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GRAPH AUTOMORPHISMS AND CELLS OF LATTICES

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Abstract. In this paper we apply the notion of cell of a lattice for dealing with graph automorphisms of lattices (in connection with a problem proposed by G. Birkhoff).

Keywords: lattice, semimodular lattice, graph automorphism, direct factor

MSC 2000: 06C10

A partially ordered set is called discrete if all its bounded chains are finite. All partially ordered sets which are dealt with in the present paper are assumed to be discrete.

Inspired by a problem proposed by Birkhoff ([1], Problem 6) the author investigated graph automorphisms of modular lattices [5] and of semimodular lattices [6].

For the references concerning graph isomorphisms of lattices cf. Ratanaprasert [11].

The notion of a cell in a lattice was introduced in the author's article [4]. It was applied for studying graph isomorphisms of semimodular lattices by Ratanaprasert and Davey [12]. Further, this notion was used for investigating graph isomorphisms of semilattices by Kolibiar [8] and Ratanaprasert [10], [11], and of directed sets by Tomková [15].

In Section 2 of the present paper we apply the notion of a cell for dealing with graph automorphisms of lattices (neither modularity nor semimodularity is assumed). We obtain a generalization of a result of [5].

Some further results (concerning semimodular lattices, graded lattices, balanced lattices and geometric lattices) are given in Sections 3 and 4.

1. Preliminaries

Let P be a partially ordered set. For $a, b \in P$ with $a \leqslant b$, the interval $[a, b]$ is the set $\{x \in P\colon\ a \leqslant x \leqslant b\}$. If $[a, b] = \{a, b\}$ and $a \neq b$, then $[a, b]$ is said to be a prime interval and we express this situation by writing $a \prec b$ or $b \succ a$.

The graph $G(P)$ is defined to be the unoriented graph whose vertex set is P and whose edges are pairs (a, b) such that either $a \prec b$ or $b \prec a$.

If P_1 and P_2 are partially ordered sets and if φ is an isomorphism of $G(P_1)$ onto $G(P_2)$, then φ is called a graph isomorphism of P_1 onto P_2. For $P_1 = P_2$ we obtain, in particular, a graph automorphism of P_1.

Let φ be a graph isomorphism of P_1 onto P_2 and let $X \subseteq P_1$. We say that X is preserved (reversed) under φ if, whenever $x_1, x_2 \in X$ and $x_1 \prec x_2$, then $\varphi(x_1) \prec \varphi(x_2)$ (or $\varphi(x_1) \succ \varphi(x_2)$, respectively).

If X is either preserved or reversed under φ, then X is called regular under φ.

For a partially ordered set P we denote by P^d the partially ordered set which is dual to P.

The direct product $P_1 \times P_2$ of partially ordered sets P_1 and P_2 is defined in the usual way. We will apply also the notion of an internal direct product decomposition of a partially ordered set (in the same sense as in [7] and [5]).

Let P, P' be partially ordered sets and let f be a graph isomorphism of P onto P'. Suppose that there exist partially ordered sets A, B and direct product representations

$$g\colon\ P \to A \times B, \quad g'\colon\ P' \to A^d \times B$$

such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & P' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ A \times B & \xrightarrow{\ i\ } & A^d \times B \end{array}$$

is commutative (where i is the identity mapping on $A \times B$). Then the graph isomorphism f is said to be given by a direct product diagram. (Cf. Kolibiar [8].)

2. Cells in lattices

Let L be a lattice. Assume that $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$, u and v are distinct elements of L such that

(i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v, \quad u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v,$

(ii) either $x_1 \vee y_1 = v$, or $x_m \wedge y_n = u$.

Then the set $C = \{u, v, x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}$ is called a cell in L. The cell C is said to be proper if either $m > 1$ or $n > 1$.

2.1. Theorem (Cf. [4]). *A graph isomorphism $f\colon L \to L'$ of lattices L and L' is given by a direct product diagram iff any proper cell in L or in L' is regular under f or f^{-1}, respectively.*

For a lattice L we denote by

$A(L)$—the set of all graph automorphisms of L;

$A_c(L)$—the set of all $\varphi \in A(L)$ such that each proper cell of L is regular under φ and under φ^{-1}.

