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BERNSTEIN-TYPE OPERATORS ON THE HALF LINE

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Abstract. We define Bernstein-type operators on the half line $[0,+\infty[$ by means of two sequences of strictly positive real numbers. After studying their approximation properties, we also establish a Voronovskaja-type result with respect to a suitable weighted norm.

Keywords: Bernstein-Chlodovsky operators, approximation process, Voronovskaja-type formula

MSC 2000: 41A10, 41A36

1. Introduction and notation

In [4] Chlodovsky introduced and studied a sequence of positive linear operators $(C_n^*)_{n\geqslant 1}$ on the space $C([0,+\infty[)$ of all real valued continuous functions on the half line $[0,+\infty[$, defined by

$$C_n^* f(x) := \begin{cases} \displaystyle\sum_{k=0}^{n} f\left(\frac{b_n k}{n}\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k} & \text{if } 0\leqslant x\leqslant b_n, \\ f(x) & \text{if } x>b_n, \end{cases} \tag{1.1}$$

where $(b_n)_{n\geqslant 1}$ is a divergent sequence of strictly positive real numbers.

Roughly speaking, the above operators, known as Bernstein-Chlodovsky operators, behave basically like the classical Bernstein ones on $[0,b_n]$, interpolating, in the meanwhile, the function f elsewhere.

A deeper analysis of their approximation properties was subsequently carried out in [8], [10] with respect to functions belonging to particular subspaces of $C([0,+\infty[)$. In this framework and without the assumption of completeness, it seems also useful to refer the reader to [6], [7], [9], [11] for general results concerning the approximation of continuous functions on unbounded intervals and for some interesting extensions of the classical Korovkin's Theorem.

The purpose of this paper is to consider a generalization of Bernstein-Chlodovsky operators (1.1) by using two sequences $(b_n)_{n\geqslant 1}$ and $(c_n)_{n\geqslant 1}$ of strictly positive real numbers, satisfying particular assumptions.

As a consequence, our definition (2.1) actually turns out to be more flexible than (1.1), allowing to state, beyond classical approximation results, a Voronovskaja-type formula, which, as far as we know, cannot be stated for the classical C_n^*.

The corresponding differential operator is a rather general second-order one degenerating at the boundaries with coefficients depending on the sequences $(b_n)_{n\geqslant 1}$ and $(c_n)_{n\geqslant 1}$, and may be readily shown to be the generator of a strongly continuous positive contraction semigroup, due to some classical results stated in [5] and [12].

It would be perhaps interesting, falling, actually, within a wide program of investigation which has been inspiring the authors and other researchers in the last years, to prove that such a semigroup may be represented in terms of powers of the operators C_n as an application of the classical Trotter representation theorem [13] (see, also, [1], Proposition 1.6.7, p. 67): this may virtually justify further analysis in the concern.

As for the notation, throughout the paper, besides $C([0,+\infty[)$, we will sometimes deal with the subspace $UC_b([0,+\infty[)$ of all bounded uniformly continuous functions on $[0,+\infty[$ which is a Banach lattice, if endowed with the sup- norm$\|\cdot\|$.

For every $\alpha > 0$ we will be mainly concerned with the weighted space

$$E_\alpha^0 := \{f \in C([0,+\infty[) \mid \exists \lim_{x\to+\infty} \frac{f(x)}{1+x^\alpha} = 0\}, \tag{1.2}$$

which becomes a Banach lattice with respect to the norm

$$\|f\|_\alpha := \sup_{x\geqslant 0} \frac{|f(x)|}{1+x^\alpha}. \tag{1.3}$$

Such spaces have been already considered in [2], [3] in which a worthy generalization of the classical Baskakov operators is studied.

As usual, for every integer $m \geqslant 1$, $C^m([0,+\infty[)$ is the vector space of all real valued m-times continuously differentiable functions on $[0,+\infty[$. For every $p \geqslant 0$, e_p is the test function defined by $e_p(x) := x^p$ $(x \geqslant 0)$, whereas, for each $x \geqslant 0$, ψ_x is the function defined by $\psi_x(t) := t - x$ $(t \geqslant 0)$.

The symbol $\omega(\cdot,\cdot)$ will denote the classical modulus of continuity, as usual.

2. The operators C_n

Let us consider two sequences $(b_n)_{n\geqslant 1}$ and $(c_n)_{n\geqslant 1}$ of strictly positive real numbers satisfying the following assumptions:

1) $b_n \to +\infty$ as $n \to +\infty$;
2) $b_n/n \to 0, \quad b_n - c_n \to 0$ as $n \to +\infty$;
3) $b_n \leqslant c_n$ for every $n \geqslant 1$.

