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SUBDIRECTLY IRREDUCIBLE MV-ALGEBRAS

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Abstract. In this note we characterize the one-generated subdirectly irreducible MV-algebras and use this characterization to prove that a quasivariety of MV-algebras has the relative congruence extension property if and only if it is a variety.

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1. Introduction and purposes

In [4] Chang introduced MV-algebras in order to give an algebraic proof of the completeness of the Łukasiewicz infinite-valued sentencial calculus. We recall that an MV-algebra is an algebraic structure $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$ such that $\langle A, \oplus, 0\rangle$ is an abelian monoid and the following identities hold: $\neg\neg x = x$; $x \oplus \neg 0 = \neg 0$; $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. The purpose of this note is to describe the subdirectly irreducible one-generated MV-algebras. This has been done by A. Romanowska and T. Traczyk for bounded commutative BCK-algebras (which are categorically equivalent to MV-algebras as shown by D. Mundici, see [13]) but not in a satisfactory way. In particular, the description of the one-generated subdirectly irreducible bounded commutative BCK-algebras (Corollary 13 of [14]) is not complete as was shown in [11]. The just mentioned Romanowska's result relies on Theorem 2 of [17]. We will go through the proofs of these results pointing out where the mistakes are and doing the necessary corrections and adjustments. We will do this in the language of MV-algebras using the translation between the category of bounded commutative BCK-algebras and the category of MV-algebras given in [13]. For all undefined notions concerning MV-algebras we ask the reader to consult [5].

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MV-algebras are also equivalent to Wajsberg algebras (see [8]). The main motivation to make a revision of the results of Romanowska and Traczyk mentioned above was to answer the question: which quasivarieties of Wajsberg algebras enjoy the relative congruence extension property? This question was partially answered in [10]. In the last section of this note we settle this problem by proving that a quasivariety of Wajsberg algebras enjoys the relative congruence extension property if and only if it is a variety.

2. Review of previous results

If we set $x \odot y = \neg(\neg x \oplus \neg y)$, $x \wedge y = (x \oplus \neg y) \odot y$ and $x \vee y = (x \odot \neg y) \oplus y$ then $\langle A, \vee, \wedge, 0, 1\rangle$ (where $1 = \neg 0$) is a bounded distributive lattice. In this lattice, $x \leqslant y$ iff $\neg x \oplus y = 1$. Remember that an $\mathbf{A} = \langle A, \rightarrow, \neg, 1\rangle$ is a Wajsberg algebra if it satisfies the following identities: $1 \rightarrow x = x$; $(x \rightarrow y) \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)] = 1$; $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$; $(\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) = 1$. An MV-algebra can be turned into a Wajsberg algebra by setting $x \rightarrow y = \neg x \oplus y$. Conversely, a Wajsberg algebra can be turned into an MV-algebra by setting $x \oplus y = \neg x \rightarrow y$.

It is known that every subdirectly irreducible MV-algebra is a chain. Indeed, an MV-algebra is finitely subdirectly irreducible iff it is a chain; see [8, Theorem 15] or [10, Proposition 2.6]. Here is an example of a MV-chain which is not subdirectly irreducible: let $\mathbf{A}$ be the MV-algebra with the universe the rational numbers in the real interval $[0, 1]$ and with operations given by $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1-x$. Fix a non-principal ultrafilter U over the set of natural numbers, ω. Consider the ultrapower $\mathbf{A}^{\omega}/U$ which is obviously a chain. According to [10, Remark 4.3], in order to show that $\mathbf{A}^{\omega}/U$ is not subdirectly irreducible, it will be enough to show that for $0 < y \in \mathbf{A}^{\omega}/U$ there exists $x \neq 0$ such that $nx < y$ for all $n \in \omega$. Remember that nx is defined for every natural number n as follows: $0x = 0$; $(n+1)x = nx \oplus x$. So, let $y = (y_0, y_1, \ldots)/U \neq 0$ and $J = \{i \in \omega \colon\ y_i = 0\}$. Since $y \neq 0$, hence $J \notin U$ and consequently, $\omega \setminus J \in U$. Let $x = (x_0, x_1, \ldots)/U$ where $x_i = y_i/i$ for $i \notin J$ and $x_i = 0$ for $i \in J$. Let $n \in \omega$. Notice that for $i > n$, $nx_i = ny_i/i < y_i$ if $i \notin J$. Let $K = \{j \in \omega \colon\ j \leqslant n\}$. As $\omega \setminus K$ is cofinite then, by [2, Example 1, p. 150], $\omega \setminus K \in U$. So $K \notin U$ and consequently $K \cup J \notin U$. It follows from this that $\omega \setminus (K \cup J) = (\omega \setminus K) \cap (\omega \setminus J) \in U$ and $nx_i < y_i$ for $i \in (\omega \setminus K) \cap (\omega \setminus J)$. So, $nx < y$.

