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MODULES WITH THE DIRECT SUMMAND SUM PROPERTY

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Abstract. The present work gives some characterizations of R-modules with the direct summand sum property (in short DSSP), that is of those R-modules for which the sum of any two direct summands, so the submodule generated by their union, is a direct summand, too. General results and results concerning certain classes of R-modules (injective or projective) with this property, over several rings, are presented.

Keywords: modules, direct summands, sum property, Artinian rings

MSC 2000: 16D10, 16D40, 16D50, 16D60, 16D70

1. Preliminaries

In [11] we have proposed the following open problem for solving: "Characterize the R-modules (the abelian groups) in which the sum of two direct summands is again a direct summand." This problem is the dual of Kaplansky's ([6, ex. 51, p. 49]) and Fuchs's ([4, problem 9, p. 96]) problems. The first solutions to this problem were obtained in [11]. The present work gives other solutions of this problem, that is, other characterizations of R-modules with the direct summand sum property (in short DSSP), that is of those R-modules for which the sum of any two direct summands, so the submodule generated by their union, is a direct summand, too. Throughout this paper we will denote by R an associative ring with unity, the modules, when not specified, will be considered left over these rings. Other (supplementary) conditions about the ring R or the R-modules will be imposed when needed.

The paper is structured in two sections: in this first section we present the definitions and the results obtained in [11] concerning the R-modules with DSSP that we need here, while in the second section the results of general character and results concerning certain classes of R-modules with DSSP are presented.

Definitions. If M is an R-module, we say that M has

1) the direct summand intersection property (in short DSIP) if the intersection of any two direct summands of M is a direct summand, too;

2) the strong direct summand intersection property (in short SDSIP) if the intersection of any number of direct summands of M is again a direct summand of M;

3) the direct summand sum property (in short DSSP) if the sum (that is the submodule of M generated by the union) of any two direct summands of M is a direct summand, too;

4) the strong direct summand sum property (in short SDSSP) if the sum (that is the submodule of M generated by the union) of any number of direct summands of M is again a direct summand of M.

Remark 1.1. If an R-module has SDSIP, it also has DSIP; the converse is generally false (see [12, p. 32]).

Remark 1.2. If an R-module has SDSSP, it also has DSSP; the converse is generally false.

Proof. Let R be a left hereditary non-Noetherian ring. Then there is an infinite family $\{M_i\}_{i\in I}$ of injective R-modules such that $\bigoplus_{i\in I} M_i$ is not injective. By Zorn's Lemma, choose such an independent family. Then the R-module $M = \prod_{i\in I} M_i$ is injective and has DSSP (see (2.11)), but $\sum_{i\in I} M_i = \bigoplus_{i\in I} M_i$ is not a direct summand in M. It follows that M does not have SDSSP. □

We will present further on the principal results obtained in solving the problem of the R-modules with DSSP, results published in [11], and those needed here.

(1.3) Let M be an R-module and let $S_M = \{T \leqslant M \mid T \text{ is a direct summand in } M\}$. If M has both DSIP and DSSP then S_M is a lattice, that is S_M is a sublattice of the lattice $S(M)$ of all submodules of M. If M has either SDSIP or SDSSP then S_M is a complete lattice, that is S_M is a complete sublattice of $S(M)$.

(1.4) Let R be a principal ideal ring, in particular a local Dedekind domain, and let M be an R-module which has a non-null divisible submodule. If M has DSIP then S_M is a complete lattice.

(1.5) Let R be an Artinian ring. Then the following statements are equivalent:

a) All injective R-modules have DSIP.

b) The ring R is (left) hereditary.

c) All injective R-modules have DSSP.

(1.6) The statement from (1.5) is not valid for all Noetherian rings; for example: the ring $\mathbb{Z}$ of integers is a hereditary Noetherian ring and there are divisible abelian groups which do not have DSIP.

(1.7) Let R be an Artinian domain. Then the following statements are equivalent:
a) All injective R-modules have SDSIP.
b) All injective R-modules have DSIP.
c) The ring R is (left) hereditary.
d) For all injective R-modules M, S_M is a complete lattice.
e) All injective R-modules have DSSP.
f) Every injective R-module M is either
  i) torsion-free, or
  ii) of torsion, and every indecomposable direct summand of M is fully invariant.

