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CONTINUOUS EXTENDIBILITY OF SOLUTIONS OF THE NEUMANN PROBLEM FOR THE LAPLACE EQUATION

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Abstract. A necessary and sufficient condition for the continuous extendibility of a solution of the Neumann problem for the Laplace equation is given.

Keywords: Neumann problem, Laplace equation, continuous extendibility

MSC 2000: 35B65, 35J05, 35J25, 31B10

1. Maximum and regularity principle

For $x, y \in \mathbb{R}^m$, $m \geqslant 2$, denote

$$h_x(y) = \begin{cases} (m-2)^{-1} A^{-1} |x-y|^{2-m} & \text{for } x \neq y,\ m > 2, \\ A^{-1} \log |x-y|^{-1} & \text{for } x \neq y,\ m = 2, \\ \infty & \text{for } x = y, \end{cases}$$

where A is the area of the unit sphere in $\mathbb{R}^m$. For a finite real Borel measure ν denote

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y) \, \mathrm{d}\nu(y),$$

the single layer potential corresponding to ν for each x for which this integral has sense.

Let H be a bounded open set in $\mathbb{R}^m$, g an arbitrary extended real-valued function defined on ∂H. We denote by $\overline{U}_g^H$ the set of all hyperharmonic functions u on H

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which are lower bounded on H and such that for any $y \in \partial H$

$$\liminf_{x \to y} u(x) \geqslant g(y).$$

We put $\underline{U}_g^H = -\overline{U}_{(-g)}^H$ and denote by $\overline{H}_g^H$ (or $\underline{H}_g^H$) the greatest lower (or least upper) bound of $\overline{U}_g^H$ (or $\underline{U}_g^H$, respectively). (Compare [3], [14].)

A function g on ∂H is said to be resolutive (relative to H), if $\overline{H}_g^H = \underline{H}_g^H$ and $|\overline{H}_g^H(x)| < \infty$ for any $x \in H$. We set $H_g^H = \overline{H}_g^H$, the generalized solution of the Dirichlet problem for the Laplace equation with the boundary condition g, provided g is resolutive. If $g \in \mathcal{C}(\partial H)$ and u is a classical solution of the Dirichlet problem for the Laplace equation with the boundary condition g then g is resolutive and $H_g^H = u$. Any bounded Baire function on ∂H is resolutive ([3], Theorem 6 and the text on p. 94).

A set $Z \subset \mathbb{R}^m$ is called a polar set if there is an open set $U \supset Z$ and a function u superharmonic on U such that $u = +\infty$ on Z.

For a compact K in $\mathbb{R}^m$ denote by $\mathcal{C}'(K)$ the Banach space of all finite real Borel measures with support in K with the total variation as a norm.

Lemma 1. *Let $H \subset \mathbb{R}^m$ be a bounded regular set, $\nu \in \mathcal{C}'(\partial H)$. Then $\mathcal{U}\nu$ is the generalized solution of the Dirichlet problem with the boundary condition $\mathcal{U}\nu/\partial H$. Let now f be a Borel measurable function on ∂H such that $\{x \in \partial H;\ \mathcal{U}\nu(x) \neq f(x)\}$ is polar. Put $f = \mathcal{U}\nu$ on H. If f is continuous and finite on ∂H then it is continuous on the closure of H. If f is bounded on ∂H then it is bounded on H and*

$$\inf_{x \in \partial H} f(x) \leqslant \inf_{x \in H} f(x) \leqslant \sup_{x \in H} f(x) \leqslant \sup_{x \in \partial H} f(x).$$

P r o o f. Suppose first that ν is nonnegative. For $z \in H$ denote by μ_z the harmonic measure corresponding to H and z. If $y \in \partial H$, $z \in H$ then

$$\int_{\partial H} h_y(x) \, \mathrm{d}\mu_z(x) = h_y(z)$$

by [19], pp. 299, 264. Using Fubini's theorem we get

$$\int \mathcal{U}\nu \, \mathrm{d}\mu_z = \int_{\partial H} \int_{\partial H} h_y(x) \, \mathrm{d}\mu_z(x) \, \mathrm{d}\nu(y) = \int_{\partial H} h_y(z) \, \mathrm{d}\nu(y) = \mathcal{U}\nu(z).$$

Thus $\mathcal{U}\nu$ is a solution of the Dirichlet problem with the boundary condition $\mathcal{U}\nu/\partial H$.

Let ν be general. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Then $\mathcal{U}\nu = \mathcal{U}\nu^+ - \mathcal{U}\nu^-$ is a solution of the Dirichlet problem with the boundary condition

$\mathcal{U}\nu/\partial H$. Since harmonic measures do not charge polar sets ([2], Lemma 4.4.5), $\mathcal{U}\nu$ is a solution of the Dirichlet problem with the boundary condition f. If f is continuous on ∂H then f is continuous on the closure of H. If f is bounded on ∂H then f is bounded on H and since harmonic measures are probability measures we get the above inequalities. □

2. Neumann problem

Suppose that $G \subset \mathbb{R}^m$ $(m \geqslant 2)$ is an open set with a non-void compact boundary ∂G. If h is a harmonic function on G such that

$$\int_H |\nabla h| \,\mathrm{d}\mathcal{H}_m < \infty$$

for all bounded open subsets H of G we define the weak normal derivative $N^G h$ of h as the distribution

$$\langle N^G h, \varphi \rangle = \int_G \nabla\varphi \cdot \nabla h \,\mathrm{d}\mathcal{H}_m$$

for $\varphi \in \mathcal{D}$ (= the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$). Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. We formulate the *Neumann problem for the Laplace equation with a boundary condition* $\mu \in \mathcal{C}'(\partial G)$ *in the sense of distributions as follows: determine a harmonic function* h *on* G *for which* $N^G h = \mu$. It is usual to look for a solution h in the form of the single layer potential $\mathcal{U}\nu$, where $\nu \in \mathcal{C}'(\partial G)$. The single layer potential $\mathcal{U}\nu$ is a harmonic function in G for which the weak normal derivative $N^G\mathcal{U}\nu$ has sense. The operator $N^G\mathcal{U}\colon \nu \mapsto N^G\mathcal{U}\nu$ is a bounded linear operator on $\mathcal{C}'(\partial G)$ if and only if $V^G < \infty$, where

$$V^G = \sup_{x \in \partial G} v^G(x),$$
$$v^G(x) = \sup\left\{ \int_G \nabla\varphi \cdot \nabla h_x \,\mathrm{d}\mathcal{H}_m ;\ \varphi \in \mathcal{D},\ |\varphi| \leqslant 1,\ \operatorname{spt}\varphi \subset \mathbb{R}^m - \{x\} \right\}$$

(see [15]). There are more geometrical characterizations of $v^G(x)$ in [15] which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \{\bigcup L_i ;\ i = 1, \ldots, k\}$, where L_i are $(m-1)$-dimensional Ljapunov surfaces (i.e. of class $C^{1+\alpha}$).

If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m ;\ (x - z)\cdot\theta < 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed *the exterior normal* of G at z in Federer's sense. If there is no exterior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$.

The set $\{y \in \mathbb{R}^m\,;\ |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial} G$.

If G has a finite perimeter (which is fulfilled if $V^G < \infty$) then $\mathcal{H}_{m-1}(\widehat{\partial} G) < \infty$ and

$$v^G(x) = \int_{\widehat{\partial} G} |n^G(y) \cdot \nabla h_x(y)| \,\mathrm{d}\mathcal{H}_{m-1}(y)$$

for each $x \in \mathbb{R}^m$. Throughout the paper we will assume that $V^G < \infty$. Then

$$N^G \mathcal{U}\nu(M) = \int_M d_G(x)\,\mathrm{d}\nu(x) + \int_{\partial G} \int_{(\partial G \cap M)} n^G(y) \cdot \nabla h_x(y)\,\mathrm{d}\mathcal{H}_{m-1}(y)\,\mathrm{d}\nu(x)$$

for each $\nu \in \mathcal{C}'(\partial G)$ and a Borel set M (see [15]).