It immediately follows that, correspondingly, $c_n \to +\infty$ and $c_n/n \to 0$ as well, and, in addition, $b_n \approx c_n$ as $n \to +\infty$.

For every $n \geqslant 1$ and for every $f \in E_\alpha^0$ we set

$$C_n f(x) := \begin{cases} \displaystyle\sum_{k=0}^{n} f\left(\frac{c_n k}{n}\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k} & \text{if } 0 \leqslant x \leqslant b_n, \\ f(c_n) & \text{if } b_n < x \leqslant c_n, \\ f(x) & \text{if } x > c_n. \end{cases} \tag{2.1}$$

Since $C_n(f) = f$ in $[c_n, +\infty[$ by definition, we may refer to C_n as to a positive linear operator acting from E_α^0 into itself. Moreover, $C_n(e_0) = e_0$ and therefore $\|C_n\| = \|C_n(e_0)\| = 1$; in addition, a very simple computation shows that

$$C_n e_1(x) = \begin{cases} \dfrac{c_n}{b_n} x & \text{if } 0 \leqslant x \leqslant b_n, \\ c_n & \text{if } b_n < x \leqslant c_n, \\ x & \text{if } x > c_n, \end{cases} \tag{2.2}$$

$$C_n e_2(x) = \begin{cases} \dfrac{c_n^2}{b_n^2} x^2 + \dfrac{c_n^2}{n b_n^2} x(b_n - x) & \text{if } 0 \leqslant x \leqslant b_n, \\ c_n^2 & \text{if } b_n < x \leqslant c_n, \\ x^2 & \text{if } x > c_n, \end{cases} \tag{2.3}$$

$$C_n \psi_x(x) = \begin{cases} \left(\dfrac{c_n}{b_n} - 1\right) x & \text{if } 0 \leqslant x \leqslant b_n, \\ c_n - x & \text{if } b_n < x \leqslant c_n, \\ 0 & \text{if } x > c_n, \end{cases} \tag{2.4}$$

and

$$C_n \psi_x^2(x) = \begin{cases} \left(\dfrac{c_n}{b_n} - 1\right)^2 x^2 + \dfrac{c_n^2}{n b_n^2} x(b_n - x) & \text{if } 0 \leqslant x \leqslant b_n, \\ (c_n - x)^2 & \text{if } b_n < x \leqslant c_n, \\ 0 & \text{if } x > c_n. \end{cases} \tag{2.5}$$

An approximation result is indicated below.

Theorem 2.1. *For every $f \in E^0_\alpha$ $(\alpha > 2)$ we have*

$$\lim_{n\to+\infty} \|C_n(f) - f\|_\alpha = 0, \tag{2.6}$$

i.e., the sequence $(C_n)_{n\geqslant 1}$ is a positive approximation process.

More precisely, for n large enough we have

$$\|C_n(f) - f\|_\alpha \leqslant 2\omega\left(f, \sqrt{(c_n - b_n)^2 + \frac{c_n^2}{nb_n}}\right). \tag{2.7}$$

P r o o f. Indeed, let us fix $n \geqslant 1$. On account of (2.2) and (2.3), we get

$$\frac{|C_n e_1(x) - x|}{1 + x^\alpha} \leqslant \begin{cases} \left(\dfrac{c_n}{b_n} - 1\right) & \text{if } 0 \leqslant x \leqslant b_n, \\ \dfrac{c_n - b_n}{1 + b_n^\alpha} & \text{if } b_n < x \leqslant c_n, \end{cases}$$

$$\frac{|C_n e_2(x) - x^2|}{1 + x^\alpha} \leqslant \begin{cases} \left(\dfrac{c_n^2}{b_n^2} - 1\right) + \dfrac{c_n^2}{nb_n} & \text{if } 0 \leqslant x \leqslant b_n, \\ \dfrac{2c_n(c_n - b_n)}{1 + b_n^\alpha} & \text{if } b_n < x \leqslant c_n. \end{cases}$$

Now observe that each member on the right-hand side in the above estimates tends to 0 as $n \to +\infty$, as a consequence of the assumptions on the sequences $(b_n)_{n\geqslant 1}$ and $(c_n)_{n\geqslant 1}$. Moreover, by definition, $|C_n e_i(x) - e_i(x)| = 0$ whenever $x \in [c_n, +\infty[$ and therefore, since obviously $C_n(e_0) = e_0$, we have just shown that

$$\lim_{n\to+\infty} \|C_n(e_i) - e_i\|_\alpha = 0 \quad \text{for} \quad i = 0, 1, 2,$$

which implies (2.6) on account of Korovkin's theorem (see, e.g., [1], Proposition 4.2.5, p. 215).

