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NON-TRANSITIVE GENERALIZATIONS OF SUBDIRECT PRODUCTS OF LINEARLY ORDERED RINGS

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Abstract. Weakly associative lattice rings (wal-rings) are non-transitive generalizations of lattice ordered rings (l-rings). As is known, the class of l-rings which are subdirect products of linearly ordered rings (i.e. the class of f-rings) plays an important role in the theory of l-rings. In the paper, the classes of wal-rings representable as subdirect products of to-rings and ao-rings (both being non-transitive generalizations of the class of f-rings) are characterized and the class of wal-rings having lattice ordered positive cones is described. Moreover, lexicographic products of weakly associative lattice groups are also studied here.

Keywords: weakly associative lattice ring, weakly associative lattice group, representable wal-ring

MSC 2000: 06F25, 06F15

0. Introduction

Weakly associative lattice groups (wal-groups) and totally semiordered groups (to-groups) are non-transitive generalizations of lattice ordered groups (l-groups) and totally ordered groups (o-groups). In contrast to l-groups and o-groups, non-trivial wal-groups and to-groups need not be torsion free and, moreover, there are many finite cases of such groups. Properties of wal-groups and to-groups, as well as of varieties of wal-groups, have been studied by the first author in [8], [9], [10], [11] and [12]. The second author introduced the notions of weakly associative lattice rings (wal-rings) and totally semiordered rings (to-rings) in [13], and developed the basic structure theory of these algebras.

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Since *wal*-rings and *to*-rings are non-transitive counterparts of lattice ordered rings (l-rings) and totally ordered rings (o-rings) and since the class of f-rings (i.e. l-rings which are isomorphic to subdirect products of o-rings) is one of the most important classes of l-rings, in the present paper we introduce and study *wal*-rings which are representable as subdirect products of *to*-rings.

We prove that the class $\mathcal{RO}_{wal}$ of such *wal*-rings is a variety of *wal*-rings. Moreover, we introduce the class $\mathcal{A}o\mathcal{RO}_{wal}$ of almost ordered representable (*ao*-representable) *wal*-rings which is closer to the class of f-rings and show that also $\mathcal{A}o\mathcal{RO}_{wal}$ is a variety. Further, the class of almost l-rings is defined and described. Moreover, we deal with lexicographic products of *wal*-groups.

For necessary results from the theory of l-groups and l-rings see e.g. [1], [4], and [6].

1. Basic notions

A *weakly associative lattice* (a *wa-lattice*) is an algebra $A = (A, \vee, \wedge)$ of signature $\langle 2, 2\rangle$ satisfying the identities

(I)	$a \vee a = a$;	$a \wedge a = a$.
(C)	$a \vee b = b \vee a$;	$a \wedge b = b \wedge a$.
(Abs)	$a \vee (a \wedge b) = a$;	$a \wedge (a \vee b) = a$.
(WA)	$((a \wedge c) \vee (b \wedge c)) \vee c = c$;	$((a \vee c) \wedge (b \vee c)) \wedge c = c$.

This notion has been introduced by E. Fried in [3] and H. L. Skala in [14] and [15]. It is obvious that the notion of a *wa*-lattice generalizes that of a lattice because the identities of associativity of the operations "$\vee$" and "$\wedge$" required for lattices are special cases of identities (WA) of weak associativity. Nevertheless, similarly as for lattices, the properties of "$\vee$" and "$\wedge$" make it possible to define a binary relation "$\leqslant$" on A also for *wa*-lattices as follows:

$$\forall\, a, b \in A;\ \ a \leqslant b \Longleftrightarrow_{\mathrm{df}}\ a \wedge b = a.$$

Then the relation "$\leqslant$" is reflexive and antisymmetric (i.e. "$\leqslant$" is a so-called *semiorder* of A and $(A, \leqslant)$ is a *semiordered set*) and for each $x, y \in A$ there exist $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$ in A. Conversely, if $(A, \leqslant)$ is a semiordered set such that any $x, y \in A$ have a supremum $\sup\{x, y\}$ and an infimum $\inf\{x, y\}$, then $(A, \sup, \inf)$ is a *wa*-lattice. Therefore we can equivalently view any *wa*-lattice as a special kind of a semiordered set.

