Ladislav Bican; Blas Torrecillas
Precovers

PRECOVERS

Ladislav Bican, Praha, and Blas Torrecillas, Almería

Abstract. Let $\mathscr{G}$ be an abstract class (closed under isomorpic copies) of left R-modules. In the first part of the paper some sufficient conditions under which $\mathscr{G}$ is a precover class are given. The next section studies the $\mathscr{G}$-precovers which are $\mathscr{G}$-covers. In the final part the results obtained are applied to the hereditary torsion theories on the category on left R-modules. Especially, several sufficient conditions for the existence of σ-torsionfree and σ-torsionfree σ-injective covers are presented.

Keywords: precover, cover, (pre)cover class of modules, hereditary torsion theory, relatively injective modules

MSC 2000: 16D90, 16S90, 16D50

Throughout this paper R denotes a ring with identity and $\sigma = (\mathscr{T}, \mathscr{F})$ a hereditary torsion theory in the category of left R modules, R-mod. An R-module M is said to be σ-injective if $\mathrm{Ext}_R^1(T, M) = 0$ for any σ-torsion module T.

In order to study the structure of a module, it is useful to approximate the module using the so-called $\mathscr{G}$-cover, where $\mathscr{G}$ is a class of left R-modules. The crucial question is the existence of such covers (cf. [9]). Associated to a torsion theory σ there exist two important classes of modules, the class of σ-torsionfree modules and the class of σ-torsionfree σ-injective modules (cf. [6], [7], [8]). In this note we consider the problem of existence of covers for a general class of modules and we apply our results to the case of the above mentioned two classes.

1. Auxiliary results

Recall that a class of modules is said to be *abstract*, if it is closed under isomorphic copies, *co-abstract*, if its members are pairwise non-isomorphic, *hereditary*, if it is abstract and closed under submodules and *inductive*, if it is closed under unions of chains. We further say that the homomorphisms $f\colon F \to M$ and $g\colon G \to M$ are *M-equivalent*, if there is an isomorphism $\pi\colon F \to G$ such that $g\pi = f$.

Lemma 1.1. *Let $F = \bigoplus_{\delta \in D} F_\delta$ be a direct sum of modules and $f\colon F \to M$ an arbitrary homomorphism. Then there is a subset $D' \subseteq D$ such that $F = U \oplus V$, where $U = \bigoplus_{\delta \in D'} F_\delta$, $V \subseteq \operatorname{Ker} f$ and for $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$, the homomorphisms $f\big|F_\delta$ and $f\big|F_\varepsilon$ are not M-equivalent.*

P r o o f.   For the sake of simplicity denote $f_\delta = f\big|F_\delta$ for every $\delta \in D$ and we define an equivalence relation $\sim$ on D in such a way that $\delta \sim \varepsilon$ if and only if the homomorphisms f_δ and f_ε are M-equivalent. In this case we denote by $\pi_{\varepsilon\delta}\colon F_\delta \to F_\varepsilon$ the isomorphism for which $f_\varepsilon \pi_{\varepsilon\delta} = f_\delta$. For each $\delta \in D$ let $D_\delta = \{\varepsilon \in D \mid \varepsilon \sim \delta\}$ be the equivalence class containing δ and $D'_\delta = D_\delta \setminus \{\delta\}$. Now for every $\varepsilon \in D'_\delta$ we set $G_{\varepsilon\delta} = \{x - \pi_{\varepsilon\delta}(x) \mid x \in F_\delta\}$ and we are going to verify that

$$\bigoplus_{\varepsilon \in D_\delta} F_\varepsilon = F_\delta \oplus \left(\bigoplus_{\varepsilon \in D'_\delta} G_{\varepsilon\delta} \right).$$

In order to show that the sum on the right is direct, let $x + \sum_{i=1}^{n} y_i \in F_\delta + \sum_{\varepsilon \in D'_\delta} G_{\varepsilon\delta}$ be such that $x \in F_\delta$, $y_i \in G_{\varepsilon_i\delta}$, where $\varepsilon_1, \dots, \varepsilon_n \in D'_\delta$ are pairwise different and $x + \sum_{i=1}^{n} y_i = 0$. There are elements $x_i \in F_\delta$ such that $y_i = x_i - \pi_{\varepsilon_i\delta}(x_i)$, $i = 1, \dots, n$, and so $x + \sum_{i=1}^{n} y_i = x + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \pi_{\varepsilon_i\delta}(x_i) = 0$, which yields $x_i = 0$ for every $i = 1, \dots, n$ and consequently $x = 0$. Now if $x + \sum_{i=1}^{n} y_i$, $x \in F_\delta$, $y_i \in F_{\varepsilon_i}$, $\{\varepsilon_1, \dots, \varepsilon_n\} \subseteq D'_\delta$, are arbitrary, then $y_i = \pi_{\varepsilon_i\delta}(x_i)$ for suitable $x_i \in F_\delta$, $i = 1, \dots, n$, hence $x + \sum_{i=1}^{n} y_i = x + \sum_{i=1}^{n} x_i - \left(\sum_{i=1}^{n} (x_i - \pi_{\delta\varepsilon_i}(x_i)) \right) \in F_\delta \oplus \left(\bigoplus_{\varepsilon \in D'_\delta} G_{\varepsilon\delta} \right)$ and the above equality is proved.