2. Modules (and rings) with DSSP

In this section we will present a series of results of general character, concerning the R-modules with DSSP. We begin our investigations with a few results analogous to those for R-modules with DSIP presented in [2], [5] and/or [12].

Remark 2.1. If the R-module M has DSSP (SDSSP), then every direct summand of M also has DSSP (respectively SDSSP).

Proof. Let M be an R-module with DSSP and let A be a direct summand in M. If T and S are two direct summands in A, then $T+S$ is a direct summand in M, but contained in A. It follows that $T+S$ is a direct summand in A and A has DSSP. The proof for SDSSP is similar. □

Proposition 2.2. *Let M be an R-module. Then M has DSSP if and only if for every pair of direct summands T and S, $\pi^{-1}(\pi(T))$ is a direct summand of M, where $\pi\colon M\to S$ is the canonical projection of M along S.*

Proof. We suppose that M has DSSP. If T and S are direct summands of M and $\pi\colon M\to S$ is the canonical projection of M along S, then $\pi^{-1}(\pi(T))=T+S'$ is a direct summand in M, where S' is a complement of S in M. Conversely, if $M=S\oplus S'=T\oplus T'$ and $\varrho\colon M\to S'$ is the canonical projection of M along S', then $\varrho^{-1}(\varrho(T))=T+S$ is a direct summand in M and thus M has DSSP. □

The converse of (2.1) is true for fully invariant direct summands.

Lemma 2.3. *Let $M = \bigoplus_{i\in I} M_i$ be an R-module, where for every $i \in I$, M_i is fully invariant in M. Then M has* DSSP *(SDSSP) if and only if for every $i \in I$, M_i has DSSP (respectively SDSSP).*

P r o o f. We suppose that M has DSSP. By virtue of (2.1), for every $i \in I$, M_i has DSSP. Conversely, we suppose that for every $i \in I$, M_i has DSSP. Let T and S be two direct summands in M, $M = S \oplus S' = T \oplus T'$. Then, according to the hypothesis, $M_i = (S \cap M_i) \oplus (S' \cap M_i) = (T \cap M_i) \oplus (T' \cap M_i)$ for every $i \in I$. It follows that $M = \bigoplus_{i\in I}[(S \cap M_i) \oplus (S' \cap M_i)] = \Big[\bigoplus_{i\in I}(S \cap M_i)\Big] \oplus \Big[\bigoplus_{i\in I}(S' \cap M_i)\Big]$, and $S = \bigoplus_{i\in I}(S \cap M_i)$. Analogously we obtain that $T = \bigoplus_{i\in I}(T \cap M_i)$. It follows that $T + S = \Big[\bigoplus_{i\in I}(S \cap M_i)\Big] + \Big[\bigoplus_{i\in I}(T \cap M_i)\Big] = \bigoplus_{i\in I}[(S \cap M_i) + (T \cap M_i)] = \bigoplus_{i\in I} D_i$, where $D_i = (S \cap M_i) + (T \cap M_i)$ is, according to the hypothesis, a direct summand in M_i. Hence $T + S$ is a direct summand in M and thus M has DSSP. The proof for SDSSP is similar. □

Corollary 2.4. *Let R be a principal ideal domain and P the set of all unassociated prime elements from R. If $M = \bigoplus_{p\in P} M_p$ is a torsion R-module, decomposed according to* [8, 6.11.3], *then M has DSSP (SDSSP) if and only if for every $p \in P$, M_p has DSSP (respectively SDSSP).*

P r o o f. Let the ring R and the R-module $M = \bigoplus_{p\in P} M_p$ be the same as in the statement. Since M_p is fully invariant in M for every $p \in P$, we can apply (2.3). □

Proposition 2.5. *If the R-module M has DSSP, then the following statements hold:*

1) *For every decomposition $M = A \oplus B$ and every homomorphism $f\colon A \to B$,* $\operatorname{Im} f$ *is a direct summand in B.*
2) *If A and B are indecomposable R-modules and $A \oplus B$ is a direct summand in M, then either*
   i) $\operatorname{Hom}(A, B) = 0$ *or*
   ii) *if $0 \neq f \in \operatorname{Hom}(A, B)$ then f is an epimorphism.*

P r o o f. 1) Let S be the submodule of M generated by the set $\{x + f(x) \mid x \in A\}$. Then $S + B = S \oplus B = A \oplus B = M$, since $S \cap B = 0$. So $S + A = A + \operatorname{Im} f = A \oplus \operatorname{Im} f$ is a direct summand in M. It follows that $\operatorname{Im} f$ is a direct summand of M, which is contained in B; so $\operatorname{Im} f$ is a direct summand in B.