A special case of a *wa*-lattice is a tournament. A semi-ordered set $(A, \leqslant)$ is said to be a *tournament* (*totally semiordered set*) if any elements $a, b \in A$ are comparable, i.e.

$$\forall\, a, b \in A;\ \ a \leqslant b\ \text{ or }\ b \leqslant a.$$

If $(G,+,\leqslant)$ is a group and $(G,\vee,\wedge) = (G,\leqslant)$ is a *wa*-lattice then the system $G = (G,+,\leqslant)$ is called a *weakly associative lattice group* (*wal-group*) if G satisfies the condition

$$(\mathrm{M}_+) \qquad \forall\, a,b,c,d \in G;\ a \leqslant b \Longrightarrow c+a+d \leqslant c+b+d.$$

If for a *wal*-group G the *wa*-lattice $(G,\leqslant)$ is a tournament, then G is called a *totally semiordered group* (*to-group*).

For basic properties of *wal*-groups and *to*-groups see [8].

If $(R,+,\cdot,\leqslant)$ is an associative ring and $(R,\vee,\wedge) = (R,\leqslant)$ is a *wa*-lattice then the system $R = (R,+,\cdot,\leqslant)$ is called a *weakly associative lattice ring* (*wal-ring*) if R satisfies the conditions

$$(\mathrm{M}_+) \qquad \forall a,b,c \in R;\ a \leqslant b \Longrightarrow a+c \leqslant b+c;$$

$$(\mathrm{M}_.) \qquad \forall a,b,c \in R;\ 0 \leqslant c \text{ and } a \leqslant b \Longrightarrow ac \leqslant bc \text{ and } ca \leqslant cb.$$

If for a *wal*-ring R the *wa*-lattice $(R,\leqslant)$ is a tournament, then R is called a *totally semiordered ring* (*to-ring*).

(For basic properties of *wal*-rings see [13].) In contrast to lattice ordered rings (l-rings) and linearly ordered rings (o-rings) (see [1]), there are non-trivial finite *wal*-rings and *to*-rings.

The class of all *wal*-rings is a variety of algebras of type $\langle +,0,-(\cdot),\cdot,\vee,\wedge\rangle$ of signature $\langle 2,0,1,2,2,2\rangle$, and l-rings form its subvariety. The variety of *wal*-rings is characterized by identities describing the varieties of all rings and all *wa*-lattices and further by the following identities:

$$a+(b\vee c)+d = (a+b+d)\vee(a+c+d),$$
$$(a\vee b)(c\vee 0) \geqslant a(c\vee 0)\vee b(c\vee 0),$$
$$(c\vee 0)(a\vee b) \geqslant (c\vee 0)a\vee(c\vee 0)b.$$

Now we recall some notions and results concerning *wal*-rings and their subrings (see [13]).

If R is a *wal*-ring then $R^+ = \{x\in R;\ 0\leqslant x\}$ is called the *positive cone* of R and its elements are *positive.*

Example 1.1. Let us consider the ring $\mathbb{Z}_3 = \{0,1,2\}$ with the addition and multiplication $\bmod 3$. We denote $R = (R,+,\cdot) = (\mathbb{Z}_3,+,\cdot)$, $\mathbb{Z}_3^+ = R^+ = \{0,1\}$. It is clear that $\mathbb{Z}_3^+$ is the positive cone of a total semiorder of the ring $\mathbb{Z}_3$.

Example 1.2. The ring $R = (\mathbb{Z}, +, \cdot)$

a) with the positive cone $R^+ = \{0, 1, 2, 4, 6, \dots\}$ is a *wal*-ring, not a *to*-ring. If $x \in R$ then we have:
   1) $x \in R^+ \Rightarrow x \vee 0 = x$;
   2) $-x \in R^+ \Rightarrow x \vee 0 = 0$;
   3) $x \notin R^+,\ \ -x \notin R^+ \Rightarrow x \vee 0 = \max\{x, 0\} + 1$,
      where $\max\{x, 0\}$ is meant in the natural ordering of $\mathbb{Z}$.

b) with the positive cone R^+ as follows:
   1) $0, 1 \in R^+$.

   Let $1 \neq n \in \mathbb{N}$.
   2) If n is the product of an odd number of prime factors (for example $12 = 2 \cdot 2 \cdot 3$), then $-n \in R^+$.
   3) If n is the product of an even number of prime factors, then $n \in R^+$.
      That means $R^+ = \{0, 1, -2, -3, 4, -5, 6, -7, -8, 9, 10, -11, -12, -13, 14, 15, 16, -17, \dots\}$. Then R^+ defines a total semi-order of the ring R. However, it is not a linear order because e.g. $4 \leqslant 1$, $1 \leqslant -2$ but $4 \geqslant -2$.

Subalgebras of *wal*-rings are called *wal-subrings*. That means if R is a *wal*-ring and $\emptyset \neq A \subseteq R$, then A is a *wal*-subring of R if A is both a subring and a *wa*-sublattice of R.