In order to establish (2.7), let us first note that by virtue of [1], Proposition 5.1.2, p. 268, a pointwise estimate

$$|C_n f(x) - f(x)| \leqslant 2\omega(f, \sqrt{C_n \psi_x^2(x)})$$

$$= \begin{cases} 2\omega\left(f, \sqrt{\left(\dfrac{c_n}{b_n} - 1\right)^2 x^2 + \dfrac{c_n^2}{nb_n^2} x(b_n - x)}\right) & \text{if } 0 \leqslant x \leqslant b_n, \\ 2\omega(f, c_n - x) & \text{if } b_n < x \leqslant c_n, \end{cases}$$

holds true for any $f \in E^0_\alpha$. The uniform estimate (2.7) now immediately follows, since a straightforward computation yields for n large enough

$$\sup_{0\leqslant x\leqslant c_n} \frac{\sqrt{C_n\psi_x^2(x)}}{1+x^\alpha} \leqslant \sqrt{(c_n-b_n)^2+\frac{c_n^2}{nb_n}}.$$

□

The following two lemmas will be very useful in the sequel.

Lemma 2.2. *Let $(\varrho_n)_{n\geqslant 1}$ be a divergent sequence of strictly positive real numbers such that*

$$\varrho_n\frac{c_n}{n} \to 2a \quad \text{and} \quad \varrho_n\left(\frac{c_n}{b_n}-1\right) \to b \quad \text{as} \quad n\to+\infty, \tag{2.8}$$

where $a>0$ $b\geqslant 0$. Then, if $\alpha\geqslant 4$, we have

$$\lim_{n\to+\infty} \frac{\varrho_n}{1+x^\alpha}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(c_n\frac{k}{n}-x\right)^4 = 0 \tag{2.9}$$

uniformly on $[0,+\infty[$.

P r o o f. For any $n\geqslant 1$ and $x\geqslant 0$ a direct computation shows that

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(c_n\frac{k}{n}-x\right)^4\\
&= x^4\left[1-4\frac{c_n}{b_n}+6\frac{c_n^2}{b_n^2}-6\frac{c_n^2}{nb_n^2}-4\frac{c_n^3(n-1)(n-2)}{n^2b_n^3}+\frac{c_n^4(n-1)^4(n-2)}{n^5b_n^4}\right]\\
&\quad + x^3\left[6\frac{c_n^2}{nb_n}-12\frac{c_n^3(n-1)}{n^2b_n^2}+3\frac{c_n^4(n-1)^2}{n^3b_n^3}-3\frac{c_n^4(n-1)}{n^3b_n^3}+3\frac{c_n^4(n-1)^4}{n^5b_n^3}\right]\\
&\quad + x^2\left[-4\frac{c_n^3}{n^2b_n}+6\frac{c_n^4(n-1)}{n^3b_n^2}+\frac{c_n^4(n-1)^3}{n^5b_n^2}\right]+x\frac{c_n^4}{n^3b_n}.
\end{aligned}$$

Let us denote by α_n, β_n, γ_n, δ_n the coefficients of the powers x^4, x^3, x^2 and x, respectively, in the above equality; since $\alpha\geqslant 4$ by assumption, for any $n\geqslant 1$ and $x\geqslant 0$ we have

$$\begin{aligned}
&\frac{\varrho_n}{1+x^\alpha}\left|\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(c_n\frac{k}{n}-x\right)^4\right|\\
&\quad\leqslant \frac{x^4}{1+x^\alpha}|\varrho_n\alpha_n|+\frac{x^3}{1+x^\alpha}|\varrho_n\beta_n|+\frac{x^2}{1+x^\alpha}|\varrho_n\gamma_n|+\frac{x}{1+x^\alpha}|\varrho_n\delta_n|\\
&\quad\leqslant |\varrho_n\alpha_n|+|\varrho_n\beta_n|+|\varrho_n\gamma_n|+|\varrho_n\delta_n|.
\end{aligned}$$

Now the assertion easily follows, because all sequences in the above last term tend to 0 as $n \to +\infty$ on account of (2.8) and the conditions on $(b_n)_{n\geqslant 1}$ and $(c_n)_{n\geqslant 1}$ stated just before the definition (2.1). $\square$

Lemma 2.3. *Under the assumptions* (2.8), *if* $\alpha > 2$, *we have*

$$\lim_{n\to+\infty} \|\varrho_n C_n(\psi_x) - be_1\|_{\alpha-1} = \lim_{n\to+\infty} \|\varrho_n C_n(\psi_x^2) - 2ae_1\|_\alpha = 0. \tag{2.10}$$