If L is a bounded linear operator on the Banach space X we denote by $\|L\|_{\mathrm{ess}}$ the essential norm of L, i.e. the distance of L from the space of all compact linear operators on X. The essential spectral radius of L is defined by

$$r_{\mathrm{ess}} L = \lim_{n \to \infty} (\|L^n\|_{\mathrm{ess}})^{1/n}.$$

If X is a complex Banach space then

$$\begin{aligned} r_{\mathrm{ess}} L &= \sup\{|\lambda|;\ \lambda I - L \text{ is not a Fredholm operator}\} \\ &= \sup\{|\lambda|;\ \lambda I - L \text{ is not a Fredholm operator with index } 0\} \end{aligned}$$

(see [12], Satz 51.8, Theorem 51.1).

Theorem ([22]). *Let $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$, where I is the identity operator, $\mu \in \mathcal{C}'(\partial G)$. Then there is a harmonic function u on G, which is a solution of the Neumann problem*

$$N^G u = \mu,$$

if and only if $\mu \in \mathcal{C}'_0(\partial G)$ (= the space of such $\nu \in \mathcal{C}'(\partial G)$ that $\nu(\partial H) = 0$ for each bounded component H of $\operatorname{cl} G$). Moreover, if $\mu \in \mathcal{C}'_0(\partial G)$ then there is a solution of this problem in the form of the single layer potential $\mathcal{U}\nu$, where $\nu \in \mathcal{C}'(\partial G)$.

Remark 1. It is well-known that the condition $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ is fulfilled for sets with a smooth boundary (of class $C^{1+\alpha}$) (see [16]) and for convex sets (see [23]). R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in $\mathbb{R}^3$ have this property (see [1], [18]). A. Rathsfeld showed in [28], [29] that polyhedral cones in $\mathbb{R}^3$ have this property. (By a polyhedral cone in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$

which is homeomorphic to $\mathbb{R}^2$) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons). N. V. Grachev and V. G. Maz'ya obtained independently an analogous result for polyhedral open sets with bounded boundary in $\mathbb{R}^3$ (see [10]). (Let us note that there is a polyhedral set in $\mathbb{R}^3$ which has not a locally Lipschitz boundary.) In [20] it was shown that the condition $r_{\rm ess}(N^G\mathcal{U}-\frac{1}{2}I)<\frac{1}{2}$ has a local character. As a conclusion we obtain that this condition is fullfiled for $G\subset\mathbb{R}^3$ such that for each $x\in\partial G$ there are $r(x)>0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain, and a diffeomorphism $\psi_x\colon\mathcal{U}(x;r(x))\to\mathbb{R}^3$ of class $C^{1+\alpha}$, where $\alpha>0$, such that $\psi_x(G\cap\mathcal{U}(x;r(x)))=D_x\cap\psi_x(\mathcal{U}(x;r(x)))$. V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with "piecewise-smooth" boundary in the general Euclidean space (see [8], [9], [11]).

In the rest of the paper we will suppose that $r_{\rm ess}(N^G\mathcal{U}-\frac{1}{2}I)<\frac{1}{2}$. Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto ∂G. Then $\mathcal{H}(\mathbb{R}^m)<\infty$ (see [22], Lemma 2).

Notation. $\mathcal{C}'_c(\partial G)$ will stand for the subspace of those $\mu\in\mathcal{C}'(\partial G)$ for which there exists a continuous function $\mathcal{U}_c\mu$ on $\mathbb{R}^m$ coinciding with $\mathcal{U}\mu$ on $\mathbb{R}^m\setminus\partial G$. It was shown in [27] that if $\nu\in\mathcal{C}'(\partial G)$ and the restriction of $\mathcal{U}\nu$ onto ∂G is finite and continuous then $\mathcal{U}\nu$ is finite and continuous in $\mathbb{R}^m$ and $\nu\in\mathcal{C}'_c(\partial G)$. If $\mu=f\mathcal{H}$, where $f\in L_p(\mathcal{H})$, $p>m-1$ then $\mu\in\mathcal{C}'_c(\partial G)$ (see [21], Remark 6).

Notation. Denote by $\mathcal{I}$ the set of all isolated points of ∂G, $\tilde G=G\cup\mathcal{I}$. Then the set $\mathcal{I}$ is finite by [22], Lemma 1. Therefore $V^{\tilde G}=V^G<\infty$, $N^{\tilde G}\mathcal{U}\nu=N^G\mathcal{U}\nu$ for $\nu\in\mathcal{C}'(\partial\tilde G)$ and $r_{\rm ess}(N^{\tilde G}\mathcal{U}-\frac{1}{2}I)=r_{\rm ess}(N^G\mathcal{U}-\frac{1}{2}I)$, because $\mathcal{C}'(\partial\tilde G)$ is a subspace of $\mathcal{C}'(\partial G)$ of a finite codimension.

Denote by $\Omega_R(x)$ the open ball with a centre x and a radius R.

Lemma 2. *Let $R>0$ be such that $\partial G\subset\Omega_R(0)$. Then $\tilde G\cap\Omega_R(0)$, $\Omega_R(0)\setminus\operatorname{cl}\tilde G$ are regular sets.*

Proof. Since the density of $\tilde G\cap\Omega_R(0)$ and the density of $\Omega_R(0)\setminus\operatorname{cl}\tilde G$ are positive at each point of the boundary of $\tilde G$ by [22], Lemma 1, the sets $\tilde G\cap\Omega_R(0)$, $\Omega_R(0)\setminus\operatorname{cl}\tilde G$ are regular (see [4], Chap. VII, §§ 2, 6, 19, Theorem 5.11, Theorem 5.10). □

Lemma 3. *G has finitely many components $G_1,\dots,G_n$ and $\operatorname{cl}G_j\cap\operatorname{cl}G_k=\emptyset$ for $j\neq k$.*

P r o o f. If we define for $f \in L_\infty(\mathcal{H})$, $x \in \partial G$

$$W^G f(x) = d_G(x) f(x) + \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}(y),$$

then W^G is a bounded linear operator on $L_\infty(\mathcal{H})$, because $V^G < \infty$. If we define for $f \in L_1(\mathcal{H})$, $x \in \partial G$

$$(N^G \mathcal{U} \mathcal{H}) f(x) = d_G(x) f(x) - \int_{\partial G} f(y) n^G(x) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}(y),$$

then $(N^G \mathcal{U} \mathcal{H})$ is a bounded linear operator on $L_1(\mathcal{H})$ (compare [17], Theorem 1). Since $N^G \mathcal{U}(f\mathcal{H}) = [(N^G \mathcal{U} \mathcal{H}) f] \mathcal{H}$ for each $f \in L_1(\mathcal{H})$ and $\{f\mathcal{H};\ f \in L_1(\mathcal{H})\}$ is a closed subspace of $\mathcal{C}'(\partial G)$, we have $r_{\mathrm{ess}}((N^G \mathcal{U} \mathcal{H}) - \frac{1}{2} I) \leqslant r_{\mathrm{ess}}(N^G \mathcal{U} - \frac{1}{2} I) < \frac{1}{2}$ by [20], Lemma 1.3 or [13], Lemma 15.