Let R be a *wal*-ring and I its ideal which is simultaneously its convex *wa*-sublattice. Then I is called a *wal-ideal* of R if it satisfies the following mutually equivalent conditions:

($\mathrm{I_a}$) $\forall\, a, b \in I,\ \ x, y \in R;\ (x \leqslant a,\ \ y \leqslant b \Longrightarrow \exists\, c \in I;\ x \vee y \leqslant c$,

($\mathrm{I_b}$) $\forall\, a, b, c \in I,\ \ x, y \in R;\ x \leqslant a,\ \ y \leqslant b \Longrightarrow (x \vee y) \vee c \in I$.

The *wal*-ideals of *wal*-rings coincide with the kernels of homomorphisms of *wal*-rings.

If I is a *wal*-ideal of R, we can define a semiorder on R/I by

$$x + I \leqslant y + I \Longleftrightarrow_{\mathrm{df}} \exists\, a \in I;\ x + a \leqslant y,$$

and R/I with this relation is a *wal*-ring.

A *wal*-ideal I of R is said to be *straightening* if it satisfies the following mutually equivalent conditions:

($\mathrm{S_a}$) $x, y \in R,\ \ 0 \leqslant x \wedge y \in I \Longrightarrow x \in I$ or $y \in I$,

($\mathrm{S_b}$) $x, y \in R,\ \ x \wedge y = 0 \Longrightarrow x \in I$ or $y \in I$,

($\mathrm{S_c}$) R/I is a *to*-ring.

A *wal*-ideal I of a *wal*-ring R is called *semimaximal* if there exists an element $a \in R$ such that I is a maximal *wal*-ideal of R with respect to the property "not containing a".

Let us recall ([1] and [4]) that an l-ring R is called a *ring of functions* (*f-ring*) if R is isomorphic to a subdirect product of linearly ordered rings (o-rings).

2. Representable wal-rings

Definition. If R is a *wal*-ring, then R is called *representable* if it is isomorphic to a subdirect product of *to*-rings.

Proposition 2.1. *Let R be a representable wal-ring. Then for any $a, b, c \in R$ we have*

(1) $c \geqslant 0 \Rightarrow (a \vee b)c = ac \vee bc,$
$$c(a \vee b) = ca \vee cb,$$
$$(a \wedge b)c = ac \wedge bc,$$
$$c(a \wedge b) = ca \wedge cb;$$
(2) $a \wedge b = 0$ *implies* $ab = 0$;
(3) *if* $a \wedge b = 0$ *and* $c \geqslant 0$, *then* $ca \wedge b = 0$ *and* $ac \wedge b = 0$;
(4) $a^2 \geqslant 0$.

The above mentioned properties of a representable *wal*-ring are obvious for a *to*-ring. They are observed by forming subdirect products. For the same reason, it is evident that a representable *wal*-ring R is an l-ring if and only if R is an f-ring.

Proposition 2.2. *A wal-ring is representable if and only if the intersection of all its straightening wal-ideals is equal to* $\{0\}$.

P r o o f. Let R be a representable *wal*-ring. Then there exists a family of surjective *wal*-homomorphisms $p_i\colon R \longrightarrow R_i$, $i \in I$ such that every R_i is totally semi-ordered and $\bigcap_{i \in I} \operatorname{Ker} p_i = \{0\}$. Hence $R/\operatorname{Ker} p_i$ $(i \in I)$ is totally semiordered and this is the case if and only if $\operatorname{Ker} p_i$ $(i \in I)$ is a straightening ideal.

The converse implication is obvious. □

Proposition 2.3. *If every semimaximal wal-ideal of a wal-ring R is straightening then R is representable.*

P r o o f. By [13, Corollary 2.2.6], the intersection of all semimaximal *wal*-ideals of a *wal*-ring is equal to $\{0\}$. □

Remark 2.4. It is obvious that we can write the property (3) from Proposition 2.1 in the following way:

$$\left.\begin{aligned}(y \vee 0)(x \vee 0) \wedge (-x \vee 0) = 0\\ (x \vee 0)(y \vee 0) \wedge (-x \vee 0) = 0\end{aligned}\right\} \quad \text{for every } x, y \in R.$$

Indeed, let the identities be fulfilled and $a \wedge b = 0$, $c \geqslant 0$. Then, by Proposition 13 of [8], $a + b = a \vee b$, hence $a = (a - b) \vee 0$ and $b = (b - a) \vee 0$. We have $0 = c((a - b) \vee 0) \wedge ((b - a) \vee 0) = ca \wedge b$. Similarly $ac \wedge b = 0$. The converse implication is obvious.