We recall here that a bounded commutative BCK-algebra is an algebra $\langle A, \ominus, 0, 1\rangle$ satisfying the following identities: $(x \ominus y) \ominus z = (x \ominus z) \ominus y$; $x \ominus (x \ominus y) = y \ominus (y \ominus x)$; $x \ominus x = 0$; $x \ominus 0 = x$; $x \ominus 1 = 0$. An MV-algebra can be turned into a BCK-algebra by setting $x \ominus y = \neg(\neg x \oplus y)$. Conversely, a bounded commutative BCK-algebra can be turned into an MV-algebra by setting $x \oplus y = 1 \ominus [(1 \ominus x) \ominus y]$ and $\neg x = 1 \ominus x$. Notice that $x \leqslant y$ iff $x \ominus y = 0$. A. Romanowska and T. Traczyk have studied the subdirectly irreducible commutative BCK-algebras and, as a particular case, the bounded ones. The following are translations of their results to the language of MV-algebras.

Lemma 1 ([16], Lemma 5.2)**.** *Let* $\mathbf{A}$ *be a non-simple subdirectly irreducible* MV*-algebra and* I_0 *its least non trivial ideal. Then* $I_0 \neq \{x \in \mathbf{A}\colon\ x \leqslant a\}$ *for all* $a \in \mathbf{A}$.

Lemma 2 ([16], Lemma 5.3)**.** *In any* MV*-algebra whose order is linear the following holds:*

(i) $0 \neq a \ominus b \leqslant a \ominus c \Rightarrow c \leqslant b$,

(ii) $0 \neq b \ominus a \leqslant c \ominus a \Rightarrow b \leqslant c$.

Lemma 3 ([16], Lemma 5.4)**.** *Let* $\mathbf{A}$ *and* I_0 *be as in Lemma* 1 *and* $a \in A \setminus I_0$. *Then for* $x, y \in I_0$ *with* $x \neq 0$ *there exists* n *such that* $y \leqslant a \ominus (a \ominus (n+1)x) \in I_0$. *In particular, with* $a = 1$, $y \leqslant (n+1)x$.

From [12] we know that if $\mathbf{A}$ is an MV-algebra then $\langle A, \oplus, \leqslant, 0\rangle$ is an ordered abelian monoid. Moreover, for $x, y \in A$, $x \leqslant y$ iff there exists $a \in A$ such that $x \oplus a = y$. In fact, $x \leqslant y$ means by definition that $y = x \oplus (y \ominus x)$. Lemma 1 says that $\mathbf{I}_0 = \langle I_0, \oplus, \leqslant, 0\rangle$ is unbounded; Lemma 2 and the comment above say that $\mathbf{I}_0$ is naturally ordered and Lemma 3 says that $\mathbf{I}_0$ is archimedean. These, together with a Theorem of Hölder and Clifford (see [9]) have as a consequence the following:

Lemma 4 ([16], Lemma 5.5)**.** *Let* $\mathbf{A}$ *and* I_0 *be as in Lemma* 1. *Then* $\mathbf{I}_0 = \langle I_0, \oplus, \leqslant, 0\rangle$ *is isomorphic to a submonoid of the additive ordered monoid of positive real numbers. More precisely, there is a submonoid* $\mathbf{C}$ *of the additive ordered monoid of positive real numbers and an isomorphism* $\varphi\colon\ \mathbf{I}_0 \longrightarrow \mathbf{C}$ *such that* $\varphi(y \ominus x) = \max(0, \varphi(y) - \varphi(x))$.