Let $D' \subseteq D$ be any set of representatives of the equivalence classes under $\sim$. Setting $U = \bigoplus_{\delta \in D'} F_\delta$ and $V = \bigoplus_{\delta \in D'} (\bigoplus_{\varepsilon \in D'_\delta} G_{\varepsilon\delta})$, we obviously have $F = U \oplus V$, where $V \subseteq \operatorname{Ker} f$ by the definition of the relation $\sim$. Finally, the M-equivalence of f_δ and f_ε means that $\delta \sim \varepsilon$, which is impossible for $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$. □

Definition 1.2. Let $\mathscr{G}$ be an abstract class of modules and let $\mathscr{G}' = \{G_\alpha \mid \alpha \in A\}$ be a co-abstract subset of $\mathscr{G}$. If $M \in R$-mod is arbitrary, then for every $\alpha \in A$ and $g \in \mathrm{Hom}(G_\alpha, M)$ we denote by $G_{\alpha g}$ an isomorphic copy of G_α. For all subsets $B \subseteq A$ and $H_\alpha \subseteq \mathrm{Hom}(G_\alpha, M)$ we take the direct sum $Y = \bigoplus\limits_{\alpha \in B} \left(\bigoplus\limits_{g \in H_\alpha} G_{\alpha g} \right)$ and denote by $\{X_\gamma \mid \gamma \in C\}$ the set of all modules from $\mathscr{G}$ which lie between Y and $E(Y)$ for some Y, where $E(Y)$ is a fixed injective envelope of Y. Now for each $\gamma \in C$ and each $g \in \mathrm{Hom}(X_\gamma, M)$ we take an isomorphic copy $X_{\gamma g}$ of X_γ together with the isomorphism $\psi_{\gamma g} \colon X_\gamma \to X_{\gamma g}$ and we finally set

(1)
$$ G = G_M = \bigoplus_{\gamma \in C} \left(\bigoplus\nolimits_{g \in \mathrm{Hom}(X_\gamma, M)} X_{\gamma g} \right). $$

Moreover, $\varphi = \varphi_M \colon G_M \to M$ will denote the natural evaluation homomorphism induced by the maps $g\psi_{\gamma g}^{-1} \colon X_{\gamma g} \to M$.

Lemma 1.3. *Let $\mathscr{G}$ be an abstract class of modules and $\mathscr{G}'$ a co-abstract subset of $\mathscr{G}$. Further, let $M \in R$-mod be an arbitrary module, let $\varphi \colon G \to M$ be as in the preceding definition and let $f \colon F \to M$ with $F \in \mathscr{G}$ be an arbitrary homomorphism. If F contains an essential submodule $F' = \bigoplus\limits_{\delta \in D'} F_\delta$, where each F_δ is isomorphic to a member of $\mathscr{G}'$, and for any $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$, the homomorphisms $f\big|F_\delta$ and $f\big|F_\varepsilon$ are not M-equivalent, then there is a homomorphism $g \colon F \to G$ such that $\varphi g = f$.*

P r o o f.   As above, we will use the brief notation $f_\delta = f\big|F_\delta$ for each $\delta \in D'$. If $\{G_\alpha \mid \alpha \in A\}$ is any list of elements of $\mathscr{G}'$ then for each $\delta \in D'$ there is an isomorphism $\theta_{\alpha\delta} \colon F_\delta \to G_\alpha$ which induces isomorphism $\varphi_{\alpha\delta} \colon F_\delta \to G_{\alpha, f_\delta \theta_{\alpha\delta}^{-1}}$. Since the equality $f_\delta \theta_{\alpha\delta}^{-1} = f_\varepsilon \theta_{\beta\varepsilon}^{-1}$ for some $\delta \neq \varepsilon$ in D' yields a contradiction $f_\delta = f_\varepsilon \theta_{\beta\varepsilon}^{-1} \theta_{\alpha\delta}$, the isomorphisms $\varphi_{\alpha\delta}$ induce isomorphism $\psi' \colon F' \to Y$, $Y = \bigoplus\limits_{\delta \in D'} G_{\alpha, f_\delta \theta_{\alpha\delta}^{-1}}$. This isomorphism extends to isomorphism $\psi \colon F \to X_\gamma$ for a suitable $X_\gamma \in \mathscr{G}$ lying between Y and its injective envelope $E(Y)$. Denoting $h = f\psi^{-1} \colon X_\gamma \to M$ and $\iota_{\gamma h} \colon X_{\gamma h} \to G$ the canonical embedding, we can take $g \colon F \to G$ as $g = \iota_{\gamma h} \psi_{\gamma h} \psi$. Then we have $\varphi g = \varphi \iota_{\gamma h} \psi_{\gamma h} \psi = h\psi_{\gamma h}^{-1} \psi_{\gamma h} \psi = h\psi = f$ and the proof is complete. $\qquad\square$

Let $\varphi \colon F \to M$ and $\psi \colon G \to M$ be homomorphisms. We define an *ordering* $\leqslant$ on the class of all pairs (F, φ) in such a way that we put $(F, \varphi) \leqslant (G, \psi)$ if and only if $F \leqslant G$ and $\psi\big|F = \varphi$.

Recall that for an abstract class $\mathscr{G}$ of modules a homomorphism $\varphi \colon G \to M$, $G \in \mathscr{G}$, is a *$\mathscr{G}$-precover* of the module M, if for each $F \in \mathscr{G}$ and each homomorphism $f \colon F \to M$ there is a homomorphism $g \colon F \to G$ such that $\varphi g = f$. A *$\mathscr{G}$-precover* φ

of M is called a $\mathscr{G}$-*cover*, if each endomorphism g of G with $\varphi g = \varphi$ is an automorphism of G.