Fix a bounded component H of G. Since $\mathcal{H}_{m-1}(\partial H) \leqslant \mathcal{H}_{m-1}(\partial G) < \infty$, the perimeter of H is finite. Since $\mathcal{H}_{m-1}(\partial G \setminus \hat{\partial} G) = 0$ by [22], Lemma 2 and $n^H(y) = n^G(y)$ for each $y \in \hat{\partial} H \cap \hat{\partial} G$, we have

$$v^H(y) = \int_{\hat{\partial} H} |n^H(x) \cdot \nabla h_y(x)| \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\hat{\partial} H \cap \hat{\partial} G} |n^G(x) \cdot \nabla h_y(x)| \, \mathrm{d}\mathcal{H}_{m-1} \leqslant v^G(y)$$

for each $y \in \partial H$. Therefore $V^H < \infty$ and $d_H(y)$ has a good meaning for each $y \in \partial H$ by [15], Lemma 2.9. Put

$$u_H(y) = \begin{cases} 1 & \text{for } y \in \hat{\partial} H \cap \hat{\partial} G, \\ 0 & \text{for } y \in \partial G \setminus \hat{\partial} H \cap \hat{\partial} G. \end{cases}$$

Since $n^G(y) = n^H(y)$ for $y \in \hat{\partial} H \cap \hat{\partial} G$ and $\mathcal{H}_{m-1}(\partial G \setminus \hat{\partial} G) = 0$, [15], Proposition 2.8 and Lemma 2.15 yield

$$\begin{aligned} W^G u_H(x) &= \frac{1}{2} u_H(x) + \int_{\hat{\partial} H \cap \hat{\partial} G} n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \\ &= \frac{1}{2} u_H(x) + \int_{\hat{\partial} H} n^H(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \\ &= \frac{1}{2} u_H(x) - d_H(x). \end{aligned}$$

If $x \in \hat{\partial} H \cap \hat{\partial} G$ then $d_H(x) = \frac{1}{2}$ and thus $W^G u_H(x) = 0$. If $d_H(x) = 0$ then $u_H(x) = 0$, therefore $W^G u_H(x) = 0$. Since $\mathcal{H}_{m-1}(\{x \in \hat{\partial} G \setminus \hat{\partial} H;\ d_H(x) > 0\} \leqslant \mathcal{H}_{m-1}(\{x \in \partial H \setminus \hat{\partial} H;\ 0 < d_H(x) \leqslant d_G(x) \leqslant \frac{1}{2}\} = 0$ by [33], Lemma 5.9.5 and $\mathcal{H}_{m-1}(\partial G \setminus \hat{\partial} G) = 0$ by [22], Lemma 2, $W^G u_H(x) = 0$ for $\mathcal{H}$-a.a. $x \in \partial G$. Since the perimeter of a nonempty open bounded set is positive (see [33], Theorem 5.4.3) and $\mathcal{H}_{m-1}(\hat{\partial} H)$ is equal to the perimeter of H (see [33],Theorem 5.81, Theorem 5.6.5) and $\mathcal{H}_{m-1}(\partial H \setminus \hat{\partial} G) = 0$, the function u_H is positive on the set $\hat{\partial} H \cap \hat{\partial} G$ of positive $\mathcal{H}$ measure.

If H_1, H_2 are different bounded components of G then $\hat{\partial} G \cap \hat{\partial} H_1 \cap \hat{\partial} H_2 = \emptyset$, because H_1, H_2 are disjoint. The set $\{u_H ;\ H$ is a bounded component of $G\}$ contains linearly independent elements of the kernel of W^G. Since $N^G(\mathcal{UH})$ is a Fredholm operator and W^G is an adjoint operator of $N^G\mathcal{UH}$, the operator W^G is a Fredholm operator as well (see [12], Satz 51.8, Theorem 27.1). Since the dimension of the kernel of W^G is greater than or equal to the number of bounded components of G and W^G is a Fredholm operator, G has only finitely many components. (Since ∂G is bounded, there is at most one unbounded component of G.) According to [22], Note 5 the codimension of the range of $N^G(\mathcal{UH})$ is equal to the number of bounded components of the closure of G. Since the dimension of the kernel of W^G is equal to the codimension of the range of $N^G(\mathcal{UH})$, because W^G is the adjoint operator of $N^G(\mathcal{UH})$ (see [12], Theorem 27.1), the number of bounded components of G is smaller than or equal to the number of bounded components of the closure of G. Therefore the number of bounded components of G is equal to the number of bounded components of the closure of G and the closures of any two different components of G are disjoint. □

Theorem 1. *Let $\nu, \mu \in \mathcal{C}'(\partial G)$, $N^G\mathcal{U}\nu = \mu$. Then the following assertions are equivalent:*

a) $\nu \in \mathcal{C}'_c(\partial G)$.

b) $\mu \in \mathcal{C}'_c(\partial G)$.

c) *There is a finite continuous extension of $\mathcal{U}\nu$ from G onto the closure of G.*

d) *There is a finite continuous extension of $\mathcal{U}\mu$ from G onto the closure of G.*

If $\partial G = \partial(\mathbb{R}^m \setminus G)$ then these assertions are equivalent to the following ones

e) *There are a polar set K and a finite continuous function f on ∂G such that $\mathcal{U}\nu = f$ on $\partial G \setminus K$.*

f) *There are a polar set K and a finite continuous function f on ∂G such that $\mathcal{U}\mu = f$ on $\partial G \setminus K$.*

Proof. Denote $\mu_{\mathcal{I}} = \mu/\mathcal{I}$, $\mu_{\tilde{G}} = \mu/(\partial G \setminus \mathcal{I})$, $\nu_{\mathcal{I}} = \nu/\mathcal{I}$, $\nu_{\tilde{G}} = \nu/(\partial G \setminus \mathcal{I})$. Since the density of G at each point of $\partial G \setminus \mathcal{I}$ is positive by [22], Lemma 1, we have $\mu_{\mathcal{I}} = \nu_{\mathcal{I}}$ by [15], Observation on p. 25. If $\mu_{\mathcal{I}} = \nu_{\mathcal{I}} \neq 0$ then none of the assertions a)–d) is true. So we can suppose that $\mu_{\mathcal{I}} = \nu_{\mathcal{I}} = 0$ and coming to $\tilde{G}$ we can suppose that $\partial G = \partial(\mathbb{R}^m \setminus G)$.

a) $\Rightarrow$ b). $\mu \in \mathcal{C}'_c(\partial G)$ by [15], Plemelj's exchange theorem 2.23.

b) $\Rightarrow$ a). This assertion is true for $m > 2$ by [21], Lemma 13. Let us suppose that $m = 2$. If we denote for $f \in \mathcal{C}(\partial G)$ (= the space of all bounded continuous functions on ∂G equipped with the maximum norm) and $x \in \partial G$

$$W^G f(x) = d_G(x) f(x) + \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}(y),$$