2) Let A and B be the same two R-modules as in the statement and let $0 \neq f \in \operatorname{Hom}(A, B)$. Then, according to the hypothesis and to what has been proved in point 1), $\operatorname{Im} f = B$. □

Remark 2.6. The converse of (2.5) 1) is generally false.

P r o o f. Indeed, let R be a Noetherian ring which is not hereditary. Then, according to (2.11), there is an injective R-module M which does not have DSSP, but which can satisfy the conditions from (2.5) 1). □

As in [12], using (2.5) we can classify some rings R in terms of which R-modules have DSSP, and we can improve these results.

Theorem 2.7. *The following statements are equivalent for a ring R:*

1) *R is Artinian semi-simple.*
2) *All R-modules have SDSSP.*
3) *All R-modules have DSSP.*
4) *All projective R-modules have DSSP.*

P r o o f. It is obvious that 1) implies 2) implies 3) implies 4). We are going to show that 4) implies 1). Let P be a projective R-module and let N be a submodule of P. Choose a free R-module F and an epimorphism $f\colon F \to N$. According to the hypothesis, $F \oplus P$ has DSSP. So $N = \operatorname{Im} f$ is a direct summand in P. It follows that any submodule of P is a direct summand in P. According to [1, 9.6], P and any quotient R-module of P are semi-simple R-modules, since any homomorphic image of a semi-simple R-module is again a semi-simple R-module (see [10, 3.6]). Since each R-module is isomorphic to a quotient module of a projective R-module, it follows that, in our case, each R-module is isomorphic to a semi-simple R-module; so R is semi-simple. In this case any R-module is injective; let M be such an R-module and let T and S be two submodules of M. Then $T \cap S$ is a submodule of M; so $T \cap S$ is a direct summand in M. It follows that $T \cap S$ is injective and M satisfies the conditions from [3, Theorem 8, p. 62]. According to [3, p. 63], R is Artinian. □

Now the result from [12, Proposition 3.b] can be improved:

Corollary 2.8. *The following statements are equivalent for a ring R:*

1) *R is Artinian semi-simple.*
2) *All R-modules have SDSSP.*
3) *All R-modules have DSSP.*
4) *All projective R-modules have DSSP.*
5) *All R-modules have SDSIP.*
6) *All R-modules have DSIP.*
7) *All injective R-modules have DSIP.*
8) *For all R-modules M, $S_M (= S(M))$ is a complete lattice.*
9) *For all R-modules M, $S_M (= S(M))$ is a lattice.*

P r o o f. The equivalence of these statements follows from (2.7), (1.3) and from [12, Proposition 3.b)]. □

Corollary 2.9. *If all projective R-modules have DSSP, then R is left hereditary.*

P r o o f. Any semi-simple ring is left hereditary according to [9, p. 73]. (Otherwise: it follows from the proof of the above theorem that any submodule of a projective R-module is, in its turn, projective; therefore R is left hereditary according to [9, 4.10]). □

Remark 2.10. The converse of (2.9) is generally false, since if R is left hereditary, then the sum of any two direct summands of a projective R-module M is a projective submodule of M, which is not necessarily a direct summand (in M); in fact not any left hereditary ring is semi-simple (see $\mathbb{Z}$).

Using [3, Proposition 10, p. 62], for injective R-modules it can be easily proved that the statements from points (1.5) b) and (1.5) c) are equivalent for any ring R. So we have the following result:

Theorem 2.11. *The following statements are equivalent for a ring R:*
a) *All injective R-modules have DSSP.*
b) *R is left hereditary.*

For Noetherian rings R, all R-modules have a unique maximal injective direct summand if and only if R is left hereditary (see [13, Theorem 2]). Now we are going to show that over any Noetherian ring, modules with DSSP have a unique direct summand of this kind, a result which is analogous to the one in [12, Proposition 5].