Lemma 1.4. *Let $\mathscr{G}$ be an abstract class of modules and $\varphi\colon F \to M$ a $\mathscr{G}$-precover of the module M. If $f\colon F \to F$ is a non-surjective monomorphism such that $\varphi f = \varphi$, then there is a $\mathscr{G}$-precover $\varphi_0\colon F_0 \to M$ of M such that $(F, \varphi) < (F_0, \varphi_0)$ and an isomorphism $\sigma\colon F \to F_0$ such that $\varphi_0 \sigma = \varphi$.*

P r o o f.   Using "standard" arguments, we can replace $f(F)$ in F by F and we obtain $F_0 = F \cup Y$, where Y is a copy of $F \setminus f(F)$. Defining σ as the identity map on Y and as f^{-1} on $f(F)$ and φ_0 as φ on Y and as φf on F, one can easily verify all the properties stated. $\qquad\square$

Lemma 1.5. *Let $\mathscr{G}$ be an abstract class of modules and*

$$
\begin{array}{ccc}
G & \xrightarrow{\ \psi\ } & M \\
{\scriptstyle f}\big\downarrow & & \big\| \\
F & \xrightarrow{\ \varphi\ } & M
\end{array}
$$

be a commutative diagram with $F, G \in \mathscr{G}$. If ψ is a $\mathscr{G}$-precover of M, then so is φ.

P r o o f   is obvious. $\qquad\square$

2. Existence of precovers

If $\mathscr{G}$ is an abstract class of modules such that every left R-module has a $\mathscr{G}$-precover, then it is usual to say that $\mathscr{G}$ is a *precover class*. In other words this means that for each $M \in R$-mod there is a module $G \in \mathscr{G}$ and a homomorphism $f\colon F \to M$ such that every homomorphism $f\colon F \to M$, $F \in \mathscr{G}$, factors through φ, i.e. $f = \varphi g$ for some homomorphism $g\colon F \to G$. Rada and Saorín [5, Theorem 3.4] observed that to ensure that every module has a $\mathscr{G}$-precover it suffices to consider any (co-abstract) subset $\mathscr{G}' \subseteq \mathscr{G}$ having the property that every homomorphism $f\colon F \to M$, $F \in \mathscr{G}$, factors through a direct sum of members of $\mathscr{G}'$. We start this section with the simple proof of this fact, namely of [5, Corollary 3.7]. Anyway, this result show that "small" classes $\mathscr{G}$ of modules are precover classes in the sense that $\mathscr{G}$ consists of all direct sums of members of a (co-abstract) subset $\mathscr{G}'$ of $\mathscr{G}$ and their isomorphic copies. For such classes it is usual to use the notation $\mathscr{G} = \mathrm{Coprod}(\mathscr{G}')$. On the other hand, large classes, e.g. $\mathscr{G} = R$-mod, are also precover classes (the identity map 1_M for every module $M \in R$-mod). So, we shall continue in this section with some sufficient conditions for precover classes.

Proposition 2.1. *If $\mathscr{G}'$ is any (co-abstract) set of modules, then $\mathscr{G} = \mathrm{Coprod}(\mathscr{G}')$ is a precover class.*

P r o o f.   We are going to verify that for every module M the homomorphism $\varphi\colon G \to M$ from Definition 1.2 is a $\mathrm{Coprod}(\mathscr{G}')$-precover of M. So, let $f\colon F \to M$, $F = \bigoplus_{\delta \in D} F_\delta$, where F_δ is an isomorphic copy of a member of $\mathscr{G}'$ for each $\delta \in D$, be arbitrary. By Lemma 1.1 there is a subset D' of D such that $F = U \oplus V$, where $U = \bigoplus_{\delta \in D'} F_\delta$ and $V \subseteq \mathrm{Ker}\, f$. By Lemma 1.3 there is $h\colon U \to G$ such that $\varphi h = f|U$ and consequently for $g = h \oplus 0\colon F \to G$ we obviously have $\varphi g = f$. $\square$

We say that a class $\mathscr{G}$ of modules is *$\mathscr{G}$-cohereditary*, if it is closed under factor-modules by submodules lying in $\mathscr{G}$. Further, submodule N of a module M is said to be *$\mathscr{G}$-pure in M*, if the factor-module M/N lies in $\mathscr{G}$.

Theorem 2.2. *Let $\mathscr{G}$ be a $\mathscr{G}$-cohereditary class of modules closed under direct sums and such that the set of $\mathscr{G}$-pure submodules of any module lying in $\mathscr{G}$ is inductive. If $\mathscr{G}'$ is a co-abstract subset of the class $\mathscr{G}$ such that each $F \in \mathscr{G}$ contains an essential submodule isomorphic to a member of $\mathrm{Coprod}(\mathscr{G}')$, then $\mathscr{G}$ is a precover class.*

P r o o f.   Let $M \in R$-mod be arbitrary and let $\varphi\colon G \to M$ be as in Definition 1.2. To show that φ is a $\mathscr{G}$-precover of the module M, let $f\colon F \to M$, $F \in \mathscr{G}$, be an arbitrary homomorphism. The hypothesis yields the existence of a maximal $\mathscr{G}$-pure submodule of F contained in $\mathrm{Ker}\, f$ and as can be easily verified, we may without loss of generality assume that $\mathrm{Ker}\, f$ contains no non-zero submodule which is $\mathscr{G}$-pure in F. By hypothesis and Lemma 1.1 the module F contains an essential submodule F' of the form $F' = U \oplus V$, where $U = \bigoplus_{\delta \in D'} F_\delta$ with no $f|F_\delta$, $f|F_\varepsilon$, $\delta, \varepsilon \in D'$, $\delta \neq \varepsilon$, M-equivalent and $V \subseteq \mathrm{Ker}\, f$. Further, $V \in \mathscr{G}$, $\mathscr{G}$ being abstract and closed under direct sums, and consequently V is $\mathscr{G}$-pure in F owing to the fact that $\mathscr{G}$ is $\mathscr{G}$-cohereditary. Thus $V = 0$, $F' = U$ is essential in F and it suffices to use Lemma 1.3. $\square$

Recall that an abstract class $\mathscr{G}$ of modules is said to be *closed under extensions*, if $G \in \mathscr{G}$ whenever there is $H \leqslant G$ such that both H and G/H belong to $\mathscr{G}$.