then W^G is a bounded linear operator on $\mathcal{C}(\partial G)$ and $N^G\mathcal{U}$ is the dual operator of W^G (see [15], Proposition 2.5, Proposition 2.20). We shall show that $\mathcal{U}_c\mu \in W^G(\mathcal{C}(\partial G))$. Since $\operatorname{Ker}(I - N^G\mathcal{U}) \cap (I - N^G\mathcal{U})(\mathcal{C}'(\partial G)) = \{0\}$ by [22], Proposition 2 and $\dim \operatorname{Ker}(I - N^G\mathcal{U}) = \operatorname{codim}(I - N^G\mathcal{U})(\mathcal{C}'(\partial G))$ because $(I - N^G\mathcal{U})$ is a Fredholm operator with index 0, the space $\mathcal{C}'(\partial G)$ is the direct sum of $(I - N^G\mathcal{U})(\mathcal{C}'(\partial G))$ and $\operatorname{Ker}(I - N^G\mathcal{U})$. Therefore $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \operatorname{Ker}(I - N^G\mathcal{U})$ and $\mu_2 \in (I - N^G\mathcal{U})(\mathcal{C}'(\partial G))$. Since $\mu_1 \in \mathcal{C}'_c(\partial G)$ by [22], Lemma 4, we get $\mathcal{U}_c\mu_1 = \mathcal{U}_c(N^G\mathcal{U}\mu_1) = W^G(\mathcal{U}_c\mu_1)$ by [15], Plemelj's exchange theorem 2.23. Since $\mu, \mu_1 \in \mathcal{C}'_c(\partial G)$, we have $\mu_2 \in \mathcal{C}'_c(\partial G)$, too. Put $\tilde{\nu} = \nu - \mu_1$. Then $N^G\mathcal{U}\tilde{\nu} = \mu_2$. Put $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Since $N^C\mathcal{U} = I - N^G\mathcal{U}$, we have $\mu_2 \in N^C\mathcal{U}(\mathcal{C}'(\partial G))$. If G is bounded then we choose $\varphi \in \mathcal{D}$ such that $\varphi = 1$ in a neigbourhood of $\operatorname{cl} G$. We get

$$\mu_2(\partial G) = \langle N^G\mathcal{U}\tilde{\nu}, \varphi \rangle = \int_G \nabla\varphi \cdot \nabla\mathcal{U}\tilde{\nu} = 0.$$

If G is unbounded we get $\mu_2(\partial G) = 0$ in a similar way using the facts that $\mu_2 \in N^C\mathcal{U}(\mathcal{C}'(\partial G))$ and C is bounded. Let $\sigma \in \mathcal{C}'(\partial G)$, $N^G\mathcal{U}\sigma = 0$. Then $\sigma \in \mathcal{C}'_c(\partial G)$ by [22], Lemma 4. Since the density of G is positive at each point of the boundary by [22], Lemma 1, we have $\mathcal{H}(\partial G) > 0$ by Isoperimetric lemma ([15], p. 50). Put $\sigma_1 = \sigma(\partial G)[\mathcal{H}(\partial G)]^{-1}\mathcal{H}$, $\sigma_2 = \sigma - \sigma_1$. Then $\mathcal{U}\sigma_1$ is finite and continuous on $\mathbb{R}^m$ by [22], Lemma 2, [15], Corollary 2.17, Lemma 2.18. Therefore $\sigma_2 \in \mathcal{C}'_c(\partial G)$. Using [22], Lemma 7 we get

$$\int_{\partial G} \mathcal{U}_c\mu_2 \, \mathrm{d}\sigma_2 = \int_G \nabla\mathcal{U}\mu_2 \cdot \nabla\mathcal{U}\sigma_2 \, \mathrm{d}\mathcal{H}_m = \int_{\partial G} \mathcal{U}_c\sigma_2 \, \mathrm{d}\mu_2.$$

If $x \in \partial G$, $\mathcal{U}|\mu_2|(x) < \infty$ then $\mathcal{U}_c\mu_2(x) = \mathcal{U}\mu_2(x)$, because $\mathcal{U}\mu_2$ is finely continuous at x (see [19], Chapter V, § 3) and $\mathbb{R}^m \setminus G$ is not a fine neighbourhood of x, because $d_G(x) > 0$ (see [4], Chap. VII, §§ 2, 6, 19, Theorem 5.11). Thus $\mathcal{U}_c\mu_2 = \mathcal{U}\mu_2$ outside the polar set $\{x;\ \mathcal{U}|\mu_2|(x) = \infty\}$. Since σ_1 does not charge polar sets (see [19], Theorem 3.1, Theorem 2.1) using Fubini's theorem we get

$$\int_{\partial G} \mathcal{U}_c\mu_2 \, \mathrm{d}\sigma_1 = \int_{\partial G} \mathcal{U}\mu_2 \, \mathrm{d}\sigma_1 = \int_{\partial G} \mathcal{U}\sigma_1 \, \mathrm{d}\mu_2 = \int_{\partial G} \mathcal{U}_c\sigma_1 \, \mathrm{d}\mu_2.$$

Denote by $H_1, \dots, H_p$ the components of G. Then there are $c_1, \dots, c_p \in \mathbb{R}$ such that $\mathcal{U}_c\sigma = c_j$ on H_j for $j = 1, \dots, p$ by [22], Lemma 12. Therefore

$$\int_{\partial G} \mathcal{U}_c\mu_2 \,\mathrm{d}\sigma = \int_{\partial G} \mathcal{U}_c\sigma \,\mathrm{d}\mu_2 = \sum_{j=1}^{p} c_j \mu_2(\partial H_j).$$

If H_j is bounded, choose $\varphi \in \mathcal{D}$ such that $\varphi = 1$ on H_j and $\varphi = 0$ on $\operatorname{cl} G \setminus H_j$. Then

$$\mu_2(\partial H_j) = \langle \mu_2, \varphi \rangle = \langle N^G \mathcal{U}\tilde{\nu}, \varphi \rangle = \int_G \nabla \mathcal{U}\tilde{\nu} \cdot \nabla\varphi \,\mathrm{d}\mathcal{H}_m = 0.$$

If H_j is unbounded then we get $\mu_2(\partial H_j) = 0$ from the facts that $\mu_2(\partial G) = 0$ and $\mu_2(\partial H_i) = 0$ for each bounded H_i. Therefore

$$\int_{\partial G} \mathcal{U}_c\mu_2 \,\mathrm{d}\sigma = 0. \tag{1}$$

Since $N^G\mathcal{U}$ is Fredholm, (1) yields that $\mathcal{U}_c\mu_2 \in W^G(\mathcal{C}(\partial G))$ by [32], Chapter VII, Theorem 3.1. Since $\mathcal{U}_c\mu_1 \in W^G(\mathcal{C}(\partial G))$, $\mathcal{U}_c\mu_2 \in W^G(\mathcal{C}(\partial G))$ we have $\mathcal{U}_c\mu \in W^G(\mathcal{C}(\partial G))$.

Put

$$\nu_0 = \mu + \sum_{j=0}^{\infty} (I - 2N^G\mathcal{U})^j (2I - N^G\mathcal{U})\mu.$$

Then $N^G\mathcal{U}\nu_0 = \mu$ by [22], Theorem 1. Put

$$\mu_j = (I - 2N^G\mathcal{U})^j (2I - N^G\mathcal{U})\mu$$

for j a nonnegative integer. According to [15], Plemelj's exchange theorem 2.23 we have $\mu_j \in \mathcal{C}'_c(\partial G)$ and

$$\mathcal{U}_c\mu_j = (I - 2W^G)^j (2I - W^G)\mathcal{U}_c\mu \text{ on } \partial G.$$

If λ is an eigenvalue of W^G, $|\lambda - \frac{1}{2}| \geqslant \frac{1}{2}$ then λ is an eigenvalue of $N^G\mathcal{U}$, because $\lambda I - N^G\mathcal{U}$, $\lambda I - W^G$ are Fredholm operators with index 0 and the kernels of these operators have the same dimension (see [32], Chapter IX, Theorem 2.1, Theorem 1.3, Chapter VII, Theorem 3.5, Chapter V, Theorem 4.1); therefore $\lambda \in \{0; 1\}$ by [22], Proposition 1. Since $\operatorname{Ker}(\lambda I - N^G\mathcal{U})^2 = \operatorname{Ker}(\lambda I - N^G\mathcal{U})$ by [22], Proposition 2 we have $\operatorname{Ker}(\lambda I - W^G)^2 = \operatorname{Ker}(\lambda I - W^G)$ by [32], Chapter V, Theorem 2.3, Chapter V, Theorem 4.1. Now [22], Proposition 3 yields that there are constants $q \in (0; 1)$, $M > 0$ such that

$$\|(I - 2W^G)^j (2I - W^G)g\|_{\mathcal{C}(\partial G)} \leqslant Mq^j \|g\|_{\mathcal{C}(\partial G)}$$

for all $g \in W^G(\mathcal{C}(\partial G))$. Since $\mathcal{U}_c\mu \in W^G(\mathcal{C}(\partial G))$ we have

$$\sum_{j=0}^{\infty} \|\mathcal{U}_c\mu_j\|_{\mathcal{C}(\partial G)} = \sum_{j=0}^{\infty} \|(I - 2W^G)^j (2I - W^G)\mathcal{U}_c\mu\|_{\mathcal{C}(\partial G)} < \infty.$$