It is known that the above mentioned identities characterize f-rings (see [4]). However, they do not characterize representable *wal*-rings.

We can consider an abelian *wal*-group $(G, +, \leqslant)$ which is not representable. The existence of such groups has been verified in [10]: Consider the abelian *wal*-group $G = (\mathbb{Z}, +, \leqslant)$ with the positive cone $G^+ = \{0, 1, 2, 4, \ldots, 2n, \ldots\}$. Since G has no straightening subgroup different from G, we conclude that G is not representable.

Then the *wal*-ring $R = (G, +, \cdot, \leqslant)$, where $x \cdot y = 0$ for every $x, y \in G$, satisfies both the identities characterizing f-rings. At the same time the *wal*-ring R is not representable. (Its *wal*-ideals coincide with *wal*-ideals of the *wal*-group $(G, +, \leqslant)$.)

Nevertheless, we will prove the following theorem.

Theorem 2.5. *The class $\mathcal{RO}_{wal}$ of all representable wal-rings is a variety of wal-rings.*

P r o o f. By Birkhoff's theorem, a nonempty class of algebras of a given type is a variety if it is closed under direct products, subalgebras and homomorphic images.

a) Obviously, the direct product of representable *wal*-rings is a representable *wal*-ring, too.

b) Let $R \in \mathcal{RO}_{wal}$ and let S be a *wal*-subring of R. Let K_β be a straightening *wal*-ideal of R. Let us denote $S_\beta = S \cap K_\beta$. It is obvious that S_β is an ideal of the ring S which is a *wa*-sublattice of the *wa*-lattice S. Let $a, b \in S_\beta$, $x \in S$, $a \leqslant x$, $x \leqslant b$. Since $a, b \in K_\beta$, we have $x \in K_\beta \cap S = S_\beta$, hence S_β is convex.

Let $a, b, c \in S_\beta$, $x, y \in S$, $x \leqslant a$, $y \leqslant b$. Then $(x \vee y) \vee c \in K_\beta \cap S = S_\beta$ and so S_β is a *wal*-ideal of S.

Let $x, y \in S$, $x \wedge y = 0$. Then $x \in K_\beta$ or $y \in K_\beta$, hence $x \in S_\beta$ or $y \in S_\beta$. That means S_β is straightening.

Now, let $\{K_\beta;\ \beta \in \Delta\}$ be the system of all straightening *wal*-ideals of R. Then $\bigcap_{\beta \in \Delta} S_\beta = \bigcap_{\beta \in \Delta} (S \cap K_\beta) \subseteq \bigcap_{\beta \in \Delta} K_\beta = \{0\}$ and so, by Proposition 2.2, S is a representable *wal*-ring.

c) Let R, R' be *wal*-rings and f a surjective *wal*-homomorphism of R onto R'. Since *wal*-rings are Ω-groups in the sense of Kurosch, we have by [7, III.2.13], if J is a *wal*-ideal of R and $J' = f(J)$ then J' is a *wal*-ideal of R'.

Suppose J is straightening. Consider $x' + J'$, $y' + J' \in R'/J'$. Let $x, y \in R$, $f(x) = x'$, $f(y) = y'$. We can assume that $x + J \leqslant y + J$. Then there exists $a \in J$

such that $x + a \leqslant y$, and consequently $x' + f(a) \leqslant y'$. We have $x' + J' \leqslant y' + J'$ because $f(a) \in J'$. Therefore J' is straightening.

Let R be representable and let $\{J_\alpha;\ \alpha \in \Gamma\}$ be the system of all straightening *wal*-ideals of R. If there exists $\beta \in \Gamma$ such that $f(J_\beta) = \{0'\}$, then $\{0'\}$ is a straightening *wal*-ideal of R', hence R' is a *to*-ring and so representable.

Let $J'_\alpha = f(J_\alpha) \neq \{0'\}$ for each $\alpha \in \Gamma$. The map f induces a bijection preserving inclusions of the set of all *wal*-ideals of R which are not contained in $\operatorname{Ker} f$ onto the set of all *wal*-ideals of R'. At the same time the *wa*-lattices R/J_α and $R'/f(J_\alpha)$ are isomorphic, hence f induces also a bijection of the set of all straightening *wal*-ideals of R onto the set of all straightening *wal*-ideals of R'. Let $J' = \bigcap_{\alpha \in \Gamma} J'_\alpha \neq \{0'\}$. Then $J = f^{-1}(J')$ is a *wal*-ideal of R which is contained in all straightening *wal*-ideals of R, hence $J = \{0\}$, a contradiction. Therefore $J' = \{0'\}$, that means R' is representable. □

Evidently, *o*-rings are special cases of *to*-rings, thus f-rings are special cases of representable *wal*-rings and they form a subvariety of the variety $\mathcal{RO}_{wal}$.