Further, let $\mathcal{C}$ and $\mathcal{C}_0$ be the classes of lattices L such that, whenever $\varphi \in A(L)$ (or $\varphi \in A_c(L)$, respectively), then φ is a lattice automorphism on L.

2.2. Theorem (Cf. [5]). *Let L be a modular lattice. Then the following conditions are equivalent:*

(i) *L belongs to $\mathcal{C}$.*

(ii) *No direct factor of L having more than one element is self-dual.*

In the present paper we prove

2.3. Theorem. *Let L be a lattice. Then the condition* (ii) *from* 2.2 *is equivalent with the condition*

(i_1) *L belongs to $\mathcal{C}_0$.*

If L is a modular lattice then no proper cell exists in L; thus $A(L) = A_c(L)$ and

$$L \in \mathcal{C} \Leftrightarrow L \in \mathcal{C}_0.$$

Hence 2.3 is a generalization of 2.2.

We prove 2.3 by means of some lemmas.

2.4. Lemma (Cf. [5]). *Let ψ be an isomorphism of L onto the direct product $A \times B$. Further, suppose that χ is an isomorphism of B onto B^d. For each $x \in L$ we put $\varphi(x) = y$, where*

$$\psi(x) = (a, b), \quad y = \psi^{-1}((a, \chi(b))).$$

Then φ is a graph automorphism of L.

By a similar argument as in the proof of Lemma 2.1 of [4] we obtain

2.5. Lemma. *Let the assumptions of* 2.4 *be satisfied. Then each proper cell of* L *is regular under the graph automorphism* φ *and under* φ^{-1}*; consequently,* $\varphi \in A_c(L)$.

2.6. Lemma. *If* L *belongs to* $\mathcal{C}_0$*, then no direct factor of* L *having more than one element is self-dual.*

P r o o f. Suppose that L belongs to $\mathcal{C}_0$. Suppose that B is a self-dual direct factor of L. There is a direct factor A of L such that the direct product decomposition

$$\psi\colon\ L \to A \times B$$

is valid. Let φ be as in 2.4. In view of 2.5, φ belongs to $A_c(L)$. By way of contradiction, assume that B has more than one element. According to Lemma 1.2 in [4], φ fails to be a lattice automorphism of L. Hence $L \notin \mathcal{C}_0$ and we have arrived at a contradiction. $\square$

2.7. Lemma. *Let the condition* (ii) *of* 2.2 *be valid. Then* L *belongs to* $\mathcal{C}_0$.

P r o o f. Let f be an arbitrary element of $A_c(L)$. Put $L' = L$. Hence f is a graph isomorphism of L onto L; moreover, all proper cells in L or in L' are regular under f or f^{-1}, respectively. Thus according to 2.1, f is given by a direct product diagram. We can apply the notation as in Section 1 (taking L instead of P).

Without loss of generality we can suppose that g and g' are internal direct product decompositions of $L = L'$ with the same central element. Then from [4], Lemma 2.1 (cf. also [7], Lemma 2.4) we obtain $A = A^d$. Therefore A is a one-element set. $\square$

From 2.6 and 2.7 we conclude that 2.3 holds.

3. Semimodular lattices

In this section we assume that L is a semimodular lattice.

3.1. Definition. Let X be a sublattice of L which is isomorphic to the lattice on Fig. 1 and let $\overline{X}$ be the convex hull of X in L. Then $\overline{X}$ is said to be an interval of type (C_0).

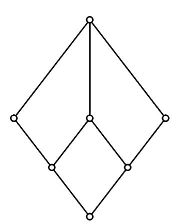

Fig. 1

If A is a direct factor of L and $y \in L$, then the component of y in the direct factor A will be denoted by $y(A)$.

Let $Y \subseteq L$. Suppose that $y_1(A) = y_2(A)$ for each $y_1, y_2 \in Y$. Then the direct factor A is said to be orthogonal to Y.

We denote by $A_1(L)$ the set of all $\varphi \in A(L)$ such that, whenever $\overline{X}$ is an interval of type (C_0) in L, then $\overline{X}$ is preserved under φ.