For each equivalence class $x/I_0 \notin \{0/I_0, 1/I_0\}$ select a representative a and fix it. Let $a \oplus I_0 = \{a \oplus t\colon\ t \in I_0\}$ and $a \ominus I_0 = \{a \ominus t\colon\ t \in I_0\}$. It is easy to see that $x/I_0 = (a \oplus I_0) \cup (a \ominus I_0)$ and $\{a\} = (a \oplus I_0) \cap (a \ominus I_0)$. $t \mapsto a \oplus t$ is an order-isomorphism from I_0 onto $a \oplus I_0$ while $t \mapsto a \ominus t$ is an anti-order isomorphism from I_0 onto $a \ominus I_0$. Let $\mathbf{S}$ be the additive subgroup of the reals with the universe $-C \cup C$ where C is the universe of the submonoid $\mathbf{C}$ from Lemma 4 and, of course, $-C = \{-c\colon\ c \in C\}$. It follows that the application $\varphi_a\colon\ x/I_0 \longrightarrow \mathbf{S}$ given by

$$\varphi_a(z) = \begin{cases} \varphi(z \ominus a) & \text{if } a \leqslant z, \\ -\varphi(a \ominus z) & \text{otherwise,} \end{cases}$$

where φ is the isomorphism from Lemma 4, is an order isomorphism. Furthermore, let $\varphi_0 (= \varphi)\colon\ 0/I_0 \longrightarrow \mathbf{C}$ and $\varphi_1\colon\ 1/I_0 \longrightarrow -\mathbf{C}$; $\varphi_1(z) = -\varphi(1 \ominus z)$. It is clear that they are order isomorphisms. Denote the equivalence class x/I_0 by $[x]$. Now consider the lexicographic order of the set $A/I_0 \times S$ and let

$$\mathbf{A}^{\mathrm{ch}} = \{([x], s) \in A/I_0 \times S\colon\ ([0], 0) \leqslant ([x], s) \leqslant ([1], 0)\}.$$

It follows that the map

$$\Phi\colon\ \mathbf{A} \longrightarrow \mathbf{A}^{\mathrm{ch}}; \quad \Phi(x) = ([x], \varphi_a(x)),$$

where a is the chosen representative of the class $[x]$, is an order isomorphism. Note that if $z \in 0/I_0 = I_0$, $\Phi(z) = ([0], \varphi(z))$ and $z \in 1/I_0$, then $\Phi(z) = ([1], -\varphi(1 \ominus z))$. By means of this order isomorphism we induce, in the obvious way, an MV-algebra structure on $\mathbf{A}^{\mathrm{ch}}$. With this MV-algebra structure $\mathbf{A}^{\mathrm{ch}}$ satisfies the following implication:

$$([x], s) \oplus ([y], r) = ([x] \oplus [y], t) \Rightarrow ([x], s+p) \oplus ([y], r) = ([x] \oplus [y], t+p).$$

Define $f\colon\ \mathbf{A}/I_0 \times \mathbf{A}/I_0 \longrightarrow S$ and $g\colon\ \mathbf{A}/I_0 \longrightarrow S$ by means of the rules

$$\begin{aligned} f([x],[y]) &= k \quad \text{where} \quad ([x],0) \oplus ([y],0) = ([x] \oplus [y], k),\\ g([x]) &= f(\neg[x],[x]). \end{aligned}$$

The following are consequences of the MV-axioms and the definitions of f and g:

(f1) $([x], s) \oplus ([y], t) = ([x] \oplus [y], f([x],[y]) + s + t)$ if $[x] \oplus [y] < [1]$. Otherwise, $([1], \min(0, s+t))$.

(f2) $f([x],[0]) = f([0],[x]) = 0$.

(f3) $f([x] \oplus [y], [z]) + f([x],[y]) = f([x], [y] \oplus [z]) + f([y],[z])$.

(f4) $\neg([x], s) = (\neg[x], g([x]) - s)$.

(f5) $f(\neg[x],[y]) + g([y]) = f(\neg[x], [x] \ominus [y]) + g([x] \ominus [y])$.