Since

$$\|\mu\|_{\mathcal{C}'(\partial G)} + \sum_{j=0}^{\infty} \|\mu_j\|_{\mathcal{C}'(\partial G)} < \infty, \quad \|\mathcal{U}_c\mu\|_{\mathcal{C}(\partial G)} + \sum_{j=0}^{\infty} \|\mathcal{U}_c\mu_j\|_{\mathcal{C}(\partial G)} < \infty,$$

[15], Lemma 4.5 yields that $\nu_0 \in \mathcal{C}_c'(\partial G)$.

Since $N^G\mathcal{U}(\nu - \nu_0) = 0$, we have $\nu - \nu_0 \in \mathcal{C}_c'(\partial G)$ by [22], Lemma 4 and thus $\nu \in \mathcal{C}_c'(\partial G)$.

c) $\Rightarrow$ e). Let f denote a finite continuous extension of $\mathcal{U}\nu$ from G onto the closure of G. Because $\mathcal{U}\nu^+$, $\mathcal{U}\nu^-$ are superharmonic functions they are continuous with respect to the fine topology (see [19], Chapter V, § 3). Denote $K = \{x \in \partial G;\ \mathcal{U}|\nu|(x) = \infty\}$. Then K is polar and $\mathcal{U}\nu(x)$ is the fine limit of $\mathcal{U}\nu$ for each $x \in \partial G \setminus K$. Thus $f(x) = \mathcal{U}\nu(x)$ for each $x \in \partial G \setminus K$, because every fine neighbourhood of x intersects G by Lemma 2, [19], Theorem 5.11, Theorem 5.10.

e) $\Rightarrow$ a). Define $f = \mathcal{U}\nu$ on $\mathbb{R}^m \setminus \partial G$. Fix $R > 0$ such that $\partial G \subset \Omega_R(0)$. Using Lemma 1 and Lemma 2 for $G \cap \Omega_R(0)$ and $M = \Omega_R(0) \setminus \operatorname{cl} G$ we get

$$f(x) = \lim_{y \to x,\, y \in \mathbb{R}^m \setminus \partial G} \mathcal{U}\nu(y) \ \text{ for } x \in \partial G.$$

Therefore $\nu \in \mathcal{C}_c'(\partial G)$. □

Lemma 4. *Let $H \subset \mathbb{R}^m$ be a bounded open set, $\mathcal{H}_{m-1}(\partial H) < \infty$, $\mu \in \mathcal{C}'(\partial H)$, let u be a solution of the Neumann problem $N^H u = \mu$, finite and continuous up to the boundary of H. Then for each $x \in H$*

$$u(x) = \mathcal{U}\mu(x) - \mathcal{D}u(x),$$

where

$$\mathcal{D}u(x) = \int_{\partial H} u(y) n^H(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y)$$

is the double layer potential corresponding to the density u.

P r o o f. Fix $x \in H$, $r > 0$ such that $\operatorname{cl} \Omega_r(x) \subset H$. Put $H(r) = H \setminus \Omega_r(x)$. Choose $\varphi \in \mathcal{D}$ such that $\varphi = 1$ in the neighbourhood of $\operatorname{cl} H(r)$ and $\varphi = 0$ in the neighbourhood of x. Green's formula yields

$$\begin{aligned} \mathcal{U}\mu(x) &= \langle N^H u, h_x\varphi \rangle \\ &= \int_{H(r)} \nabla h_x \cdot \nabla u \, \mathrm{d}\mathcal{H}_m + \int_{\partial \Omega_r(x)} h_x(y) n^{\Omega_r(x)}(y) \cdot \nabla u(y) \, \mathrm{d}\mathcal{H}_{m-1}(y). \end{aligned}$$

Since $\mathcal{H}_{m-1}(\partial H) < \infty$ there is a positive konstant K such that for each positive integer k there are balls $\Omega_{r_1}(x_1), \ldots, \Omega_{r_j}(x_j)$ such that $\partial H \subset (\Omega_{r_1}(x_1) \cup \ldots \cup \Omega_{r_j}(x_j))$, $r_1^{m-1} + \ldots + r_j^{m-1} \leqslant K$, $\max(r_1, \ldots, r_j) \leqslant \frac{1}{k}$, $\operatorname{dist}(x_i, \partial H) \leqslant \frac{1}{k}$ for $i = 1, \ldots, j$; put $H_k(r) = H(r) \setminus (\Omega_{r_1}(x_1) \cup \ldots \cup \Omega_{r_j}(x_j))$. Then $\mathcal{H}_m(H(r) \setminus H_k(r)) \to 0$ as $k \to \infty$, $\mathcal{H}_{m-1}(H_k(r)) \leqslant L \equiv (K + r^{m-1})\mathcal{H}_{m-1}(\partial\Omega_1(0))$.

Fix $\varepsilon > 0$. Since $\operatorname{cl} H$ is compact, there is a polynomial p such that $|u - p| \leqslant \varepsilon$ on $\operatorname{cl} H$. Using Green's formula we get

$$\begin{aligned}
&\left| \int_{H(r)} \nabla h_x(y) \cdot \nabla u(y) \, \mathrm{d}\mathcal{H}_m(y) - \int_{\partial H(r)} u(y) n^{H(r)}(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \right| \\
&\quad = \left| \lim_{k\to\infty} \int_{H_k(r)} \nabla h_x(y) \cdot \nabla u(y) \, \mathrm{d}\mathcal{H}_m(y) \right. \\
&\qquad \left. - \int_{\partial H(r)} u(y) n^{H(r)}(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \right| \\
&\quad = \left| \lim_{k\to\infty} \int_{\partial H_k(r)} u n^{H_k(r)} \cdot \nabla h_x \, \mathrm{d}\mathcal{H}_{m-1} \int_{\partial H(r)} u n^{H(r)} \cdot \nabla h_x \, \mathrm{d}\mathcal{H}_{m-1} \right| \\
&\quad \leqslant \left| \lim_{k\to\infty} \int_{\partial H_k(r)} p(y) n^{H_k(r)}(y) \cdot h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \right. \\
&\qquad \left. - \int_{\partial H(r)} p(y) n^{H(r)}(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \right| + \frac{\varepsilon 2L}{r^{m-1}A} \\
&\quad = \left| \lim_{k\to\infty} \int_{H_k(r)} \nabla p \cdot \nabla h_x \mathcal{H}_m - \int_{H(r)} \nabla p \cdot \nabla h_x \mathcal{H}_m \right| \\
&\qquad + \frac{\varepsilon 2L}{r^{m-1}A} = \frac{\varepsilon 2L}{r^{m-1}A}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{H(r)} \nabla h_x \cdot \nabla u \, \mathrm{d}\mathcal{H}_m &= \int_{\partial H(r)} u(y) n^{H(r)} \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y), \\
\mathcal{U}\mu(x) &= \int_{\partial H(r)} u n^{H(r)} \cdot \nabla h_x \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\Omega_r(x)} h_x n^{\Omega_r(x)} \cdot \nabla u \, \mathrm{d}\mathcal{H}_{m-1}.
\end{aligned}$$

If $r \to 0$ we get

$$\mathcal{U}\mu(x) = \int_{\partial H} u(y) n^H(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) + u(x).$$

$\square$

Lemma 5. *Let $\mu \in \mathcal{C}'(\partial G)$, let u be a solution of the Neumann problem $N^G u = \mu$, finite and continuous up to the boundary of G. Then $\mu \in \mathcal{C}'_c(\partial G)$.*

Proof. Let G be bounded. Then $u = \mathcal{U}\mu - \mathcal{D}u$. Since u is continuous and finite on ∂G, the double layer potential $\mathcal{D}u$ is continuously extendible to the closure of G (see [15], Chapter 2). Therefore $\mathcal{U}\mu = \mathcal{D}u + u$ is continuously extendible to the closure of G. Hence $\mu \in \mathcal{C}_c'(\partial G)$ by Theorem 1.

