, x^0, \lambda^\varepsilon) - A(t, x^0, \lambda^0)}{\varepsilon},$$

$$g'_x(t, x^0, \lambda^0) = w - \lim_{\varepsilon \to 0} \frac{g(t, x^\varepsilon, \lambda^0) - g(t, x^0, \lambda^0)}{x^\varepsilon - x^0}$$

*and*

$$g'_\lambda(t, x^0, \lambda^0; \lambda - \lambda^0) = w - \lim_{\varepsilon \to 0} \frac{g(t, x^0, \lambda^\varepsilon) - g(t, x^0, \lambda^0)}{\varepsilon}.$$

Proof. Let $(\lambda^0, B^0), (\lambda, B) \in Q_m \times \mathscr{P}_{a,b}$. Since $Q_m \times \mathscr{P}_{a,b}$ is a closed convex subset of $Q_m \times \mathscr{L}(V, V^*)$, we have $(\lambda^\varepsilon, b^\varepsilon) \in Q_m \times \mathscr{P}_{a,b}$, where $\lambda^\varepsilon = \lambda^0 + \varepsilon(\lambda - \lambda^0)$, $B^\varepsilon = B^0 + \varepsilon(B - B^0)$, $x^\varepsilon = x(\lambda^\varepsilon, B^\varepsilon)$ and $x^0 = x(\lambda^0, B^0)$ for $0 \leqslant \varepsilon \leqslant 1$. Using (2.1) and defining $\varphi^\varepsilon \equiv (x^\varepsilon - x^0)/\varepsilon$, we obtain

$$(4.2)\qquad \dot{\varphi}^\varepsilon + \frac{A(t, x^\varepsilon, \lambda^\varepsilon) - A(t, x^0, \lambda^\varepsilon)}{\varepsilon} + B^\varepsilon \varphi^\varepsilon - \frac{g(t, x^\varepsilon, \lambda^\varepsilon) - g(t, x^0, \lambda^\varepsilon)}{\varepsilon}$$
$$= -\frac{A(t, x^0, \lambda^\varepsilon) - A(t, x^0, \lambda^0)}{\varepsilon} + (B^0 - B)x^0 + \frac{g(t, x^0, \lambda^\varepsilon) - g(t, x^0, \lambda^0)}{\varepsilon}$$
$$\varphi^\varepsilon(0) = 0.$$

Scalar multiplying both sides of the first equation of (4.2) by $\varphi^\varepsilon$ and using the assumptions, we have

$$\begin{aligned}(4.3)\quad \frac{1}{2}\|\varphi^\varepsilon(t)\|_H^2 &+ \int_0^t \langle A'_x(\sigma, x^0(\sigma)+\varepsilon\mu_1(x^\varepsilon(\sigma)-x^0(\sigma)),\lambda^\varepsilon)\varphi^\varepsilon(\sigma),\varphi^\varepsilon(\sigma)\rangle\,\mathrm{d}\sigma \\ &+ \int_0^t \langle B^\varepsilon\varphi^\varepsilon(\sigma),\varphi^\varepsilon(\sigma)\rangle\,\mathrm{d}\sigma \\ &= \int_0^t \langle (B^0-B)x^0(\sigma),\varphi^\varepsilon(\sigma)\rangle\,\mathrm{d}\sigma \\ &+ \int_0^t \langle -A'_\lambda(\sigma,x^0(\sigma),\lambda^0+\varepsilon\mu_2(\lambda-\lambda^0);\lambda-\lambda^0),\varphi^\varepsilon(\sigma)\rangle\,\mathrm{d}\sigma \\ &+ \int_0^t \langle g'_x(\sigma,x^0(\sigma)+\varepsilon\nu_1(x^\varepsilon(\sigma)-x^0(\sigma)),\lambda^\varepsilon)\varphi^\varepsilon(\sigma),\varphi^\varepsilon(\sigma)\rangle\,\mathrm{d}\sigma \\ &+ \int_0^t \langle g'_\lambda(\sigma,x^0(\sigma),\lambda^0+\varepsilon\nu_2(\lambda-\lambda^0);\lambda-\lambda^0),\varphi^\varepsilon(\sigma)\rangle\,\mathrm{d}\sigma,\end{aligned}$$

where $\mu_1,\mu_2,\nu_1,\nu_2\in[0,1]$.