Theorem 2.3. *Let $\mathscr{G}$ be an abstract, $\mathscr{G}$-cohereditary and inductive class of modules closed under direct sums and extensions. If $\mathscr{G}'$ is a co-abstract subset of the class $\mathscr{G}$ such that each $F \in \mathscr{G}$ contains an essential submodule isomorphic to a member of $\mathrm{Coprod}(\mathscr{G}')$, then $\mathscr{G}$ is a precover class.*

P r o o f.   Let $M \in R$-mod be arbitrary, $\varphi\colon G \to M$ as in Definition 1.2 and let $f\colon F \to M$, $F \in \mathscr{G}$, be an arbitrary homomorphism. By hypothesis there is an

essential submodule $F' \in \mathscr{G}$ of F which can be by virtue of Lemma 1.1 written in the form $F' = U \oplus V$, $V \subseteq \operatorname{Ker} f$, $V \in \mathscr{G}$. The class $\mathscr{G}$ is inductive and so there is a submodule $V' \subseteq \operatorname{Ker} f$ maximal with respect to $V \subseteq V'$ and $V' \in \mathscr{G}$. By hypothesis, the factor-module $\overline{F} = F/V'$ belongs to $\mathscr{G}$ and f induces $\overline{f} \colon \overline{F} \to M$ naturally in such a way that $\overline{f}\pi = f$, π being the canonical projection $F \to \overline{F}$. Similarly to the case of F there is an essential submodule $\overline{F}' = \overline{U} \oplus \overline{V}$ of $\overline{F}$ with $\overline{V} \subseteq \operatorname{Ker} \overline{f}$. Then $\overline{V} = \tilde{V}/V'$, where $\tilde{V} \subseteq \operatorname{Ker} f$ and $\tilde{V} \in \mathscr{G}$ owing to the fact that $\mathscr{G}$ is closed under extensions. Now the maximality of V' yields $\overline{V} = 0$ and an application of Lemma 1.3 gives the existence of a homomorphism $\bar{g} \colon \overline{F} \to G$ with $\varphi \bar{g} = \overline{f}$, from which the assertion follows easily. $\qquad\qquad\square$

Proposition 2.4. *Let $\mathscr{G}$ be an abstract class of modules closed under injective hulls. If M is an injective module, then a homomorphism $\varphi \colon G \to M$ is a $\mathscr{G}$-precover of M if and only if for every $H \in \mathscr{G}$, H injective, and every homomorphism $f \colon H \to M$ there is a homomorphism $g \colon H \to G$ such that $\varphi g = f$.*

P r o o f.   Only the sufficiency requires verification. So, let $F \in \mathscr{G}$ and $h \colon F \to M$ be arbitrary. If $i \colon F \to E(F) = H$ is the canonical embedding, then there is $f \colon H \to M$ with $fi = h$, M being injective. By hypothesis, there is $g \colon H \to G$ such that $\varphi g = f$. Thus $\varphi g i = fi = h$ and we are through. $\qquad\qquad\square$

We say that a co-abstract set $\mathscr{G}'$ is *closed under injective hulls*, if $\mathscr{G}'$ with each its element contains a copy of its injective hull.

Theorem 2.5. *Let $\mathscr{G}$ be a hereditary class of modules closed under direct sums and injective hulls and let $\mathscr{G}'$ be a co-abstract subset of $\mathscr{G}$ closed under injective hulls. If, for each $F \in \mathscr{G}$, F injective, the set of $\mathscr{G}$-pure submodules is inductive and F contains an essential submodule isomorphic to a member of $\operatorname{Coprod}(\mathscr{G}')$, then every injective module has a $\mathscr{G}$-precover.*

P r o o f.   Let $M \in R\text{-mod}$ be injective and let $\varphi \colon G \to M$ be as in Definition 1.2. By Proposition 2.4 it suffices to test the homomorphism φ by the injective elements of $\mathscr{G}$ only. So, let $f \colon F \to M$, $F \in \mathscr{G}$ injective, be an arbitrary homomorphism. By hypothesis and Lemma 1.1 there is an essential submodule $F' = U \oplus V$ of F such that $V \subseteq \operatorname{Ker} f$ and $U = \bigoplus_{\delta \in D'} F_\delta$ where no different $f|_{F_\delta}$, $f|_{F_\varepsilon}$, $\delta, \varepsilon \in D'$, are M-equivalent.

If $V = 0$ then an application of Lemma 1.3 finishes the proof. Assuming $V \neq 0$ we shall adopt the notation of Lemma 1.1 and its proof. So, there are $\delta \neq \varepsilon$ in D with $\delta \sim \varepsilon$ and consequently $G_{\varepsilon\delta} \cong F_\delta$ is isomorphic to a member of $\mathscr{G}'$. Moreover, since F is injective, we may assume that F_δ and consequently $G_{\varepsilon\delta}$ are also injective. Then $F = G_{\varepsilon\delta} \oplus L$ where L can be taken as the injective hull of the direct sum of

all remaining $F_{\delta'}$ and $G_{\varepsilon'\delta'}$ and so $L \in \mathscr{G}$ by the hypotheses. Thus the isomorphism $F/G_{\varepsilon\delta} \cong L$ shows that $G_{\varepsilon\delta}$ is a $\mathscr{G}$-pure submodule of F contained in $\operatorname{Ker} f$. So, the hypothesis yields the existence of a maximal $\mathscr{G}$-pure submodule K of F contained in $\operatorname{Ker} f$. Denoting $\overline{F} = F/K$ and $\pi\colon F \to \overline{F}$ the canonical projection, there is a natural homomorphism $\overline{f}\colon \overline{F} \to M$ with $\overline{f}\pi = f$. Now if i is the embedding of $\overline{F}$ into its injective envelope $H = E(\overline{F})$ then the injectivity of M yields the extension $f^*\colon H \to M$ of $\overline{f}$, $f^*i = \overline{f}$. By hypothesis and Lemma 1.1 the module H contains an essential submodule $H' = \overline{U} \oplus \overline{V}$ with $\overline{V} \cong \operatorname{Ker} f^*$. Now it remains to verify that $\overline{V} = 0$, since in that case Lemma 1.3 yields the existence of $\overline{g}\colon H \to G$ with $\varphi\overline{g} = f^*$ and consequently $\varphi\overline{g}i\pi = f^*i\pi = \overline{f}\pi = f$.