3. The variety of *ao*-representable *wal*-rings

We could see that representable *wal*-rings are a non-transitive generalization of f-rings and in addition, an l-ring is an f-ring if and only if it is a representable *wal*-ring. Nevertheless, the class $\mathcal{RO}_{wal}$ of all representable *wal*-rings is still rather a large extension of the class $\mathcal{RO}_l$ of all f-rings because the notion of a *to*-ring is a considerable generalization of that of an *o*-ring. Therefore, in this part we will deal with subdirect products of *to*-rings with total semiorders very close to linear orders.

A tournament $(T, \leqslant)$ is said to be *circular* if

(a) there exist $a, b, c \in T$ such that $a < b < c < a$, and

(b) whenever $x, y, z \in T$ satisfy $x < y < z < x$, then there exists no $w \in T$ such that $w < \{x, y, z\}$ or $w > \{x, y, z\}$.

Definition. A *to*-group G is called *circular* if the tournament $(G, \leqslant)$ is circular. A *to*-ring R is called *circular* if the tournament $(R, \leqslant)$ is circular.

Definition. A *to*-group G is called an *almost o-group* (*ao-group*) if G is either an *o*-group or a circular *to*-group. A *to*-ring R is called an *almost o-ring* (*ao-ring*) if R is either an *o*-ring or a circular *to*-ring.

The circular *to*-groups and the *ao*-groups have been introduced and studied in [9] and [11].

Proposition 3.1. *Let R be a to-ring. Then R is an ao-ring if and only if R^+ is a linearly ordered set.*

Proof. Let R be a circular *to*-ring, $a, b, c \in R^+ \setminus \{0\}$, $a < b < c$. Consider $a > c$. Then $a < b < c < a$ and $0 < \{a, b, c\}$, a contradiction. Thus $a < c$, therefore the restriction of $<$ to R^+ is transitive.

Conversely, let R^+ be a linearly ordered set and let R be not a linearly ordered ring. Then there exist $a, b, c, d \in R$ such that $a < b < c < a$ and, for example, $d < \{a, b, c\}$. Then $-d+a < -d+b < -d+c < -d+a$ and $0 < \{-d+a, -d+b, -d+c\}$. Hence R^+ is not a linearly ordered set, a contradiction. Similarly for $d > \{a, b, c\}$. It follows that R is circular. □

Example 3.2.

a) It is obvious that every linearly ordered ring is an *ao*-ring.
b) Let us consider the ring $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition and multiplication $\bmod 3$ and $\mathbb{Z}_3^+ = \{0, 1\}$. Then $(\mathbb{Z}_3, +, \cdot)$ is an *ao*-ring, not an *o*-ring because e.g. $0 < 1 < 2 < 0$.

By Example 3.2, it is seen that there exist *ao*-rings both with an upper unbounded positive cone and with a positive cone having the greatest element. Now we will investigate *ao*-rings with the greatest positive element which are simultaneously integral domains.

Let R be an integral *ao*-domain containing the greatest element $a \neq 0$ in R^+. Since always $a^2 \in R^+$, we have $a^2 \leqslant a$.

a) Let $a^2 = a$. Then $(2a)^2 = 4a^2 = 4a$, therefore $4a \geqslant 0$, thus $4a \leqslant a$. That means $a \leqslant -2a$.
   First, let us suppose that $a = -2a$. Then $3a = 0$ and so $4a = a$. Simultaneously we get $4a^2 - a = 0$, therefore $a(4a-1) = 0$. As R is an integral domain, we have $4a = 1$, that means $a = 1$. That is why R has characteristic 3 in this case. Now let $a < -2a$ hold. Then $-2a < 0$. At the same time $0 < a$, therefore $a < 2a$, and so $2a < 0$, a contradiction.
b) Let $a^2 < a$ and let R be finite. As $0 < a^2 < a$, we get $0 \leqslant \ldots \leqslant a^n \leqslant a^{n-1} \leqslant \ldots \leqslant a^2 < a$, thus there exists $n \in \mathbb{N}$ such that $a^{n-1} \neq 0$ and $a^n = 0$, a contradiction with the assumption that R is an integral domain.

Therefore we get the following proposition.

Proposition 3.3.

a) *Let a non-trivial ao-ring R be an integral domain. If R^+ has the greatest element a and if $a^2 = a$, then R has characteristic 3. In addition, the element a is equal to the element 1.*
b) *Every non-trivial finite integral ao-domain has characteristic 3.*

Definition. A *wal*-ideal I of a *wal*-ring R is called an *ao-straightening wal-ideal* of R if R/I is an *ao*-ring.