Using assumptions $\mathrm{H}(A)_1$, $\mathrm{H}(g)_1$ and $\mathscr{P}_{a,b}$ in (4.3), we obtain

$$\begin{aligned}(4.4)\quad \frac{1}{2}\|\varphi^\varepsilon(t)\|_H^2 &+ \beta\int_0^t \|\varphi^\varepsilon(\sigma)\|_V^2\,\mathrm{d}\sigma \\ &\leqslant a\int_0^t \|\varphi^\varepsilon(\sigma)\|_H^2\,\mathrm{d}\sigma + \int_0^t \|(B^0-B)x^0(\sigma)\|_{V^*}\|\varphi^\varepsilon(\sigma)\|_V\,\mathrm{d}\sigma \\ &+ \int_0^t \|A'_\lambda(\sigma,x^0(\sigma),\lambda^0+\varepsilon\mu_2(\lambda-\lambda^0);\lambda-\lambda^0)\|_{V^*}\|\varphi^\varepsilon(\sigma)\|_V\,\mathrm{d}\sigma \\ &+ \int_0^t \|g'_\lambda(\sigma,x^0(\sigma),\lambda^0+\varepsilon\nu_2(\lambda-\lambda^0);\lambda-\lambda^0)\|_{V^*}\|\varphi^\varepsilon(\sigma)\|_V\,\mathrm{d}\sigma.\end{aligned}$$

Using the inequality $ab=\sqrt{\frac{\beta}{3}}a\sqrt{\frac{3}{\beta}}b\leqslant\frac{1}{2}(\frac{\beta}{3}a^2+\frac{3}{\beta}b^2)$ in (4.4), we have

$$\begin{aligned}(4.5)\qquad \|\varphi^\varepsilon(t)\|_H^2 &+ \beta\int_0^t \|\varphi^\varepsilon(\sigma)\|_V^2\,\mathrm{d}\sigma \\ &\leqslant 2a\int_0^t \|\varphi^\varepsilon(\sigma)\|_H^2\,\mathrm{d}\sigma + \frac{3}{\beta}\int_0^t \|(B^0-B)x^0(\sigma)\|_{V^*}^2\,\mathrm{d}\sigma \\ &+ \frac{3}{\beta}\int_0^t \|A'_\lambda(\sigma,x^0(\sigma),\lambda^0+\varepsilon\mu_2(\lambda-\lambda^0);\lambda-\lambda^0)\|_{V^*}^2\,\mathrm{d}\sigma \\ &+ \frac{3}{\beta}\int_0^t \|g'_\lambda(\sigma,x^0(\sigma),\lambda^0+\varepsilon\nu_2(\lambda-\lambda^0);\lambda-\lambda^0)\|_{V^*}^2\,\mathrm{d}\sigma.\end{aligned}$$

Using Gronwall's lemma in (4.5), we conclude that

$$
\begin{aligned}
(4.6)\qquad & \|\varphi^\varepsilon(t)\|_H^2 + \beta\int_0^t \|\varphi^\varepsilon(\sigma)\|_V^2 \,\mathrm{d}\sigma \\
&\leqslant \frac{3}{\beta}\exp(2|a|T)\biggl(\int_0^t \|(B^0-B)x^0(\sigma)\|_{V^*}^2 \,\mathrm{d}\sigma \\
&\quad + \int_0^t \|A'_\lambda(\sigma, x^0(\sigma), \lambda^0+\varepsilon\mu_2(\lambda-\lambda^0);\lambda-\lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma \\
&\quad + \int_0^t \|g'_\lambda(\sigma, x^0(\sigma), \lambda^0+\varepsilon\nu_2(\lambda-\lambda^0);\lambda-\lambda^0)\|_{V^*}^2 \,\mathrm{d}\sigma\biggr)
\end{aligned}
$$

for all $\varepsilon\in[0,1]$. Since $\varepsilon\mu_2, \varepsilon\nu_2\in[0,1]$, it follows from assumptions $\mathrm{H}(A)_1$ and $\mathrm{H}(g)_1$ and the definition of $\mathscr{P}_{a,b}$ that the right terms in (4.6) are well defined. This shows that $\{\varphi^\varepsilon,\ 0\leqslant\varepsilon\leqslant 1\}$ is contained in a bounded subset of $L_\infty(I,H)\cap L_2(I,V)$. Since $L_2(I,V)$ is a reflexive Banach space, we can extract a subsequence $\{\varphi^n\}\equiv\{\varphi^{\varepsilon_n}\}\subset\{\varphi^\varepsilon\}$ with $\varepsilon_n\in[0,1]$ and $\varepsilon_n\to 0$, and a $\varphi^0\in L_2(I,V)$ such that $\varphi^n\xrightarrow{w}\varphi^0$ in $L_2(I,V)$. This proves that the Gâteaux differential of $x$ exists and is given by $\hat{x}(\lambda^0,B^0;\lambda-\lambda^0,B-B^0)\equiv\varphi^0$. It remains to show that $\varphi^0$ is a solution of (4.1). Indeed, since