Theorem 2.12. *Let M be a module over a Noetherian ring R. If M has DSSP, then M has a unique maximal injective direct summand.*

P r o o f. According to Zorn's Lemma, we can choose a maximal independent set $\{E_i\}_{i\in I}$ of indecomposable injective submodules of M. Since R is Noetherian, $E = \bigoplus_{i\in I} E_i$ is injective too and so E is a direct summand in M. We claim that E contains all injective submodules of M. Let F be an injective submodule of M. According to the hypothesis, $E+F$ is a direct summand in M. Suppose that $F \not\subseteq E$. Then $E+F = E\oplus G$ with $G \neq 0$—a direct summand in M. It follows that $F\setminus E \subseteq G$. Let $x \in F \setminus E$ and let F_1 be the least direct summand of F which contains x. Then F_1 is not a direct summand in E, but F_1 has a direct summand in G. In this case the set $\{E_i\}_{i\in I}$ does not contain all indecomposable direct summands of F_1; so we have obtained a contradiction to the choice of $\{E_i\}_{i\in I}$.

It follows that $F \subseteq E$ and E is the unique maximal injective direct summand of M. □

Now we prove the following

Proposition 2.13. *Let R be a commutative Artinian ring and let E_1 and E_2 be two indecomposable injective R-modules such that E_1 is isomorphic to E_2 and $E_1 \oplus E_2$ has DSSP. Then there is a prime ideal P of R such that for every $0 \neq x \in E_1$, $\mathrm{Ann}(x) = P$. ($\mathrm{Ann}(x)$ is the annihilator of x.)*

P r o o f. Let $f\colon E_1 \to E_2$ be an isomorphism of R-modules. We suppose that there are $x, y \in E_1 \setminus \{0\}$ such that $\mathrm{Ann}(x) \neq \mathrm{Ann}(y)$. We consider $a \in \mathrm{Ann}(x) \setminus \mathrm{Ann}(y)$ and define $g\colon E_1 \to E_2$ by: for every $m \in E_1$, $g(m) = f(am)$. It is obvious that g is a homomorphism of R-modules. According to the hypothesis and to (2.5) 1), $\mathrm{Im}\, g$ is a direct summand in E_2, so either $\mathrm{Im}\, g = 0$ or $\mathrm{Im}\, g = E_2$. Let us remark that $g(x) = f(ax) = f(0) = 0$ and $g(y) = f(ay) \neq 0$. Hence g is neither null nor a monomorphism. It follows that $\mathrm{Im}\, g = E_2$, so g is an epimorphism. Then $f^{-1}g$ is an epimorphism, too. Since R is Artinian, according to [10, p. 120] E_1 is a Noetherian R-module. According to the hypothesis and to [8, 6.5.8] it follows that $f^{-1}g$ is an automorphism; so g is a monomorphism and $\ker g = 0$, which is impossible, since $\ker g \neq 0$. Hence all elements of $E_1 \setminus \{0\}$ have the same annihilator; let it be P. So $P = \mathrm{Ann}(E_1 \setminus \{0\})$. Let $m \in E_1 \setminus \{0\}$ and let us suppose that $rs \in P$, and $r \notin P$. Then $rm \neq 0$ and $P \subseteq \mathrm{Ann}(rm)$ for every $m \in E_1 \setminus \{0\}$. But $\mathrm{Ann}(rm) = P$ and since $rsm = 0$, it follows that $s \in P$. Therefore P is a prime ideal of R. □

Now, for Artinian rings, the result from [12, Proposition 6] can be improved in the following way:

Theorem 2.14. *Let R be a commutative Artinian ring and let E be an injective R-module. The following statements are equivalent:*

1) *E has DSIP.*
2) *E has SDSIP.*
3) *E has SDSSP.*
4) *E has DSSP.*

P r o o f. According to [10, p. 78], [12, Proposition 6], [7, 1.4.47] and (1.2), we have that 1) is equivalent to 2) which is equivalent to 3) which implies 4). So we are going to show only that 4) implies 3). Let E be an injective R-module with DSSP. Then $E = \bigoplus_{i \in I} E_i$, where for every $i \in I$, E_i is an indecomposable injective R-module of E. Let $J_i = \{k \in I \mid E_k \cong E_i\}$. Then we obtain the following equivalence relationship over I, denoted by "$\approx$": $i_1 \approx i_2$ if and only if $E_{i_1} \cong E_{i_2}$, and $\{J_i\}_{i \in I}$ is the partition corresponding to "$\approx$" over I. So $E = \bigoplus_{i \in I} E_i^*$, where $E_i^* =$