Definition. A *wal*-ring R is called *ao-representable* if it is isomorphic to a subdirect product of *ao*-rings.

Obviously, every *ao*-straightening *wal*-ideal is also straightening and every *ao*-representable *wal*-ring is also representable.

Proposition 3.4. *A wal-ring is ao-representable if and only if the intersection of all its ao-straightening wal-ideals is equal to* $\{0\}$.

P r o o f. The proof is similar to that of Proposition 2.2. □

Theorem 3.5. *The class* $\mathcal{A}o\mathcal{RO}_{wal}$ *of all ao-representable wal-rings is a variety of wal-rings.*

P r o o f. Similarly as in Theorem 2.5, we will use Birkhoff's characterization of a variety as a class of algebras of a given type closed under direct products, subalgebras and homomorphic images. Let us denote $\mathcal{W} = \mathcal{A}o\mathcal{RO}_{wal}$.

a) Evidently, the direct product of *wal*-rings belonging to $\mathcal{W}$ is also contained in $\mathcal{W}$.

b) Let $R \in \mathcal{W}$ be a subdirect product of *ao*-rings R_α $(\alpha \in \Gamma)$ and let S be a *wal*-subring of R. Let K_β be any *ao*-straightening *wal*-ideal of R. Let us denote $S_\beta = S \cap K_\beta$. By the proof of Theorem 2.5, S_β is a straightening *wal*-ideal of S.

Let $\{K_\beta;\ \beta \in \Delta\}$ be the system of all *ao*-straightening *wal*-ideals of R. Then $\bigcap\limits_{\beta\in\Delta} S_\beta = \bigcap\limits_{\beta\in\Delta} (S \cap K_\beta) \subseteq \bigcap\limits_{\beta\in\Delta} K_\beta = \{0\}$, hence, by Proposition 3.4, $S \in \mathcal{W}$.

c) Let R, R' be *wal*-rings and let f be a surjective *wal*-homomorphism of R onto R'. For any *wal*-ideal J of R put $J' = f(J)$. If J is a straightening *wal*-ideal of R then, by the proof of Theorem 2.5, J' is a straightening *wal*-ideal of R'. Let now J be an *ao*-straightening *wal*-ideal of R. Consider $x' + J'$, $y' + J'$, $z' + J' \in (R'/J')^+$ such that $x' + J' \leqslant y' + J'$, $y' + J' \leqslant z' + J'$. Let $x, y, z \in R$ be such that $x' = f(x)$, $y' = f(y)$, $z' = f(z)$ and $x + J$, $y + J$, $z + J \in (R/J)^+$. Since R/J is a *to*-ring, $x + J$ and $y + J$ are comparable. If $x + J \geqslant y + J$ then $x' + J' \geqslant y' + J'$, hence $x' + J' = y' + J'$. Thus $x' + J' \leqslant z' + J'$. Similarly for $y + J \geqslant z + J$. Therefore we can suppose $x + J \leqslant y + J$ and $y + J \leqslant z + J$. Since R/J is an *ao*-ring by Proposition 3.1, we have $x + J \leqslant z + J$, hence $x' + J' \leqslant z' + J'$, too. Therefore, by Proposition 3.1, J' is an *ao*-straightening *wal*-ideal of R'.

Let now $R \in \mathcal{W}$ and let $\{J_\alpha;\ \alpha \in \Gamma\}$ be the system of all *ao*-straightening *wal*-ideals of R. If there exists $\beta \in \Gamma$ such that $f(J_\beta) = \{0'\}$, then $\{0'\}$ is an *ao*-straightening *wal*-ideal of R' and hence R' is an *ao*-ring.

Let $J'_\alpha = f(J_\alpha) \neq \{0'\}$ for each $\alpha \in \Gamma$. As f induces a bijection preserving inclusions of the set of all *wal*-ideals of R which are not contained in $\operatorname{Ker} f$ onto the set of all *wal*-ideals of R' and at the same time the *wa*-lattices R/J_α and $R'/f(J_\alpha)$ are isomorphic, hence f induces also a bijection of the set of all *ao*-straightening *wal*-ideals of R onto the set of all *ao*-straightening *wal*-ideals of R'. Let $J' = \bigcap\limits_{\alpha\in\Gamma} J'_\alpha \neq \{0'\}$. Then $J = f^{-1}(J')$ is a *wal*-ideal of R which is contained in all *ao*-straightening *wal*-ideals of R, hence $J = \{0\}$, a contradiction. Therefore $J' = \{0'\}$, and hence, by Proposition 3.4, R' is *ao*-representable. □