$$
\begin{aligned}
\frac{A(t,x^n,\lambda^n)-A(t,x^0,\lambda^0)}{\varepsilon} &\xrightarrow{w} A'_x(t,x^0,\lambda^0)\varphi^0 \ \text{ in } \ L_2(I,V^*),\\
B^n\varphi^n = B^0\varphi^n+\varepsilon_n(B-B^0)\varphi^n &\xrightarrow{w} B^0\varphi^0 \ \text{ in } \ L_2(I,V^*),\\
\frac{A(t,x^0,\lambda^n)-A(t,x^0,\lambda^0)}{\varepsilon} &\xrightarrow{w} A'_\lambda(t,x^0,\lambda^0;\lambda-\lambda^0) \ \text{ in } \ L_2(I,V^*),\\
\frac{g(t,x^n,\lambda^n)-g(t,x^0,\lambda^0)}{\varepsilon} &\xrightarrow{w} g'_x(t,x^0,\lambda^0)\varphi^0 \ \text{ in } \ L_2(I,V^*),\\
\frac{g(t,x^0,\lambda^n)-g(t,x^0,\lambda^0)}{\varepsilon} &\xrightarrow{w} g'_\lambda(t,x^0,\lambda^0;\lambda-\lambda^0) \ \text{ in } \ L_2(I,V^*),
\end{aligned}
$$

it follows from (4.2) that $\dot{\varphi}^n\in L_2(I,V^*)$ and $\dot{\varphi}^n\xrightarrow{w}\eta$ in $L_2(I,V^*)$ for suitable $\eta\in L_2(I,V^*)$, and that is the distributional derivative of $\varphi^0$. Hence $\varphi^0$ satisfies the equality

$$
\begin{aligned}
&\dot{\varphi}^0 + A'_x(t,x^0,\lambda^0)\varphi^0 + B^0\varphi^0 - g'_x(t,x^0,\lambda^0)\varphi^0\\
&= -A'_\lambda(t,x^0,\lambda^0;\lambda-\lambda^0) + (B^0-B)x^0 + g'_\lambda(t,x^0,\lambda^0;\lambda-\lambda^0)
\end{aligned}
$$

in the sense of vector-valued distributions in $V^*$. Since $\varphi^0\in L_2(I,V)$ and $\dot{\varphi}^0\in L_2(I,V^*)$, it is clear that $\varphi^0\in C(I,H)$ and $\varphi^0(0)$ is well defined and equals $\varphi^n(0)=0$ for all $n$. Hence $\varphi^0$ satisfies the differential equation (4.1) and one may identify $\varphi^0$ as $e$. This completes the proof. $\square$

With the help of Lemma 4.1, we derive the following necessary conditions for optimality.

**Theorem 4.1.** *Suppose that assumptions* $\mathrm{H}(A)_1$, $\mathrm{H}(g)_1$ *and* $\mathrm{H}(f)_1$ *hold. Consider system* (2.1) *and the identification problem* (P) *with*

$$J(\lambda, B) = \int_I f(t, x(\lambda, B)(t), \lambda)\,\mathrm{d}t.$$

*Then in order that* $(\lambda^0, B^0) \in Q_m \times \mathcal{P}_{a,b}$ *be the optimal pair for the unknown parameter and the unknown operator, it is necessary that there exist a pair* $\{x^0, z^0\} \in C(I, H) \times C(I, H)$ *satisfying the system of equations*

$$\text{(4.7)} \qquad \begin{cases} \dot{x} + A(t, x, \lambda^0) + B^0 x = g(t, x, \lambda^0), \\ x(0) = x_0, \quad \lambda^0 \in Q_m, \quad B^0 \in \mathscr{P}_{a,b}, \end{cases}$$

*the adjoint equation*

$$\text{(4.8)} \qquad \begin{cases} -\dot{z} + (A'_x(t, x^0, \lambda^0))^* z + (B^0)^* z - (g_x(t, x^0, \lambda^0))^* z = f'_x(t, x^0, \lambda^0), \\ z(T) = 0, \forall t \in [0, T) \end{cases}$$