Proving indirectly let us assume that $\overline{V} \neq 0$. As in the case $V \neq 0$ we can find a non-zero submodule $L \subseteq \operatorname{Ker} f^*$ which is $\mathscr{G}$-pure in H. Then $0 \neq L \cap \overline{F} \subseteq \operatorname{Ker} \overline{f}$ and $\frac{\overline{F}}{L \cap \overline{F}} \cong \frac{\overline{F}+L}{L} \leqslant \frac{H}{L}$ yields that $L \cap \overline{F}$ is $\mathscr{G}$-pure in $\overline{F}$, $\mathscr{G}$ being a hereditary class of modules. Thus we have obtained a $\mathscr{G}$-pure submodule $0 \neq L \cap \overline{F} = S/K$ of $\overline{F} = F/K$ contained in $\operatorname{Ker} \overline{f}$. Hence $S \subseteq \operatorname{Ker} f$, $K \subset S$ and S is $\mathscr{G}$-pure in F since $F/S \cong F/K/S/K \in \mathscr{G}$. This contradicts the maximality of K and completes the proof of the theorem. $\square$

Theorem 2.6. *If $\mathscr{G}$ is a hereditary class of modules, then every module has a $\mathscr{G}$-precover if an only if every injective module has a $\mathscr{G}$-precover.*

Proof. Only the sufficiency requires verification. So, let $M \in R\text{-mod}$ be arbitrary and let $\beta\colon M \to E(M)$ be its injective hull. If $\varphi\colon F \to E(M)$ is a $\mathscr{G}$-precover of $E(M)$, then it is easy to see that in the pullback diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\ \psi\ } & M \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\
F & \xrightarrow{\ \varphi\ } & E(M)
\end{array}
$$

the homomorphism f is injective, hence $G \in \mathscr{G}$ by hypothesis and it is easy to see that the homomorphism $\psi\colon G \to M$ is a $\mathscr{G}$-precover of the module M. $\square$

3. Precovers that are covers

Theorem 3.1. *Let $\mathscr{G}$ be an abstract class of modules. If $\varphi\colon F \to M$ is a $\mathscr{G}$-cover of the module M then in every commutative diagram*

$$(*)\qquad
\begin{array}{ccc}
F & \xrightarrow{\ \varphi\ } & M \\
{\scriptstyle f}\big\downarrow & & \big\| \\
G & \xrightarrow{\ \psi\ } & M
\end{array}
$$

where ψ is a $\mathscr{G}$-precover of M, the homomorphism f is injective. The converse holds if the class $\mathscr{G}$ is inductive.

P r o o f. Since φ is a $\mathscr{G}$-precover of the module M, there is a homomorphism $g\colon G \to F$ such that $\varphi h = \psi$. But then $\varphi h f = \psi f = \varphi$ yields that $h\varphi$ is an automorphism of F and consequently f is a monomorphism.

To prove the converse we will say, for the sake of brevity, that a $\mathscr{G}$-precover φ of the module M has the property $(*)$ if it satisfies the condition of the theorem, i.e. if for any $\mathscr{G}$-precover $\psi\colon G \to M$ of M, every homomorphism $f\colon F \to G$ making the diagram $(*)$ commmutative is injective.

Take any set X with $F \subseteq X$, $|F| < |X|$, and consider the family $\Sigma = \{(F_0, \varphi_0)\}$, where $F_0 \subseteq X$ and $\varphi_0\colon F_0 \to M$ is a $\mathscr{G}$-precover of the module M having the property $(*)$. Since $(F, \varphi) \in \Sigma$, Σ is non-empty and we can define the natural order $\leqslant$ on Σ in such a way that $\{(F_0, \varphi_0)\} \leqslant \{(F_1, \varphi_1)\}$ if and only if $F_0 \subseteq F_1$ and $\varphi_1\big|F_0 = \varphi_0$.

Let us verify, that Zorn's lemma can be applied. If $\{(F_i, \varphi_i) \mid i \in I\} \subseteq \Sigma$ is any chain, set $F^* = \bigcup_{i \in I} F_i$ and define $\varphi^*\colon f^* \to M$ via $\varphi^*(x) = \varphi_i(x)$ whenever $x \in F_i$. Obviously, $F^* \subseteq X$ and $F^* \in \mathscr{G}$ by the hypothesis. To show that φ^* is a $\mathscr{G}$-precover of M it suffices to apply Lemma 1.5 to the commutative diagram

$$
\begin{array}{ccc}
F_i & \xrightarrow{\ \varphi_i\ } & M \\
{\scriptstyle \iota_i}\big\downarrow & & \big\| \\
F^* & \xrightarrow{\ \varphi^*\ } & M
\end{array}
$$

with the inclusion map ι_i, $i \in I$. In order to verify the property $(*)$, consider the commutative diagram

$$
\begin{array}{ccccc}
F_i & \xrightarrow{\ \iota_i\ } & F^* & \xrightarrow{\ \varphi^*\ } & M \\
{\scriptstyle g_i}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\| \\
G & =\!\!=\!\!= & G & \xrightarrow{\ \psi\ } & M
\end{array}
$$

and assume that $\operatorname{Ker} g \neq 0$. Then $\operatorname{Ker} g \cap F_i \neq 0$ for a suitable $i \in I$. Since $\varphi^* \iota_i = \varphi_i$ and $\psi g_i = \psi g \iota_i = \varphi^* \iota_i = \varphi_i$, the homomorphism g_i is injective by the property $(*)$, owing to the fact that $(F_i, \varphi_i) \in \Sigma$. On the other hand, $F_i \cap \operatorname{Ker} g_i = 0$, which contradicts the choice of $i \in I$, and consequently $(F^*, \varphi^*) \in \Sigma$.