If G is unbounded, fix $R > 0$ such that $\partial G \subset \Omega_R(0)$. Put $H = G \cap \Omega_R(0)$. Then $V^H < \infty$, $r_{\rm ess}(N^H\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. If we put

$$\tilde{\mu}(M) = \int_{\partial\Omega_R(0)\cap M} \frac{x}{|x|} \cdot \nabla u(x)\,\mathrm{d}\mathcal{H}_{m-1}(x)$$

for a Borel measurable set M then $N^H u = \mu + \tilde{\mu}$. Since u is finite and continuous on $\operatorname{cl} H$, $\mu + \tilde{\mu} \in \mathcal{C}_c'(\partial H)$. Since $\mathcal{U}\tilde{\mu}$ is continuous in a neighbourhood of ∂G by [15], Lemma 2.18, we have $\mu \in \mathcal{C}_c'(\partial G)$. □

Lemma 6. *Let G be unbounded, let w be a solution of the Neumann problem in the sense of distributions with the null boundary condition. Suppose that there are $q \geqslant 1$, $R > 0$ such that $|\nabla w| \in L_q(G \setminus \Omega_R(0))$. Then there is a real number a such that $w - a = O(|x|^{1-m})$, $|\nabla w| = O(|x|^{-m})$ as $|x| \to \infty$.*

Proof. Fix $x_0 \in \mathbb{R}^m \setminus \operatorname{cl} G$. Then [31], Chapter I, Theorem 3.5 yields that there are real numbers a, b and a harmonic function v on a neighbourhood of 0 with $v(0) = 0$ such that

$$w(x) = a + bh_{x_0} + |x - x_0|^{2-m} v\Big(\frac{x - x_0}{|x - x_0|^2}\Big).$$

Fix $R > 0$ such that $\partial G \subset \Omega_R(x_0)$. If $\varphi \in \mathcal{D}$, $\varphi = 1$ on $\Omega_R(x_0)$ then

$$\begin{aligned} 0 = \langle N^G w, \varphi\rangle &= \langle N^{G\cap\Omega_R(x_0)} w, \varphi\rangle + \langle N^{G\setminus\Omega_R(x_0)} w, \varphi\rangle \\ &= -\int_{\partial\Omega_R(x_0)} n^{\Omega_R(x_0)} \cdot \nabla w\,\mathrm{d}\mathcal{H}_{m-1} \\ &= b - \int_{\partial\Omega_R(x_0)} n^{\Omega_R(x_0)}(x) \cdot \nabla\Big[|x - x_0|^{2-m} v\Big(\frac{x - x_0}{|x - x_0|^2}\Big)\Big]\,\mathrm{d}\mathcal{H}_{m-1}(x). \end{aligned}$$

Since $|\nabla[|x - x_0|^{2-m} v((x - x_0)/|x - x_0|^2)]| = O(|x|^{-m})$ as $|x| \to \infty$ by [31], Chapter I, Corollary and Remark 3.6, we get $b = 0$ taking $R \to \infty$. Therefore $|\nabla w(x)| = O(|x|^{-m})$, $|w(x) - a| = O(|x|^{1-m})$ as $|x| \to \infty$. □

Theorem 2. *Denote by $G_1, \ldots, G_k$ all components of G. If $\mu \in \mathcal{C}_0'(\partial G)$ then there is a solution of the Neumann problem in the sense of distributions with the boundary* *condition μ, which is continuous up to the boundary, if and only if $\mu \in \mathcal{C}'_c(\partial G)$. If G is bounded then the general form of this solution is*

$$u = \mathcal{U}\nu + \sum_{j=1}^{k} c_j \chi_{G_j}, \tag{2}$$

where

$$\nu = \mu + 2\sum_{j=0}^{\infty} (I - 2N^G\mathcal{U})^j (I - N^G\mathcal{U})\mu, \tag{3}$$

χ_{G_j} are characteristic functions of G_j and c_j are arbitrary constants. If G is unbounded then (2) *is a general form of solutions continuously extendible to the boundary of G for which there are $R > 0$, $p \geqslant 1$ such that $|\nabla u| \in L_p(G \setminus \Omega_R(0))$.*

Proof. If $\mu \in \mathcal{C}'_c(\partial G)$, then u given by (2) is a solution of the Neumann problem with the boundary condition μ, which is continuous up to the boundary (see Theorem 1 and [22], Theorem 1).

If u is a continuous (up to the boundary) solution of the Neumann problem with the boundary condition μ, then $\mu \in \mathcal{C}'_c(\partial G)$ by Lemma 5. Put $w = u - \mathcal{U}\nu$. Then w is a solution of the Neumann problem in the sense of distributions with the zero boundary condition, continuous up to the boundary.

Suppose that G is bounded. Then $w = -\mathcal{D}w$ on G by Lemma 4. Since $V^G < \infty$, $-\mathcal{D}w$ has a limit $W^{\mathbb{R}^m \setminus G} w$ on the boundary, where

$$W^{\mathbb{R}^m\setminus G} w(x) = d_{\mathbb{R}^m\setminus G}(x) w(x) - \int_{\partial G} w(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \tag{4}$$

by [15], Remark 2.24. If we denote for $f \in \mathcal{C}(\partial G)$ and $x \in \partial G$

$$W^G f(x) = d_G(x) f(x) + \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y), \tag{5}$$

then $W^G w = 0$. Since $N^G\mathcal{U}$ is a Fredholm operator, the codimension of the range of $N^G\mathcal{U}$ is equal to k by [22], Theorem 1 and $N^G\mathcal{U}$ is the adjoint operator of W^G by [15], Proposition 2.20, the dimension of the kernel of W^G is equal to k by [12], Theorem 27.1. In a similar way as for w we get that $W^G \chi_{\partial G j} = W^G \chi_{\mathrm{cl}\, G_j} = 0$. Since $\chi_{G_1}, \dots, \chi_{G_k}$ form a base of the kernel of W^G and $W^G w = 0$, w is constant on ∂G_j for each $j = 1, \dots, k$. Since w is harmonic and continuous, it is constant on G_j for each $j = 1, \dots, k$. So, u has the form (2).