3.2. Theorem (Cf. [6]). *The following conditions are equivalent:*

(i) *If $\varphi \in A_1(L)$, then φ is a lattice automorphism of L.*

(ii) *If A is a self-dual direct factor of L such that A is orthogonal to each interval of type (C_0) in L, then A is a one-element set.*

3.3. Lemma. *Let C be a proper cell in L. Then (under the notation as in Section* 2*) we have*

(i_1) $x_m \wedge y_n = u$;

(ii_1) *C is a subset of an interval of type (C_0) in L.*

P r o o f of (i_1). By way of contradiction, suppose that (i_1) fails to hold. Then, in view of the definition of a cell, the relation $x_1 \vee y_1 = v$ is valid. Since either $m > 1$ or $n > 1$, we have a contradiction with the semimodularity of L. $\square$

P r o o f of (ii_1). Put $z = x_1 \vee y_1$. If $z \leqslant x_m$, then we would have $x_m \wedge y_n \geqslant y_1$, which is impossible in view of (i_1). Similarly, $z \nleqslant y_m$. Hence, in view of semimodularity of L, the element z is incomparable with both x_m and y_n. Since $x_1 \prec z$, $y_1 \prec z$, $x_m \prec v$ and $y_n \prec v$, we obtain

$$x_m \wedge z = x_1, \quad x_m \vee z = v,$$
$$y_n \wedge z = y_1, \quad y_n \vee z = v.$$

Thus $X = \{u, v, x_1, x_m, y_1, y_n\}$ is a sublattice of L which is isomorphic to the lattice in Fig. 1 and $\overline{X}$ is an interval of type (C_0) in L. Moreover, $C \subseteq \overline{X}$. $\square$

3.4.1. Lemma. *Let $\varphi \in A_1(L)$, $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in L$ be such that $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v$, $u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v$. Suppose that*

$$\text{(a)} \qquad \varphi(u) \prec \varphi(x_1) \prec \varphi(x_2) \prec \ldots \prec \varphi(x_m) \prec \varphi(v).$$

Then $\varphi(u) \prec \varphi(y_1) \prec \varphi(y_2) \prec \ldots \prec \varphi(y_n) \prec \varphi(v)$.

P r o o f. It suffices to apply an analogous argument as in the proof of Lemma 2.3 in [4]. $\square$

Analogously we have

3.4.2. Lemma. *Let the same assumptions as in* 3.4.1 *be satisfied with the distinction that instead of* (a) *the relation*

(b) $$\varphi(u) \succ \varphi(x_1) \succ \varphi(x_2) \succ \ldots \succ \varphi(x_m) \succ \varphi(v)$$

holds. Then $\varphi(u) \succ \varphi(y_1) \succ \varphi(y_2) \succ \ldots \succ \varphi(y_n) \succ \varphi(v)$.

3.5. Lemma. *Let C be a proper cell in L and $\varphi \in A(L)$. Then C cannot be reversed under φ.*

Proof. For the cell C we apply the same notation as in the proof of 3.3. By way of contradiction, suppose that C is reversed under φ. Then the mapping φ (reduced to the set $[u, v]$) is a dual lattice isomorphism of the interval $[u, v]$ onto the interval $[\varphi(v), \varphi(u)]$ of L. Then according to 3.3 we have

$$\varphi(x_m) \wedge \varphi(y_n) = \varphi(v),$$
$$\varphi(x_m) \vee \varphi(y_n) = \varphi(u),$$
$$\varphi(x_n) \succ \varphi(v), \quad \varphi(y_n) \succ \varphi(v)$$

and some of the relations $\varphi(x_m) \prec \varphi(u)$, $\varphi(y_n) \prec \varphi(u)$ fails to hold. This contradicts the semimodularity of L. □

3.6. Lemma. $A_1(L) = A_c(L)$.

Proof. The relation $A_c(L) \subseteq A_1(L)$ is a consequence of 3.3. Let X be a sublattice of L isomorphic to the lattice in Fig. 1; let u and v be the least element and the greatest element of X, respectively. Then there exists a proper cell C with elements denoted as in Section 2. Let $\varphi \in A_1(L)$. Thus C is regular under φ. Hence C is either preserved or reversed under φ.

In view of 3.4.1 and 3.4.2, the interval $[u, v]$ of L is either preserved or reversed under φ. Then according to 3.5, the first possibility must occur. Thus $[u, v] = \overline{X}$ is preserved under φ. Hence $A_c(L) \subseteq A_1(L)$. □

The following theorem sharpens the implication (i) ⇒ (ii) from 3.2.