(f6) If $[x] > [y]$ then $f([x] \ominus [y], [y]) + g([x] \ominus [y]) = f(\neg[x],[y]) + g([x])$.

Suppose now that $\mathbf{A}/I_0$ is finite so that it is isomorphic to $\mathbf{L}_n$, the subalgebra of $[0,1]$ with the universe $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, for some n. Let $A/I_0 = \{c_0, c_1, \ldots, c_{n-1}\}$ so that c_i corresponds to $i/(n-1)$. Let $u = -(f(c_1,c_1) + f(c_2,c_1) + \ldots + f(c_{n-2},c_1))$ and consider the MV-algebra $\mathbf{L}^C_{n,u}$ with the universe

$$\Big\{\Big(\frac{i}{n-1}, s\Big) \in L_n \times S\colon\ (0,0) \leqslant \Big(\frac{i}{n-1}, s\Big) \leqslant (1,u)\Big\}$$

and the MV-operations defined as follows:

$$\Big(\frac{i}{n-1}, s\Big) \oplus \Big(\frac{j}{n-1}, r\Big) = \begin{cases} (\frac{i+j}{n-1}, s+r) & \text{if } i+j < n-1,\\ (1, \min\{u, s+r\}) & \text{otherwise;} \end{cases}$$

$$\neg\Big(\frac{i}{n-1}, s\Big) = \Big(1 - \frac{i}{n-1}, u - s\Big).$$

Using the properties (f1) through (f6) it can be proved that the rule

$$\psi(c_i, s) = \left(\frac{i}{n-1}, s - \sum_{k=0}^{i-2} f(c_k, c_1)\right)$$

defines an isomorphism from $\mathbf{A}^{\mathrm{ch}}$ onto $\mathbf{L}^C_{n,u}$. All these allow us to state the following

Proposition 1. *A non-simple subdirectly irreducible* MV*-algebra* $\mathbf{A}$ *such that* $\mathbf{A}/I_0$ *is finite, where* I_0 *is the least non-trivial ideal of* $\mathbf{A}$*, is isomorphic to an* MV*-algebra of the form* $\mathbf{L}^C_{n,u}$ *for some* n*.*

The details of the verification of the statements made above are left to the reader. The ideas were taken from [15]. We just want to call attention to the fact that we have avoided making use of an assertion made in the proof of Lemma 1.1 in [15] about the uniqueness of de Morgan complementation in a certain chain of which we doubt. Actually, the referred lemma is supposed to be valid in the more general context of commutative (not necessarily bounded) BCK-algebras.

3. One-generated s.i. MV-algebras

We now restate in a correct way Theorem 2 of [17] in the language of MV-algebras and review its proof in order to point out where the correction has been made. Indeed the statement in the following theorem encompasses also Corollary 13 of [14].

Theorem 1. *A subdirectly irreducible one-generated* MV*-algebra* $\mathbf{A}$ *is either ismorphic to a subalgebra of* $[0,1]$ *in which case it is simple, or it is isomorphic to* $\mathbf{L}^{\omega}_{n,u}$ *for some* n *and* u *with* $\gcd(n,u)=1$*.*

Revision of the proof. Let e be a generator of $\mathbf{A}$. Without loss of generality we may assume that $e \leqslant \neg e = 1 \ominus e$. Put $e_1 = 1, e_0 = e$ and for $j \geqslant 0$ define

$$e_{j+1} = e_{j-1} \ominus n_j e_j$$

if $0 < e_{j-1} \ominus n_j e_j \leqslant e_j$ for some n_j; otherwise, e_{j+1} is not defined. There are essentially three cases:

Case 1: $e_{k+1} = e_k$ for some k. (This case encompasses cases 1 and 3 of the original proof of Traczyk.) In this case e_k happens to be an atom of $\mathbf{A}$ and, as a result, $\mathbf{A}$ is simple and finite.

Case 2: $e_{k+1} < e_k$ for any k. In this case, as was shown by Traczyk, $\mathbf{A}$ happens to be simple, infinite and atomless but this is not a contradiction as it is asserted in [17]. What happens is that $\mathbf{A}$ is isomorphic to a subalgebra of $[0,1]$ generated by an irrational number. It is worth mentioning here that there are uncountably many of these algebras as was proved in [11].