Suppose now that G is unbounded and there are $R > 0$, $p \geqslant 1$ such that $|\nabla u| \in L_p(G \setminus \Omega_R(0))$. According to Lemma 6 there is a real number a such that $|\nabla w(x)| =$

$O(|x|^{-m})$, $|w(x)-a| = O(|x|^{1-m})$ as $|x| \to \infty$. Fix $x_0 \in \mathbb{R}^m \setminus \operatorname{cl} G$, $R > 0$ such that $\partial G \subset \Omega_R(x_0)$. According to Lemma 4 we have for $x \in G \cap \Omega_R(x_0)$

$$w(x) - a = \int_{\partial\Omega_R(x_0)} h_x n^{\Omega_R(x_0)} \cdot \nabla w \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial\Omega_R(x_0)} (w-a) n^{\Omega_R(x_0)} \cdot \nabla h_x \, \mathrm{d}\mathcal{H}_{m-1} - \mathcal{D}(w-a)(x).$$

Tending $R \to \infty$ we get $w(x) - a = -\mathcal{D}(w-a)(x)$ in G. Since $V^G < \infty$, $-\mathcal{D}(w-a)$ has the limit $W^{\mathbb{R}^m \setminus G}(w-a)$ (given by (4)) on the boundary. Therefore $W^G(w-a) = 0$ ($W^G f$ is given by (5)). Since $N^G\mathcal{U}$ is a Fredholm operator, the codimension of the range of $N^G\mathcal{U}$ is equal to $k-1$ by [22], Theorem 1 and $N^G\mathcal{U}$ is the adjoint operator of W^G, the dimension of the kernel of W^G is equal to $k-1$ by [12], Theorem 27.1. In a similar way as for w we get that $W^G\chi_{\partial G j} = W^G\chi_{\operatorname{cl} G_j} = 0$ for each bounded component G_j of G. Since $\{\chi_{G_j};\ G_j \text{ bounded}\}$ form a base of the kernel of W^G and $W^G(w-a) = 0$, w is constant on ∂G_j for each $j = 1, \ldots, k$ and $(w-a) = 0$ on the boundary of the unbounded component of G. Since $(w-a)$ is harmonic, continuous on $\operatorname{cl} G$ and $(w(x)-a)$ tends to 0 as $|x|$ tends to infinity, w is constant on G_j for each $j = 1, \ldots, k$. So, u has the form (2). □

Remark 2. If G is unbounded then the space of all solutions of the Neumann problem in the sense of distributions with the zero boundary condition, which are continuously extendible onto the closure of G, has infinite dimension. For a positive integer j put

$$f_j(x_1, \ldots, x_m) = \sum_{k=0}^{j} \binom{2j}{2k} (-1)^{j-k} x_1^{2k} x_2^{2j-2k}.$$

Then f_j are harmonic functions in $\mathbb{R}^m$. According to Theorem 2 there are $\nu_j \in \mathcal{C}_c'(\partial G)$ such that $\mathcal{U}\nu_j$ is a solution of the Neumann problem in the sense of distributions with the boundary condition $\frac{\partial f_j}{\partial n}\mathcal{H}$. Then $u_j = f_j - \mathcal{U}\nu_j$ are solutions of the Neumann problem in the sense of distributions with the zero boundary condition, which are continuously extendible onto the closure of G. Since $\lim u_j(x_1, \ldots, x_m)/x_1^j \to 1$ as $x_1 \to \infty$, the functions u_j are linearly independent.

Lemma 7. *Let $\nu \in \mathcal{C}_c'(\partial G)$. If $m > 2$ then $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$. If $m = 2$ then $|\nabla \mathcal{U}\nu| \in L_{2,\mathrm{loc}}(\mathbb{R}^m)$ and $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$ if and only if $\nu(\mathbb{R}^m) = 0$.*

Proof. If $m > 0$ or $m = 2$ and $\nu(\mathbb{R}^m) = 0$ then $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$ by [22], Lemma 2, Lemma 6. Let now $m = 2$, $\nu(\mathbb{R}^m) \neq 0$. Choose $x \in G$, $r > 0$ such that $\Omega_{2r}(x) \subset G$. Put $H = G \setminus \operatorname{cl} \Omega_r(x)$ and let μ be the restriction of $\mathcal{H}_1$ onto $\partial\Omega_r(x)$. Fix a constant c such that $\nu(\mathbb{R}^m) - c\mu(\mathbb{R}^m) = 0$. Since $\nu - c\mu \in \mathcal{C}_c'(\partial H)$

by [15], Lemma 2.18, we have $|\nabla \mathcal{U}\nu - c\nabla \mathcal{U}\mu| \in L_2(\mathbb{R}^m)$ (see [22], Lemma 6). Easy calculation yields that there are constants c_1, c_2 such that $\mathcal{U}\mu = c_1$ in $\Omega_r(x)$ and $\mathcal{U}\mu = c_1 + c_2 \log(|x|/r)$ on $\mathbb{R}^m \setminus \Omega_r(x)$. Since $|\nabla \mathcal{U}\mu| \in L_{2,\text{loc}}(\mathbb{R}^m) \setminus L_2(\mathbb{R}^m)$ we have got the assertion of the lemma. □

Notation. Denote by $W^{1,2}(G)$ the collection of all functions $f \in L_2(G)$ the distributional gradient of which belongs to $[L_2(G)]^m$.

Lemma 8. *Let $\nu \in \mathcal{C}_c'(\partial G)$. If G is bounded then $\mathcal{U}\nu \in W^{1,2}(G)$. If G is unbounded and $m > 4$ then $\mathcal{U}\nu \in W^{1,2}(G)$; if $3 \leqslant m \leqslant 4$ then $\mathcal{U}\nu \in W^{1,2}(G)$ if and only if $\nu(\mathbb{R}^m) = 0$.*

Proof. $\mathcal{U}\nu \in W^{1,2}(G)$ for G bounded because $|\nabla \mathcal{U}\nu| \in L_{2,\text{loc}}(\mathbb{R}^m)$ and $\mathcal{U}\nu$ is continuously extendible to $\operatorname{cl} G$. Let now G be unbounded, $m > 2$. The assertion follows from the facts that $|\nabla \mathcal{U}\nu| \in L_2(\mathbb{R}^m)$, $\mathcal{U}\nu$ is continuously extendible to $\operatorname{cl} G$ and $\mathcal{U}\nu(x) = \nu(\mathbb{R}^m)|x|^{2-m} + O(|x|^{1-m})$ for $|x| \to \infty$. □

Throughout the rest of paper we will suppose that $\mathcal{D}$ is dense in $W^{1,2}(G)$. According to [33], Theorem 2.3.2 this condition is fulfilled if $\{f/G;\ f \in W^{1,2}(\mathbb{R}^m)\} = W^{1,2}(G)$.

Definition. Let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathcal{D}(G) = \{\varphi \in \mathcal{D};\ \operatorname{spt} \varphi \subset G\}$. We say that $u \in W^{1,2}(G)$ is a weak solution of the Neumann problem for the Laplace equation with the boundary condition L if

$$\int_G \nabla u \cdot \nabla v \,\mathrm{d}\mathcal{H}_m = L(v)$$

for each $v \in W^{1,2}(G)$.

Lemma 9. *Let $\mu \in \mathcal{C}_c'(\partial G)$. If G is bounded suppose that $\mu(\partial G) = 0$. Then there is a unique bounded linear functional L_μ on $W^{1,2}(G)$ such that*

$$L_\mu(\varphi) = \int_{\partial G} \varphi \,\mathrm{d}\mu \tag{6}$$

for each $\varphi \in \mathcal{D}$.