*and the inequality*

$$\text{(4.9)} \quad \int_I \langle -A'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0), z^0(t)\rangle_{V,V^*}\,\mathrm{d}t + \int_I \langle (B^0 - B) x^0(t), z^0(t)\rangle_{V,V^*}\,\mathrm{d}t \\ + \int_I \langle g'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0), z^0(t)\rangle_{V,V^*}\,\mathrm{d}t + \int_I f'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0)\,\mathrm{d}t \geqslant 0$$

*for all* $\lambda \in Q_m$, $B \in \mathscr{P}_{a,b}$.

Proof. Since $(\lambda, B) \to x(\lambda, B)$ has a (weak) Gâteaux differential on $Q_m \times \mathscr{P}_{a,b}$, it follows that $J(\cdot,\cdot)$ as defined above also has a Gâteaux differential. Denote $x^0 \equiv x(\lambda^0, B^0)$. Then in order that $J(\cdot,\cdot)$ attain its minimum at $(\lambda^0, B^0) \in Q_m \times \mathscr{P}_{a,b}$, it is necessary that

$$J'(\lambda^0, B^0; \lambda - \lambda^0, B - B^0) \equiv \lim_{\varepsilon \to 0} \frac{J(\lambda^\varepsilon, B^\varepsilon) - J(\lambda^0, B^0)}{\varepsilon} \geqslant 0$$

for all $(\lambda, B) \in Q_m \times \mathscr{P}_{a,b}$. Using the result of Lemma 4.1, it follows from the above that

$$\text{(4.11)} \qquad J'(\lambda^0, B^0; \lambda - \lambda^0, B - B^0) \\ = \int_I \langle f'_x(t, x^0(t), \lambda^0), \varphi^0(t)\rangle\,\mathrm{d}t + \int_I f'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0)\,\mathrm{d}t \geqslant 0$$

for all $(\lambda, B) \in Q_m \times \mathscr{P}_{a,b}$, where $\varphi^0(t)$ is the Gâteaux differential as given by Lemma 4.1. Using (4.1) and (4.11), we obtain the adjoint equation (4.8). Reversing the flow of time $t \to T-t$, it follows from Theorem 2.1 that the system (4.8) also has a unique weak solution $z^0 \in L_2(I, V) \cap C(I, H)$. Utilizing (4.8), (4.11) and integrating by parts, we obtain

$$\begin{aligned}(4.12)\quad &\int_I \langle \dot{\varphi}^0(t) + A'_x(t, x^0(t), \lambda^0)\varphi^0(t) + B^0\varphi^0(t) - g'_x(t, x^0(t), \lambda^0)\varphi^0(t), z^0(t)\rangle_{V^*,V}\,\mathrm{d}t \\ &\quad + \int_I f'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0)\,\mathrm{d}t \geqslant 0, \ \ \forall\,(\lambda, B) \in Q_m \times \mathscr{P}_{a,b}.\end{aligned}$$

From (4.8) and (4.12), we obtain

$$\begin{aligned}&\int_I \langle f'_x(t, x^0(t), \lambda^0), \varphi^0(t)\rangle_{V^*,V}\,\mathrm{d}t + \int_I f'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0)\,\mathrm{d}t \\ &= \int_I \langle \dot{\varphi}^0(t) + A'_x(t, x^0(t), \lambda^0)\varphi^0(t) + B^0\varphi^0(t) - g'_x(t, x^0(t), \lambda^0)\varphi^0(t), z^0(t)\rangle_{V^*,V}\,\mathrm{d}t \\ &\quad + \int_I f'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0)\,\mathrm{d}t \\ &= -\int_I \langle A'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0), z^0(t)\rangle_{V^*,V}\,\mathrm{d}t + \int_I \langle (B - B^0)x^0(t), z^0(t)\rangle_{V^*,V}\,\mathrm{d}t \\ &\quad + \int_I \langle g'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0), z^0(t)\rangle_{V^*,V}\,\mathrm{d}t + \int_I f'_\lambda(t, x^0(t), \lambda^0; \lambda - \lambda^0)\,\mathrm{d}t \geqslant 0\end{aligned}$$

for all $\lambda \in Q_m$, $B \in \mathscr{P}_{a,b}$. Hence we obtain (4.9), which completes the proof. □

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