Case 3: There is a k such that $e_k < e_{k-1} \ominus n e_k$ for all n. In this case, e_{k+1} cannot be defined and e_k is an atom of $\mathbf{A}$. Indeed $\mathbf{A}$ is not simple, $e_k \in I_0$ and $\mathbf{A}/I_0$ is finite; see case 4 of the proof in the referred paper. So, by Proposition 1, $\mathbf{A}$ is isomorphic to $\mathbf{L}^{\omega}_{n,u}$ for some n. Notice that in this case, since e_k is an atom, it generates $\mathbf{I}_0$ and therefore $\mathbf{I}_0$ has to be isomorphic to ω (as a submonoid of the reals).

Following [7] we denote by $\mathbf{MV}^{\omega}_n$ the variety generated by $\mathbf{L}^{\omega}_{n,0}$. In [6] it is proved that an MV-algebra belongs to this variety iff it satisfies the identities

$$(nx^{n-1})^2 = 2x^n,$$
$$(px^{p-1})^n = nx^p \text{ for } 1 < p < n-1 \text{ and } p \text{ is not a divisor of } (n-1),$$

where $x^n = \neg(\neg x \oplus (n-1)(\neg x))$. It is routine to check that $\mathbf{L}^{\omega}_{n,u}$ satisfies these identities and therefore it belongs to $\mathbf{MV}^{\omega}_n$. That $\gcd(n,u) = 1$ follows now from Theorem 1.8 (iii) of [7].

An element x of an MV-chain is said to be of finite order if there is a natural number n such that $nx = 1$. In this case, the *order* of an element is the least natural number with that property. If $nx < 1$ for all n then x is said to be of infinite order.

Corollary 1. *The smallest generator of a one-generated subdirectly irreducible* MV-*algebra is always of finite order except in the case of* $\mathbf{L}^{\omega}_{2,0}$ *which, by the way, is isomorphic to* $\mathbf{L}^{\omega}_{2,u}$ *for any* u.

P r o o f. It is clear that the only elements of infinite order in $\mathbf{L}^{\omega}_{n,u}$ are those in I_0 and they do not generate $\mathbf{L}^{\omega}_{n,u}$ unless $n = 2$. It is also clear that every element of $[0,1]$ is of finite order. □

To conclude this section we note that if $\mathbf{A}/I_0$ is isomorphic to a subalgebra of $[0,1]$ generated by an irrational number and $\mathbf{I}_0$ is isomprphic to ω then any one-generated subalgebra of $\mathbf{A}$ is simple (in fact, isomorphic to a subalgebra of $[0,1]$ generated by an irrational number) or isomorphic to $\mathbf{L}^{\omega}_{2,0}$.

4. Relative congruence extension property

Let $\mathcal{K}$ be a quasivariety and $\mathbf{A}$ an algebra in $\mathcal{K}$. A congruence Θ of $\mathbf{A}$ is called a $\mathcal{K}$-congruence if $\mathbf{A}/\Theta$ is in $\mathcal{K}$. We denote the set of all $\mathcal{K}$-congruences of $\mathbf{A}$ by $\mathrm{Con}_{\mathcal{K}}(\mathbf{A})$. This set forms an algebraic lattice, the meet operation of which is the set theoretic intersection and the join operation is the smallest $\mathcal{K}$-congruence containing the set theoretic union. $\mathcal{K}$ is *relative congruence distributive* (RCD for short) if $\mathrm{Con}_{\mathcal{K}}(\mathbf{A})$ is a distributive lattice for each $\mathbf{A} \in \mathcal{K}$. For $(a,b) \in A^2$ let $\Theta_A^{\mathcal{K}}(a,b)$ denote the smallest $\mathcal{K}$-congruence containing (a,b). A congruence like this is called a *principal* $\mathcal{K}$-congruence. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and Θ is a $\mathcal{K}$-congruence of $\mathbf{B}$ then we say that Θ can be extended to $\mathbf{A}$ if there is a $\mathcal{K}$-congruence Θ' of $\mathbf{A}$ such that $\Theta = \Theta' \cap B^2$; Θ' is called the *extension* of Θ to $\mathbf{A}$. $\mathcal{K}$ is said to enjoy the *relative congruence extension property* (RCEP for short) if for every algebra $\mathbf{A} \in \mathcal{K}$, any $\mathcal{K}$-congruence of any subalgebra of $\mathbf{A}$ can be extended to $\mathbf{A}$.