P r o o f. According to Theorem 2 and Theorem 1 there is $\nu \in \mathcal{C}'_c(\partial G)$ such that $N^G\mathcal{U}\nu = \mu$. Fix $\psi \in \mathcal{D}$ such that $\psi = 1$ in a neighbourhood of ∂G. If $\varphi \in \mathcal{D}$ then

$$\begin{aligned}\int_{\partial G} \varphi \,\mathrm{d}\mu &= \int_{\partial G} \psi\varphi \,\mathrm{d}N^G\mathcal{U}\nu = \int_G \nabla(\psi\varphi)\cdot\nabla\mathcal{U}\nu \,\mathrm{d}\mathcal{H}_m \\ &\leqslant \sup|\psi| \Big(\int_{G\cap \operatorname{spt}\psi} |\nabla\varphi|^2 \,\mathrm{d}\mathcal{H}_m\Big)^{1/2} \Big(\int_{G\cap \operatorname{spt}\psi} |\nabla\mathcal{U}\nu|^2 \,\mathrm{d}\mathcal{H}_m\Big)^{1/2} \\ &\quad + \sup|\nabla\psi| \Big(\int_{G\cap \operatorname{spt}\psi} |\varphi|^2 \,\mathrm{d}\mathcal{H}_m\Big)^{1/2} \Big(\int_{G\cap \operatorname{spt}\psi} |\nabla\mathcal{U}\nu|^2 \,\mathrm{d}\mathcal{H}_m\Big)^{1/2} \\ &\leqslant C\|\varphi\|_{W^{1,2}(G)},\end{aligned}$$

where

$$C = 2(\sup|\psi| + \sup|\nabla\psi|)\Big(\int_{G\cap \operatorname{spt}\psi} |\nabla\mathcal{U}\nu|^2 \,\mathrm{d}\mathcal{H}_m\Big)^{1/2} < \infty$$

by Lemma 7. According to the Hahn-Banach theorem there is a bounded linear functional L_μ on $W^{1,2}(G)$ such that (6) holds. Since $\mathcal{D}$ is dense in $W^{1,2}(G)$, the functional L_μ is unique. □

Theorem 3. *Let $\mu \in \mathcal{C}'_0(\partial G) \cap \mathcal{C}'_c(\partial G)$. If G is unbounded suppose moreover that $m > 2$ and $\mu(\mathbb{R}^m) = 0$ for $3 \leqslant m \leqslant 4$. Then there is a weak solution $u \in W^{1,2}(G)$ of the Neumann problem for the Laplace equation with the boundary condition L_μ. If $G_1, \ldots, G_k$ are all components of G then the general solution of this problem has the form (2), where ν is given by (3) and $c_j = 0$ for G_j unbounded while c_j is arbitrary constant for G_j bounded.*

P r o o f. Let ν be given by (3). Then $N^G\mathcal{U}\nu = \mu$ and $\nu \in \mathcal{C}'_C(\partial G)$ by Theorem 1, Theorem 2. If $\mu(\mathbb{R}^m) = 0$ then $\nu(\mathbb{R}^m) = 0$, because $N^G\mathcal{U}\mu(\mathbb{R}^m) = 0$ by [22], Lemma 9. According to Lemma 8 we have $\mathcal{U}\nu \in W^{1,2}(G)$. For a fixed $v \in W^{1,2}(G)$ choose $\varphi_n \in \mathcal{D}$ such that $\varphi_n \to v$ in $W^{1,2}(G)$ as $n \to \infty$. Then

$$L_\mu(v) = \lim_{n\to\infty} \int \varphi_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_G \nabla\varphi_n \cdot \nabla\mathcal{U}\nu \,\mathrm{d}\mathcal{H}_m = \int_G \nabla v \cdot \nabla\mathcal{U}\nu \,\mathrm{d}\mathcal{H}_m.$$

$\mathcal{U}\nu$ is a weak solution of the Neumann problem for the Laplace equation with the boundary condition L_μ. If u has the form (2), where $c_j = 0$ for G_j unbounded, then u is a weak solution of the Neumann problem for the Laplace equation with the boundary condition L_μ.

Let $u \in W^{1,2}(G)$ be a weak solution of the Neumann problem for the Laplace equation with the boundary condition L_μ. Since $u - \mathcal{U}\nu \in W^{1,2}(G)$ we have

$$0 = \int_G \nabla u \cdot \nabla(u - \mathcal{U}\nu)\,\mathrm{d}\mathcal{H}_m - \int_G \nabla\mathcal{U}\nu \cdot \nabla(u - \mathcal{U}\nu)\,\mathrm{d}\mathcal{H}_m = \int_G |\nabla(u - \mathcal{U}\nu)|^2 \,\mathrm{d}\mathcal{H}_m.$$

Since $(u - \mathcal{U}\nu)$ is locally constant on G, u has the form (2). □

Theorem 4. *Let L be a bounded linear functional on $W^{1,2}(G)$ and let $\mu \in \mathcal{C}'(\partial G)$ be such that $L(\varphi) = \int \varphi \,\mathrm{d}\mu$ for each $\varphi \in \mathcal{D}$. If $u \in W^{1,2}(G)$ is a weak solution of the Neumann problem for the Laplace equation with the boundary condition L, then u is continuously extendible to the closure of G if and only if $\mu \in \mathcal{C}'_c(\partial G)$.*

Proof. Since $N^G u = \mu$, [22], Theorem 1 yields that $\mu \in \mathcal{C}'_0(\partial G)$. If u is continuously extendible to the closure of G then $\mu \in \mathcal{C}'_c(\partial G)$ by Theorem 2. Suppose now that $\mu \in \mathcal{C}'_c(\partial G)$. If G is bounded put $\tilde{G} = G$, $\tilde{\mu} = \mu$. If G is unbounded fix $R > 0$ such that $\partial G \subset \Omega_R(0)$ and put $\tilde{G} = G \cap \Omega_R(0)$, $\tilde{\mu} = \mu + \frac{\partial u}{\partial n}(\mathcal{H}_{m-1}/\partial\Omega_R(0))$. Since $V^G < \infty$ we have $V^{\tilde{G}} < \infty$. Since $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ and $(N^H\mathcal{U} - \frac{1}{2}I)$ is compact for each bounded open set H with a smooth boundary, [20], Theorem 2.3 yields that $r_{\mathrm{ess}}(N^{\tilde{G}}\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. Since $N^{\tilde{G}}u = \tilde{\mu}$, [22], Theorem 1 yields that $\mu \in \mathcal{C}'_0(\partial G)$. If G is unbounded then $\frac{\partial u}{\partial n}(\mathcal{H}_{m-1}/\partial\Omega_R(0)) \in \mathcal{C}'_c(\partial\tilde{G})$ by [21], Remark 6 and therefore $\tilde{\mu} \in \mathcal{C}'_c(\partial\tilde{G})$. Since u is a weak solution of the Neumann problem for the Laplace equation on $\tilde{G}$ with the boundary condition $L_{\tilde{\mu}}$, Theorem 3 and Theorem 2 yield that u is continuously extendible to the closure of $\tilde{G}$. □

Remark on the Boundary Element Method. Let H be a bounded domain in $\mathbb{R}^m$ ($m = 2$ or 3) with a piecewise-smooth boundary, let f be a bounded measurable function on ∂H. We want to solve the Neumann problem for the Laplace equation with the boundary condition f. Denote by $\mathcal{H}$ the surface measure on ∂H. Since $\mathcal{U}(f\mathcal{H})$ is a continuous function in $\mathbb{R}^m$ (see [15], Lemma 2.18), there is $u \in \mathcal{C}(\partial H)$ which is a solution of the Neumann problem for the Laplace equation with the boundary condition $f\mathcal{H}$ in the sense of distributions (see Theorem 2). According to Lemma 4

$$u(x) = \mathcal{U}(f\mathcal{H})(x) - \mathcal{D}u(x)$$

for each $x \in H$. Using the boundary behaviour of a double layer potential with a continuous density ([15], Chapter 2), we get for $x \in \partial H$

$$u(x) = \mathcal{U}(f\mathcal{H})(x) - \mathcal{D}u(x) + d_{\mathbb{R}^m \setminus \mathrm{cl}\, H}(x)u(x).$$

Therefore, the equation

$$d_H(x)u(x) + \mathcal{D}u(x) = \mathcal{U}(f\mathcal{H})(x),$$

which is the starting point of the boundary element method, holds and there is a continuous solution u of this equation.

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