4. Almost l-rings

Let R be a *wal*-ring. It is obvious that its positive cone R^+ is closed under addition if and only if R is an l-ring. If a *wal*-ring R is not an l-ring, then R^+ need not even be a *wa*-sublattice of R. For instance, for a *wal*-ring $\mathbb{Z}$ such that $\mathbb{Z}^+ = \{0, 1, 2, 4, 6, \dots, 2n, \dots\}$ we have $1, 4 \in \mathbb{Z}^+$, but $5 = 1 \vee 4 \notin \mathbb{Z}^+$. However, it is seen that for every representable *wal*-ring R, R^+ is its *wa*-sublattice and, moreover, in the case of an *ao*-representable *wal*-ring, R^+ is a lattice. (Then we can say briefly that R^+ is a sublattice of R.) Evidently, each l-ring also has the same property. Denote by $\mathcal{PLO}_{wal}$ the class of all *wal*-rings with the property "R^+ is a sublattice of R". Then $\mathcal{PLO}_{wal}$ contains, among others, the varieties $\mathcal{A}o\mathcal{RO}_{wal}$ of all *ao*-representable *wal*-rings and $\mathcal{O}_l$ of all l-rings as proper subclasses. Now we characterize the *wal*-rings belonging to $\mathcal{PLO}_{wal}$.

Definition. a) We say that a *wal*-ring R is *circular* if there exist elements $a, b, c \in R$ such that $a < b < c$, and $a \nleqslant c$ and if R satisfies the condition

(R_1^+) If $x, y, z \in R$ are such that $x < y < z$ and $x \nleqslant z$,
then there is no $w \in R$ satisfying $w < \{x, y, z\}$ or $\{x, y, z\} < w$.

b) A *wal*-ring R is called an *almost l-ring* (an *al-ring*) if R is either an l-ring or a circular *wal*-ring.

Denote by $\mathcal{A}l\mathcal{O}_{wal}$ the class of all *al*-rings. It is obvious that each *ao*-ring belongs to $\mathcal{A}l\mathcal{O}_{wal}$.

Theorem 4.1. *Let R be a wal-ring. Then its positive cone R^+ is a sublattice of R if and only if R^+ is a wa-sublattice of R and R is an al-ring.*

Proof. a) Let R^+ be a sublattice of R. Let us suppose that R is not an l-ring. Then the relation $\leqslant$ is not transitive, thus there exist elements $a, b, c \in R$

such that $a < b$, $b < c$ and at the same time $a > c$ or $a \parallel c$. Suppose that there exists $w \in R$ such that $w < \{a, b, c\}$. Then $-w + a$, $-w + b$, $-w + c \in R^+ \setminus \{0\}$ and $-w + a < -w + b$, but $-w + a > -w + c$ or $-w + a \parallel -w + c$, hence R^+ is not a lattice, a contradiction. Similarly for $\{a, b, c\} < w$. Therefore R is an *al*-ring.

b) Let R be an *al*-ring and let R^+ be a *wa*-sublattice of R. Suppose that R^+ is not a lattice. Then the restriction of the relation $\leqslant$ to R^+ is not transitive, thus there exist $a, b, c \in R^+ \setminus \{0\}$ such that $a < b < c$ and $a \nleqslant c$, a contradiction with the assumption that R is circular. Therefore R^+ is a sublattice of R. □

Remark 4.2. By [8, Proposition 1.9] in any *wal*-group, and then in any *wal*-ring, the quasi-identity $(x \vee z = y \vee z,\ \ x \wedge z = y \wedge z) \Longrightarrow x = y$ is satisfied. Thus, if R^+ is a sublattice of R then a lattice R^+ is distributive.

As an immediate consequence of Theorem 4.1 we get the following result.

Theorem 4.3. *The classes of wal-rings $\mathcal{PLO}_{wal}$ and $\mathcal{A}l\mathcal{O}_{wal}$ coincide and $\mathcal{A}l\mathcal{O}_{wal}$ is a variety of wal-rings determined by the identities*

(1) $((x \vee 0) \vee (y \vee 0)) \wedge 0 = 0$;

(2) $(x \vee 0) \vee ((y \vee 0) \vee (z \vee 0)) = ((x \vee 0) \vee (y \vee 0)) \vee (z \vee 0)$;

(3) $(x \vee 0) \wedge ((y \vee 0) \wedge (z \vee 0)) = ((x \vee 0) \wedge (y \vee 0)) \wedge (z \vee 0)$.