By $\mathbf{A} \leqslant_{\mathrm{SD}} \prod\limits_{i \in I} \mathbf{A}_i$ we mean that $\mathbf{A}$ is isomorphic to a subdirect product of the family $\{\mathbf{A}_i \colon i \in I\}$. In this case the isomorphism is called a *subdirect embedding.* An algebra $\mathbf{A} \in \mathcal{K}$ is said to be *relatively subdirectly irreducible* (RSI for short) or *$\mathcal{K}$-subdirectly irreducible* if it cannot be subdirectly embedded into a direct product of a family of algebras in $\mathcal{K}$ unless the composite of the embedding with one of the projections is an isomorphism. It can be shown that $\mathbf{A} \in \mathcal{K}$ is $\mathcal{K}$-subdirectly irreducible iff there exists a least non-zero $\mathcal{K}$-congruence of $\mathbf{A}$. Such a congruence is called the $\mathcal{K}$-monolith of $\mathbf{A}$.

In [10] it is proved that a non-RCD quasivariety of Wajsberg algebras which does not generate the whole variety of Wajsberg algebras, does not enjoy the RCEP. In the next proposition it is proved that the last condition can be dropped and the result still holds. In what follows we denote the variety of Wajsberg algebras by $\mathcal{W}$ and $\Theta_A^{\mathcal{W}}(a,b)$ is written shortly as $\Theta_A(a,b)$.

Proposition 2. *Let $\mathcal{K}$ be a non-*RCD *quasivariety of Wajsberg algebras that generates the whole variety $\mathcal{W}$. Then $\mathcal{K}$ does not enjoy the* RCEP.

Proof. We assume $\mathcal{K}$ does not have the RCEP and look for a contradiction. Just as in the proof of Proposition 4.10 of [10] there exist non-zero elements a, b in some RSI member $\mathbf{A}$ of $\mathcal{K}$ such that

(i) $\Upsilon = \Theta_A^{\mathcal{K}}(0,a)$,

(ii) $\Theta_A(0,a) \cap \Theta_A(0,b) = \Delta_A$, where Υ denotes the monolith of $\mathbf{A}$.

Observe that (ii) implies that $a \not\equiv 0\Theta_A(0,b)$ and (i) together with Proposition 2.3 of [3] implies that $a \equiv 0\Theta_S^{\mathcal{K}}(0,b)$ where $\mathbf{S}$ denotes the subalgebra of $\mathbf{A}$ generated by

$\{a, b\}$. Consider now a subdirect representation

$$\mathbf{S} \leqslant_{\mathrm{SD}} \prod_{i \in I} \mathbf{L}_i = \mathbf{L}$$

where $\{L_i \colon i \in I\}$ is a family of subdirectly irreducible members of $\mathcal{W}$. Observe that since $a \wedge b = 0$ (this follows from (ii)), we have

$$a_i \neq 0 \Rightarrow b_i = 0, \tag{1}$$

$$b_i \neq 0 \Rightarrow a_i = 0. \tag{2}$$

Set $c = a \vee b$ so that $c_i \in \{a_i, b_i\}$ for each $i \in I$. We claim that $\mathbf{L}_i$ is generated by c_i. To see it we argue like this: it is clear that the algebra generated by c_i is a subalgebra of $\mathbf{L}_i$. Now, since the representation is subdirect, the canonical projection π_i restricted to $\mathbf{S}$ is an homomorphism onto $\mathbf{L}_i$. So, for $d \in \mathbf{L}_i$, there is a term operation $p(x, y)$ such that $\pi_i(p(a, b)) = d$. But, due to (1) and (2), $p(a, b)_i = \pi_i(p(a, b))$ is in the subalgebra of $\mathbf{L}_i$ generated by c_i.