$\bigoplus_{k\in J_i} E_k$. Since $\operatorname{Hom}(E^*_{i_1}, E^*_{i_2}) = \operatorname{Hom}\Big(\bigoplus_{k\in J_{i_1}} E_k, \bigoplus_{l\in J_{i_2}} E_1\Big)$ is isomorphically embedded in $\operatorname{Hom}\Big(\bigoplus_{k\in J_{i_1}} E_k, \prod_{l\in J_{i_2}} E_l\Big) = \prod_{k\in J_{i_1}} \prod_{l\in J_{i_2}} \operatorname{Hom}(E_k, E_1) = 0$, according to [4, 43.1], [4, 43.2] and (2.5) 2) we obtain that for every i_1 and i_2 which are not equivalent, $E^*_{i_1}$ and $E^*_{i_2}$ are fully invariant. According to (2.3), it suffices to show that each E^*_i has SDSSP. So, for every $i \in I$, E^*_i is a direct sum of isomorphic indecomposable injective submodules. If E^*_i is indecomposable, then it has SDSSP. If E^*_i is not indecomposable, then there is a prime ideal P of R such that $E_k = E(R/P)$ for every $k \in J_i$ and $\operatorname{Ann}(x) = P$ for every $x \in E_k \setminus \{0\}$ according to [10, Theorem 2.32, Corollary] and (2.13). Then, for every $k \in J_i$, E_k is a torsion-free injective module over the domain R/P. It follows that for every $k \in J_i$, E_k is isomorphic to the quotient field of R/P. Under these conditions $E^*_i = \bigoplus_{k\in J_i} E_k = \bigoplus_{k\in J_i} E(R/P)$ is a vector space over this field and thus E^*_i has SDSSP, too. □

Remark 2.15. Let M be an indecomposable R-module and let $M^* = M \oplus M$. Then the following statements hold:

i) If M^* has DSIP, then each $0 \neq f \in \operatorname{End}(M)$ is a monomorphism.
ii) If M^* has DSSP, then each $0 \neq f \in \operatorname{End}(M)$ is an epimorphism.
iii) If M^* has both DSIP and DSSP, then $\operatorname{End}(M)$ is a division ring.

P r o o f. Let M be an R-module as in the statement.

i) If M^* has DSIP, then, according to [5, 1.4], for every endomorphism f of M, $\ker f$ is a direct summand in M. So, either $\ker f = 0$ or $\ker f = M$, that is either f is a monomorphism or $f = 0$.

ii) We can apply (2.5) 2) for $A = B = M$.

iii) The statement of this point follows from what we have proved in points i) and ii).

□

From (1.7), (2.14) and (2.15) we obtain

Corollary 2.16. *The following statements are equivalent for a commutative Artinian ring R:*

1) *R is semi-simple.*
2) *All R-modules have SDSSP.*
3) *All R-modules have DSSP.*
4) *All projective R-modules have DSSP.*
5) *All R-modules have SDSIP.*
6) *All R-modules have DSIP.*
7) *All injective R-modules have DSIP.*

8) *All injective R-modules have SDSIP.*
9) *All injective R-modules have DSSP.*
10) *All injective R-modules have SDSSP.*
11) *The ring R is left hereditary.*
12) *For all R-modules M, S_M is a complete lattice.*
13) *For all R-modules M, S_M is a lattice.*
14) *For all injective R-modules M, S_M is a complete lattice.*
15) *For all injective R-modules M, S_M is a lattice.*
16) *Every injective R-module M is either*
    i) *torsion-free and for every indecomposable direct summand A of M,* $\mathrm{End}(A)$ *is a division ring, or*
    ii) *of torsion, and every indecomposable direct summand of M is fully invariant.*

At the end of this section we are going to see under what conditions the ring $E = \mathrm{End}(M)$ of all endomorphisms of an R-module M has DSSP. To this aim, we will first prove the following technical result:

Lemma 2.17. *If π_1, π_2 and π are three idempotent endomorphisms of an R-module M such that $\pi_1 M + \pi_2 M = \pi M$, then $\pi_1 E + \pi_2 E = \pi E$, where $E = \mathrm{End}(M)$.*

Proof. First, we remark that for every idempotent $\alpha \in E$, $\alpha(M) = (\alpha E)M$. Since $\pi_1 M + \pi_2 M = \pi M$, it follows that $\pi_1 M \subseteq \pi M$ and $\pi_2 M \subseteq \pi M$. Then $(\pi_1 E)M \subseteq (\pi E)M$ and $(\pi_2 E)M \subseteq (\pi E)M$. It follows that $\pi_1 E \subseteq \pi E$ and $\pi_2 E \subseteq \pi E$; therefore

$$\pi_1 E + \pi_2 E \subseteq \pi E. \tag{1}$$

Since $(\pi_1 E)M + (\pi_2 E)M = (\pi E)M$, it follows that

$$\pi E \subseteq \pi_1 E + \pi_2 E. \tag{2}$$

From the relationships (1) and (2) we obtain the desired equality.