5. Lexicographic products of *wal*-groups

The construction called a lexicographic product is very important in the theory of l-groups. This construction can be generalized to *wal*-groups as well.

Definition. Let $\{H_\alpha;\ \alpha \in \Gamma\}$ be a collection of *wal*-groups with a linearly ordered index set. Consider all elements $a = (a_\alpha)$ of the direct product of groups H_α such that the set Γ_a of indices α such that $a_\alpha \neq 0$ (the support of the element a) is well-ordered. We can define a semiorder by declaring $a > 0$ if and only if $a_{\alpha_0} > 0$ for the smallest element α_0 of its support. The semiordered group obtained in this way will be called the *lexicographic product* $\overrightarrow{\prod\limits_{\alpha \in \Gamma}} H_\alpha$ of *wal*-groups H_α.

Remark 5.1. Let us show that it does not make sense to introduce a similar notion for *wal*-rings. Namely, let S, T be non-trivial *wal*-rings and let $R = S \overrightarrow{\times} T$ and suppose $0 < s \in S$, $0 < t \in T$. Then $(0, t), (s, -t) \in R^+$ and $(0, t) \cdot (s, -t) = (0, -t^2) \notin R^+$, hence R is not even a semiordered ring.

Now we will study lexicographic products of *wal*-groups, *to*-groups and *ao*-groups.

Theorem 5.2. a) *Let Γ be a well-ordered set and let $\{G_\alpha;\ \alpha\in\Gamma\}$ be a system of wal-groups. Then their lexicographic product $G=\overrightarrow{\prod_{\alpha\in\Gamma}} G_\alpha$ is a wal-group if and only if all G_α $(\alpha\in\Gamma)$ are to-groups or Γ has the greatest element β, G_β is a wal-group and all G_α for $\alpha<\beta$ are to-groups.*

b) *G is a to-group if and only if all G_α $(\alpha\in\Gamma)$ are to-groups.*

Proof. The proof is the same as the proof of an analogous proposition for l-groups in [5] and hence it is omitted. □

Theorem 5.3. *Let $\{G_\alpha;\ \alpha\in\Gamma\}$ be a system of non-trivial to-groups with a well-ordered index set $(\Gamma,\prec)$, where α_1 is the least element of Γ. Then the lexicographic product $G=\overrightarrow{\prod_{\alpha\in\Gamma}} G_\alpha$ is an ao-group if and only if G_{α_1} is an ao-group and all the other groups G_α $(\alpha\neq\alpha_1,\ \alpha\in\Gamma)$ are o-groups.*

Proof. By Theorem 5.2, G is always a *to*-group for any *to*-groups G_α.

a) Let G_{α_1} be an *ao*-group and let G_α be *o*-groups for all $\alpha\in\Gamma$, $\alpha\neq\alpha_1$. If $x\in G_{\alpha_1}$ then denote by K_x the set of all $a=(a_\alpha)$ in G such that $a_{\alpha_1}=x$. Then the semiorder of K_x induced by the semiorder of G is a linear order. We have $G^+=L\cup\bigcup(K_x;\ x\in G^+_{\alpha_1}\setminus\{0\})$, where $L=\{a\in G;\ a_{\alpha_1}=0$ and $a_{\gamma(a)}>0$ for the least element $\gamma(a)\in\Gamma_a\}$.

The semiordered set L is isomorphic to a subset of the lexicographic product of linearly ordered sets G_α, $\alpha\in\Gamma$, $\alpha\neq\alpha_1$, and therefore L is a linearly ordered set. At the same time by [11] or by the proof of Proposition 3.1, $G^+_{\alpha_1}\setminus\{0\}$ is a linearly ordered set, hence $K=\bigcup(K_x;\ x\in G^+_{\alpha_1}\setminus\{0\})$, as the ordinal sum of linearly ordered sets is a linearly ordered set, too.

In this way, G^+ is the ordinal sum of linearly ordered sets L and K therefore G is an *ao*-group.

b) Conversely, let there exist $\alpha\in\Gamma$, $\alpha\neq\alpha_1$, such that G_α is not an *o*-group. Then there exist $y_1,y_2\in G_\alpha$ such that $0<y_1<y_2<0$. Let $0<x\in G_{\alpha_1}$. Consider $a,b,c\in G$ such that $a_{\alpha_1}=b_{\alpha_1}=c_{\alpha_1}=x$ and $a_\alpha=0$, $b_\alpha=y_1$, $c_\alpha=y_2$. Then $a<b<c<a$, hence G^+ is not linearly ordered. Therefore G is not an *ao*-group. □

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