Case 1: all the $\mathbf{L}_i$'s are members of $\mathcal{K}$. Let $J = \{i \in I \colon b_i \neq 0\}$ and let Ψ be the kernel of the projection of $\mathbf{L}$ on $\prod_{i \in I \setminus J} \mathbf{L}_i$. Since all the $\mathbf{L}_i$'s are in $\mathcal{K}$ we have $\Psi \in \mathrm{Con}_{\mathcal{K}} \mathbf{L}$. Clearly $\Theta^{\mathcal{K}}_{L}(0, b) \subseteq \Psi$ because $(0, b) \in \Psi$. Since $\mathbf{S} \leqslant \mathbf{L}$ and we are assuming that $\mathcal{K}$ enjoys RCEP, there exists $\Psi' \in \mathrm{Con}_{\mathcal{K}} \mathbf{L}$ such that $\Theta^{\mathcal{K}}_{S}(0, b) = \Psi' \cap (S \times S)$. But then we have a contradiction because $a \not\equiv 0 \Psi$ whereas $a \equiv 0 \Theta^{\mathcal{K}}_{L}(0, b)$; just remember that $a \equiv 0 \Theta^{\mathcal{K}}_{S}(0, b)$ and use Proposition 2.3 of [3] or the main result of [1].

Case 2: L_i is not in $\mathcal{K}$ for some $i \in I$. Suppose, without loss of generality, that $c_i = a_i$. If $c_i = 0$ then $\mathbf{L}_i \cong \mathbf{L}_2 \in \mathcal{K}$, which is not the case. So, $a_i \neq 0$. By Corollary 1, a_i, the element which generates $\mathbf{L}_i$, is of finite order unless $\mathbf{L}_i$ is isomorphic to $\mathbf{L}^{\omega}_{2,0}$, the case that will be considered later. Say that the order of a_i is n and set $d = na$ so that $d_i = (na)_i = 1$. Let $\mathbf{S}'$ be the subalgebra of $\mathbf{S}$ generated by $\{d, b\}$. Since $\mathbf{S}$ is embedded in $\mathbf{L}$, so is $\mathbf{S}'$. Notice that $d \wedge b = 0$ so that (1) and (2) still hold with d instead of a. Let Θ_i be the kernel of the composite of the i^{th} projection of $\mathbf{S}' \leqslant \mathbf{L}$ onto $\mathbf{L}_2$ with the embedding of $\mathbf{S}'$ into $\mathbf{L}$. Since $\mathbf{S}'/\Theta_i \cong \mathbf{L}_2 \in \mathcal{K}$, we have $\Theta_i \in \mathrm{Con}_{\mathcal{K}} \mathbf{S}'$. Since by hypothesis $\mathcal{K}$ enjoys RCEP, there is a congruence $\Phi \in \mathrm{Con}_{\mathcal{K}} \mathbf{S}$ such that $\Phi \cap (S' \times S') = \Theta_i$. Clearly, $\Upsilon \cap S^2 \subseteq \Phi$. So, $(0, a) \in \Phi$ which implies $(0, d) \in \Phi$. Then, $(0, d) \in \Phi \cap (S' \times S') = \Theta_i$ which means $d_i = 0$. But this is a contradiction since we saw above that $d_i = 1$. To complete the proof, let see what happens if $\mathbf{L}_i \cong \mathbf{L}^{\omega}_{2,0}$. Since, by hypothesis, every member of $\mathcal{W}$ is an homomorphic image of some member of $\mathcal{K}$, there exists $\mathbf{A} \in \mathcal{K}$ and an onto homomorphism $f \colon \mathbf{A} \longrightarrow \mathbf{L}^{\omega}_{2,0}$. Let $a \in A$ be such that $f(a) = (0, 1)$. Let $b = a \wedge \neg a$. Clearly, $f(b) = (0, 1)$ and f restricted to the subalgebra of $\mathbf{A}$ generated by b establishes an isomorphism between such a subalgebra and $\mathbf{L}^{\omega}_{2,0}$ against the hypothesis that $\mathbf{L}_i$ is not in $\mathcal{K}$. This completes the proof. □

Theorem 2. *A subquasivariety of $\mathcal{W}$ has the* RCEP *if and only if it is a variety.*

Proof. By Proposition 2 and Propositions 4.7 and 4.10 of [10].

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