Now we are going to verify that φ is a $\mathscr{G}$-cover of the module M. Proving indirectly, let us assume that there exists an endomorphism f of the module F such that $\varphi f = \varphi$, f is injective, but not surjective. By Lemma 1.4 and Zorn's lemma there is a maximal element (F^*, φ^*) of Σ such that $(F, \varphi) < (F^*, \varphi^*)$. By the property $(*)$ there exists a monomorphism $f_1\colon F^* \to F$ with $f_1\varphi = \varphi^*$. Now the composition of f_1 with the

inclusion map $\iota\colon F \to F^*$ yields an injective non-surjective endomorphism ιf_1 of F^* such that $\varphi^*\iota f_1 = \varphi f_1 = \varphi^*$. To obtain the final contradiction with the maximality of (F^*, φ^*) it suffices now to apply Lemma 1.4. $\qquad \square$

As a consequence of this theorem we can easily derive the result [9, Theorem 2.2.8] on the existence of $\mathscr{G}$- covers.

Corollary 3.2. *Let $\mathscr{G}$ be an abstract class of modules closed under direct limits. If a module M has a $\mathscr{G}$-precover, then it has a $\mathscr{G}$-cover.*

P r o o f. Using [9, Lemma 2.2.10] we see that there exists a $\mathscr{G}$-precover of M having the property $(*)$ and Theorem 3.1 applies. $\qquad \square$

Theorem 3.3. *Let $\mathscr{G}$ be an abstract and inductive class of modules and let $\varphi\colon F \to M$ be a $\mathscr{G}$-precover of the module M. If each endomorphism f of F with $\varphi f = \varphi$ is injective and $f(F)$ is essential in F, then φ is a $\mathscr{G}$-cover of M.*

P r o o f. Similarly as in the preceding proof we shall consider a set X with $F \subseteq X$, $|F| < |X|$ and the family $\Sigma = \{(F_0, \varphi_0)\}$ with $F \subseteq F_0 \subseteq X$, F essential in F_0, and $\varphi_0\colon F_0 \to M$ a $\mathscr{G}$-precover of the module M. The collection Σ is nonempty since $(F, \varphi) \in \Sigma$, and it is ordered by the relation $\leqslant$ where $(F_0, \varphi_0) \leqslant (F_1, \varphi_1)$ if and only if $F_0 \subseteq F_1$ and $\varphi_1\big|F_0 = \varphi_0$.

If $\{(F_i, \varphi_i) \mid i \in I\}$ is a chain in Σ, then we set $F^* = \bigcup_{i \in I} F_i$ and $\varphi^*(x) = \varphi_i(x)$ whenever $x \in F_i$. Then $F^* \in \mathscr{G}$ by hypothesis, φ^* is a $\mathscr{G}$-precover of M by Lemma 1.5 and so $(F^*, \varphi^*) \in \Sigma$, F being obviously essential in F^*.

Consider the commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\ \varphi\ } & M \\
\iota \downarrow & & \| \\
F^* & \xrightarrow{\ \varphi^*\ } & M \\
f_1 \downarrow & & \| \\
F & \xrightarrow{\ \varphi\ } & M
\end{array}
$$

where (F^*, φ^*) is a maximal element of Σ, ι is the embedding and f_1 is an arbitrary homomorphism making the bottom square commutative. Now $\varphi f_1\iota = \varphi^*\iota = \varphi$ and consequently $f_1\iota$ is injective by hypothesis. Further, for $y \in \operatorname{Ker} f_1 \cap \iota(F)$ we have $y = \iota(x)$ for some $x \in F$, and so $f_1(y) = f_1\iota(x) = 0$ yields $y = 0$, which means that $\operatorname{Ker} f_1 \cap \iota(F) = 0$. Thus $\operatorname{Ker} f_1 = 0$, $\iota(F)$ being essential in F^*. Moreover, $\operatorname{Im}(f_1\iota)$ is essential in F by hypothesis and so is $\operatorname{Im} f_1$ in view of the obvious inclusion $\operatorname{Im}(f_1\iota) \subseteq \operatorname{Im} f_1$.

Our next step is to show that f_1 is an epimorphism. If not, then ιf_1 is a non-surjective monomorphism of F^* such that $\varphi^* \iota f_1 = \varphi f_1 = \varphi^*$ and consequently Lemma 1.4 yields a contradiction with the maximality of (F^*, φ^*).