3.7. Theorem. *Let the condition* (i) *from* 3.2 *be valid. Then we have*

(ii_2) *if A is a self-dual direct factor of L, then A is a one-element set.*

Proof. This is a consequence of 3.6 and 2.3. □

3.8. Theorem. *Let the condition* (ii) *from* 3.2 *be valid. Then the condition* (ii_2) *from* 3.7 *holds.*

P r o o f. In view of 3.2, the condition (i) from 3.2 holds. Then 3.7 yields that (ii_2) is valid. □

4. Graded, balanced and geometric lattices

The remarks contained in this section can be considered as an addendum to the author's paper [5]. All lattices dealt with in the present section are assumed to be of finite length.

In A), B) and D) we apply the results of Duffus and Rival [2], of Lee [9], or of Stern [14], respectively, concerning graph isomorphisms of some types of lattices.

We will say that a pair of lattices (L, L') satisfies the condition (α) if each graph isomorphism of L onto L' is given by a direct product diagram.

A) A lattice L will be said to be of type (DR) if

(i) L is graded,

(ii) every element of L is a join of atoms and a meet of coatoms.

Lattices of such type were studied by Duffus and Rival [2].

4.1. Theorem (Cf. [2]). *Let L and L' be graded lattices and let L be of type* (RD). *Then the pair (L, L') satisfies the condition (α).*

By the same method as in [5] (with the only distinction that instead of [3] we now apply 4.1) we conclude

4.2. Theorem. *Let L be a graded lattice of type* (DR). *Then the following conditions are equivalent:*

(i) *Each graph automorphism of L is a lattice automorphism.*

(ii) *No direct factor of L having more than one element is self-dual.*

B) A semimodular lattice of finite length is called geometric if every element of L is a join of atoms (cf. [9]).

4.3. Theorem (Cf. [9]). *Let L and L' be lattices with isomorphic graphs. If L is geometric, then the pair (L, L') satisfies the condition (α).*

Analogously as in A) (by applying 4.3) we obtain

4.4. Theorem. *Let L be a geometric lattice and let the conditions* (i) *and* (ii) *be as in* 4.2. *Then* (i) *and* (ii) *are equivalent.*

4.5. Corollary (Cf. [9]). *Let L be a geometric lattice. Suppose that L is directly indecomposable and that it is not self-dual. Then every graph automorphism of L is a lattice automorphism.*

C) From 2.2 we immediately obtain

4.6. Corollary (Cf. [9]). *Let L be a non-self-dual, directly indecomposable modular lattice. Then every graph automorphism of L is a lattice automorphism.*

D) For a lattice L we denote by

$\mathcal{J}(L)$—the set of all elements x of L such that x fails to be the least element of L and x is join-irreducible;

$M(L)$—the set of all elements y of L such that y fails to be the greatest element of L and y is meet-irreducible.

Let $j \in \mathcal{J}(L)$ and $m \in M(L)$. We denote by j' the unique element of L such that $j' \prec j$; further, let m^* be the unique element of L with $m \prec m^*$.

Assume that $j \nleqq m$. The arrow-relations between $\mathcal{J}(L)$ and $M(L)$ are defined by

$$j \nearrow m \Leftrightarrow j \vee m = m^*, \quad j \swarrow m \Leftrightarrow j \wedge m = j'.$$

(Cf. Wille [16].)

The lattice L is called balanced if for all $j \in \mathcal{J}(L)$ and $M(L)$ we have

$$j \nearrow m \Leftrightarrow j \swarrow m.$$

(Cf. Reuter [13].)

4.7. Theorem (Cf. [14]). *Let L and L' be lattices which are graded and balanced. Then the pair (L, L') satisfies the condition (α).*

If L is a modular lattice or if it is of type (RD), then L is gradded and balanced (cf. [14]). Hence 4.7 is a generalization of the author's result from [3], and of the result of Duffus and Rival (cf. 4.1).

Similarly as in A), from 4.7 we conclude

4.8. Theorem. *Let L be a balanced lattice. Let* (i) *and* (ii) *be as in* 4.3. *Then the conditions* (i) *and* (ii) *are equivalent.*

This generalizes 4.2 and also the result $(*)$ from [5].

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