□

Now, we can prove a result analogous to [2, Theorem].

Theorem 2.18. *An R-module M has DSSP if and only if*
(i) *$E = \mathrm{End}(M)$ has DSSP, as a right E-module, and*
(ii) *for all idempotents π and ϱ in E, $\pi M + \varrho M = (\pi E + \varrho E)M$.*

P r o o f. We suppose that M has DSSP. Then, for every π_1 and π_2-idempotents in E, there is a π-idempotent in E such that $\pi_1 M + \pi_2 M = \pi M$. Then, according to (2.17), $\pi_1 E + \pi_2 E = \pi E$ and $\pi_1 M + \pi_2 M = \pi M = (\pi E)M = (\pi_1 E + \pi_2 E)M$.

Conversely, we suppose that the statements (i) and (ii) hold and let T and S be two direct summands of M. If $\pi_1\colon\ M \to T$ and $\pi_2\colon\ M \to S$ are the canonical projections of M along T and S respectively, then $\pi_1 E$ and $\pi_2 E$ are direct summands in E. According to the hypothesis, there is an idempotent $\pi \in E$ such that $\pi_1 E + \pi_2 E = \pi E$. Then $\pi M = (\pi E)M = (\pi_1 E + \pi_2 E)M = \pi_1 M + \pi_2 M = T + S$ is a direct summand in M. Therefore M has DSSP. □

For the rings with DSSP we have

Proposition 2.19. *If a ring R has DSSP as a right R-module, then the following statements hold:*

(i) *For every idempotent $e \in R$ and every $r \in (1-e)Re$, the right ideal rR is projective.*

(ii) *For every idempotent $e \in R$ and every $r, s \in (1-e)Re$, $rR + sR = (r+s)R \oplus L$, where L is a direct summand in R with the property that $rL = sL = 0$.*

P r o o f. (i) We observe that in this case $R = \operatorname{End}_R(R_R)$. If $e = e^2 \in R$ and $r \in (1-e)Re$, then $r^2 = 0$ (which can be checked immediately) and there is a direct decomposition of R which assumes the form $R = I \oplus J$ with $rR = rI \subseteq J$ and $rJ = 0$. According to the hypothesis and to (2.5) 1), rI is a direct summand in J. If $R = I \oplus rI \oplus K$, where K is a direct summand in J with the property that $rK = 0$, then rR is a direct summand in R. It follows that rR is a projective ideal of R.

(ii) According to what we have proved in point (i), for every $e \in R$ and every $r, s \in (1-e)Re$, the ideals rR and sR are direct summands in R. It can be easily proved that then $r + s \in (1-e)Re$ and

$$rs = sr = 0; \tag{3}$$

so $(r+s)R$ is a direct summand, too (in R), contained in the direct summand $rR+sR$. It follows that

$$rR + sR = (r+s)R \oplus L, \tag{4}$$

where L is a direct summand in R. From the relationships (3) and (4) we obtain that $rL = sL = 0$.

□

Let M and N be two R-modules. If we denote by $S_M(N)$ the M-socle of N, that is the sum of all homomorphic images of M in N, then (2.19) and [2, p. 523] yield

Corollary 2.20. *Let M be an R-module. If the ring $E = \mathrm{End}_R(M)$ has* DSSP *as a right E-module, then the following statements hold:*

(i) *For every $\pi = \pi^2 \in E$ and every $\varepsilon \in (1-\pi)E\pi$, $S_M(\ker\varepsilon)$ is a direct summand in M.*

(ii) *For every $\pi = \pi^2 \in E$ and every $\sigma, \tau \in (1-\pi)E\pi$, $\sigma E + \tau E = (\sigma+\tau)E \oplus L$, where L is a direct summand in E with the property that $\sigma L = \tau L = 0$.*

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