To complete the proof it suffices to consider the commutative diagram

$$
\begin{array}{ccc}
F^* & \xrightarrow{\varphi^*} & M \\
{\scriptstyle f_1}\downarrow & & \| \\
F & \xrightarrow{\varphi} & M \\
{\scriptstyle f}\downarrow & & \| \\
F & \xrightarrow{\varphi} & M
\end{array}
$$

Since $\varphi f f_1 = \varphi f_1 = \varphi^*$, $f f_1$ is an epimorphism by the preceding part, and so is f, as we wished to show. $\qquad\square$

Let $\mathscr{G}$ be an abstract class of modules. We say that a proper submodule N of a module M is *almost $\mathscr{G}$-pure in M*, if the factor-module M/N contains a non-zero submodule from $\mathscr{G}$. Furthermore, we say that a module U is *almost $\mathscr{G}$-hereditary*, if every non-zero submodule of U contains a non-zero submodule from $\mathscr{G}$. Finally, the class $\mathscr{G}$ is called *almost $\mathscr{G}$-hereditary*, if every module $U \in \mathscr{G}$ is almost $\mathscr{G}$-hereditary.

Theorem 3.4. *Let $\mathscr{G}$ be an abstract inductive class of modules and let $\varphi : F \to M$ be a $\mathscr{G}$-precover of the module M. If F is almost $\mathscr{G}$-hereditary and $\operatorname{Ker} \varphi$ contains no non-zero submodule almost $\mathscr{G}$-pure in F, then φ is a $\mathscr{G}$-cover of M.*

P r o o f.   The idea of the proof is to verify that any endomorphism f of the module F such that $\varphi f = \varphi$ is injective with essential image and then apply the preceding theorem.

First, $F/\operatorname{Ker} f \cong \operatorname{Im} f \leqslant F$ yields that $\operatorname{Im} f$ contains a non-zero element from $\mathscr{G}$, hence $\operatorname{Ker} f$ is almost $\mathscr{G}$-pure in F and $\operatorname{Ker} f = 0$ by hypothesis, owing to the obvious inclusion $\operatorname{Ker} f \subseteq \operatorname{Ker} \varphi$. Continuing indirectly, let us suppose that $f(F)$ is not essential in F. Thus there is a non-zero submodule K of F with $f(F) \cap K = 0$ and we may without loss of generality assume that $K \in \mathscr{G}$, F being almost $\mathscr{G}$-hereditary. Setting $S = \{x - f(x) \mid x \in K\}$ we have $x - f(x) = 0$ if and only if $x = f(x) \in K \cap f(F) = 0$, and the mapping $g\colon K \to S$ given by $g(x) = x - f(x)$, $x \in K$, is an isomorphism. Thus $S \cong K$ lies in the class $\mathscr{G}$ and obviously $S \subseteq \operatorname{Ker} \varphi$. Further, $S \cap f(F) = 0$ since for $x - f(x) = f(y)$, $x \in K$, $y \in F$, we have $x = f(x + y) \in K \cap f(F) = 0$, and so $\operatorname{Im} f \cong \frac{f(F) \oplus S}{S} \leqslant \frac{F}{S}$. By hypothesis, $\operatorname{Im} f$ contains a non-zero submodule from $\mathscr{G}$, hence $0 \neq S \subseteq \operatorname{Ker} \varphi$ is almost $\mathscr{G}$-pure in F, which contradicts the hypothesis. $\qquad\square$

4. Applications

Recall that a hereditary torsion theory $\sigma = (\mathscr{T}, \mathscr{F})$ for the category R-mod consists of two abstract classes $\mathscr{T}$ and $\mathscr{F}$, the σ-*torsion class* and the σ-*torsionfree class*, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathscr{T}$ and $F \in \mathscr{F}$, the class $\mathscr{T}$ is closed under submodules, factor modules, extensions and direct sums, the class $\mathscr{F}$ is closed under submodules, extensions and direct product and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$. With each hereditary torsion theory we associate a Gabriel filter of left ideals $\mathscr{L} = \{I \leqslant R \mid R/I \in \mathscr{T}\}$ and the torsion part $\sigma(M) = T$ of the module M consists of all elements $a \in M$ with $(0 : a) \in \mathscr{L}$. The torsion theory σ is said to be of *finite type*, if the filter $\mathscr{L}$ contains a cofinal subset of finitely generated left ideals. For more details see e.g. [3] or [1]. The following two consequences of the above theory can be found in [7, Theorem] and [2, Theorem 1].

Corollary 4.1. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory of finite type for R-mod, then every module has a σ-torsionfree cover.*

P r o o f. First we show that every R-module has a σ-torsionfree precover. Since the class $\mathscr{F}$ is hereditary, it suffices by virtue of Theorem 2.6 to show that every injective module has an $\mathscr{F}$-precover. For this reason we are going to verify the hypotheses of Theorem 2.5. Clearly, $\mathscr{F}$ is closed under direct sums and injective hulls. If $\mathscr{F}'$ is a co-abstract set consisting of injective hulls of cyclic modules from $\mathscr{F}$, then obviously every injective module $F \in \mathscr{F}$ contains an essential submodule isomorphic to a member of $\operatorname{Coprod}(\mathscr{F}')$. Since the set of $\mathscr{F}$-pure submodules of any module is inductive by [3, Proposition 6.18], the proof of this part is complete. Now if $\psi \colon G \to M$ is an $\mathscr{F}$-precover of the module M, then $\operatorname{Ker} \psi$ contains a maximal $\mathscr{F}$-pure submodule K of G. Denoting $F = G/K$ and $\varphi \colon F \to M$ the homomorphism naturally induced by ψ, Lemma 1.5 yields that φ is a σ-precover of M and Theorem 3.4 applies. $\square$

Corollary 4.2. *Over any commutative domain every module has a torsionfree cover.*

P r o o f is obvious. $\square$

Let $\sigma = (\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for R-mod and let $\mathscr{I}$ denote the class of σ-*torsionfree σ-injective modules*.

Lemma 4.3. *If σ is a hereditary torsion theory of finite type, then a σ-torsionfree module M is σ-injective if the induced map $\operatorname{Hom}(R, M) \to \operatorname{Hom}(I, M)$ is an epimorphism for every finitely generated left ideal I from $\mathscr{L}$.*

P r o o f.   In view of the relative Baer's criterion, we can investigate the commutative diagram

$$
\begin{array}{ccccc}
I & \xrightarrow{\ i\ } & J & \xrightarrow{\ \iota\ } & R \\
{\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
M & =\!\!=\!\!= & M & =\!\!=\!\!= & M
\end{array}
$$

with $J \in \mathscr{L}$ and $f\colon J \to M$ given. By hypothesis there is a finitely generated left ideal $I \in \mathscr{L}$ with the inclusion map $i\colon I \to J$ and a homomorphism $g\colon R \to M$ such that $g\iota i = f'$, where $f' = f \mid I = fi$. For an arbitrary $j \in J$ we have $K = (I : j) \in \mathscr{L}$ and for each $k \in K$ we have $k(f - g\iota)(j) = (f - g\iota)(kj) = 0$ since $kj \in I$ and $f\big|I = g\iota i$. Hence $K(f - g\iota)(j) = 0$, which means $(f - g\iota)(j) \in \sigma(M) = 0$ and consequently $f = g\iota$, as desired.   □

Lemma 4.4.   *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory of finite type, then the class $\mathscr{I}$ of all σ-torsionfree σ-injective modules is inductive.*

P r o o f.   See [3, Proposition 42.9].   □

The following two corollaries partly generalize some results from [8, Corollary 2.10] and [4, Proposition 2], respectively. Recall that a hereditary torsion theory σ is called *exact* if $M \in \mathscr{I}$ implies $E(M)/M \in \mathscr{I}$ and that σ is called *perfect* if it is exact and of finite type.

Corollary 4.5.   *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a perfect torsion theory for R-mod, then every module has a σ-torsionfree σ-injective cover.*

P r o o f.   Using Theorem 2.3 we first show that every module has an $\mathscr{I}$-precover. The class $\mathscr{I}$ of all σ-torsionfree σ-injective modules is $\mathscr{I}$-cohereditary by [3, Proposition 44.1], it is inductive by Lemma 4.4 and it is easy to see that it is closed under direct sums and extensions. Taking any co-abstract subset $\mathscr{I}'$ of $\mathscr{I}$ consisting of elements which are essential extensions of σ-torsionfree cyclic modules, then using [3, Proposition 10.11] it is a routine to check that each $F \in \mathscr{I}$ contains an essential submodule isomorphic to a member of $\mathrm{Coprod}(\mathscr{I}')$. Thus every module has an $\mathscr{I}$-precover and by virtue of inductivity and Lemma 1.5 we may assume that for an arbitrary module M there exists an $\mathscr{I}$-precover $\varphi\colon F \to M$ such that $\mathrm{Ker}\,\varphi$ contains no non-zero submodule $\mathscr{I}$-pure in F. Considering the diagram $(*)$ in Theorem 3.1, the isomorphism $F/\mathrm{Ker}\,f \cong \mathrm{Im}\,f$ yields that $\mathrm{Ker}\,f$ is σ-closed in F and consequently $\mathscr{I}$-pure in F by [3, Proposition 10.11]. In view of the obvious inclusion $\mathrm{Ker}\,f \subseteq \mathrm{Ker}\,\varphi$, $\mathrm{Ker}\,f = 0$ and φ is the $\mathscr{I}$-cover of M by Theorem 3.1.   □

Corollary 4.6. *If $\sigma = (\mathscr{T}, \mathscr{F})$ is a centrally splitting torsion theory for R-mod, then every module has a σ-torsionfree σ-injective cover.*

P r o o f.   The corresponding radical filter $\mathscr{L}$ has the smallest element $I = Re$, e being a central idempotent, and so σ is obviously of finite type. Let $M \in \mathscr{F}$ be arbitrary. Assuming $\sigma(E(M)/M) = K/M \neq 0$, for each $x \in K \setminus M$ we have $Ix \subseteq M$, i.e. $ex \in M$. Moreover, $e(x - ex) = 0$, so $I(x - ex) = 0$ and $x = ex$, M being σ-torsionfree. Hence $x \in M$, which is a contradiction proving that every σ-torsionfree module is σ-injective; an application of Corollary 4.1 completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

References

[1] *L. Bican, T. Kepka and P. Němec*: Rings, Modules, and Preradicals. Marcel Dekker, New York, 1982.

[2] *E. Enochs*: Torsion free covering modules II. Proc. Amer. Math. Soc. *114* (1963), 884–889.

[3] *J. Golan*: Torsion Theories. Pitman Monographs and Surveys in Pure an Applied Matematics, vol. 29, Longman Scientific and Technical, 1986.

[4] *J. R. García Rozas and B. Torrecillas*: On the existence of covers by injective modules relative to a torsion theory. Comm. Algebra *24* (1996), 1737–1748.

[5] *J. Rada and M. Saorín*: Rings characterized by (pre)envelopes and (pre)covers of their modules. Comm. Algebra *26* (1998), 899–912.

[6] *M. Teply*: Torsionfree injective modules. Pacific J. Math. *28* (1969), 441–453.

[7] *M. Teply*: Torsion-free covers II. Israel J. Math. *23* (1976), 132–136.

[8] *B. Torrecillas*: T-torsionfree T-injective covers. Comm. Algebra *12* (1984), 2707–2726.

[9] *J. Xu*: Flat Covers of Modules. Springer Verlag, Berlin-Heidelberg-New York, 1996.

Authors' addresses: L . B i c a n, KA MFF UK, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `bican@karlin.mff.cuni.cz`; B . T o r r e c i l l a s, Department of Algebra and Analysis, Universidad de Almería, 04071 Almería, Spain, e-mail: `btorreci@ual.es`.