P r o o f. Let us choose $n \geqslant 1$; then, on account of (2.4) and (2.5), we get the estimates

$$\frac{|\varrho_n C_n \psi_x(x) - bx|}{1+x^{\alpha-1}} \leqslant \begin{cases} \left|\varrho_n\left(\dfrac{c_n}{b_n}-1\right) - b\right| & \text{if } 0 \leqslant x \leqslant b_n, \\ \dfrac{\varrho_n(c_n-b_n)+bc_n}{1+b_n^{\alpha-1}} & \text{if } b_n < x \leqslant c_n, \end{cases}$$

$$\frac{|\varrho_n C_n \psi_x^2(x) - 2ax|}{1+x^{\alpha}} \leqslant \begin{cases} \varrho_n\left(\dfrac{c_n}{b_n}-1\right)^2 + \dfrac{\varrho_n c_n^2}{nb_n^2} + \left|\dfrac{\varrho_n c_n^2}{nb_n} - 2a\right| & \text{if } 0 \leqslant x \leqslant b_n, \\ \dfrac{\varrho_n(c_n-b_n)^2+2ac_n}{1+b_n^{\alpha}} & \text{if } b_n < x \leqslant c_n, \end{cases}$$

and all terms on the right-hand sides tend to 0 as $n \to +\infty$. Now, in order to find out an estimate for $x > c_n$, let us first observe that the function $g(x) := x/(1+x^{\alpha-1})$ $(x \geqslant 0)$ attains its maximum at a point, say x_0, in $]0,+\infty[$. Of course there exists $k \in \mathbb{N}$ such that $c_n > x_0$ for any $n \geqslant k$ and g is strictly decreasing in $[c_n,+\infty[$. It immediately follows that for $n \geqslant k$ and $x \in [c_n,+\infty[$

$$\frac{|\varrho_n C_n \psi_x(x) - bx|}{1+x^{\alpha-1}} = bg(x) \leqslant \frac{bc_n}{1+c_n^{\alpha-1}},$$

where again the term on the right-hand side tends to 0 as $n \to +\infty$. Arguing similarly for $C_n\psi_x^2(x)$ gives (2.10). $\square$

Now we are ready to prove our main result, which states a Voronovskaja-type formula for the operators C_n.

Theorem 2.4. *For any* $f \in C^2([0,+\infty[) \cap E_\alpha^0$ $(\alpha \geqslant 4)$ *such that* $f'' \in UC_b([0,+\infty[)$ *we have*

$$\lim_{n\to+\infty} \varrho_n(C_n f(x) - f(x)) = axf''(x) + bxf'(x) \quad \textit{in} \quad E_\alpha^0, \tag{2.11}$$

$(\varrho_n)_{n\geqslant 1}$, *a and b being the same as those appearing in Lemma* 2.2.

Proof. First of all, let us note that if $f \in C^2([0,+\infty[) \cap E^0_\alpha$ with $f'' \in UC_b([0,+\infty[)$, because of the identity

$$f'(x) = f'(0) + \int_0^x f''(s)\,\mathrm{d}s \quad (x \geqslant 0), \tag{1}$$

for a suitable constant $K > 0$ one has

$$\frac{|f'(x)|}{1+x} \leqslant K \quad (x \geqslant 0). \tag{2}$$

Moreover, if $|f''(x)| \leqslant M$ for every $x \geqslant 0$, then obviously

$$|f'(x) - f'(y)| \leqslant M|x-y| \quad (x, y \geqslant 0). \tag{3}$$

We will show that (2.11) holds true on each of the intervals $[0, b_n]$, $]b_n, c_n]$, and $]c_n, +\infty[$, as suggested by the definition of our operators C_n.

To start with, fix $n \geqslant 1$ and note that if $x \in [0, b_n]$, by virtue of Taylor's formula, for any $k = 0, 1, \ldots, n$ there exists $d_{n,k,x}$ lying between x and $c_n k/n$ such that

$$f\Big(\frac{c_n k}{n}\Big) - f(x) = f'(x)\Big(c_n\frac{k}{n} - x\Big) + \frac{f''(x)}{2}\Big(c_n\frac{k}{n} - x\Big)^2 + \frac{f''(d_{n,k,x}) - f''(x)}{2}\Big(c_n\frac{k}{n} - x\Big)^2.$$

After setting

$$\mu\Big(x, \frac{c_n k}{n}\Big) := \frac{f''(d_{n,k,x}) - f''(x)}{2}, \tag{4}$$

we may therefore write

$$\varrho_n(C_n f(x) - f(x)) = \varrho_n f'(x) C_n \psi_x(x) + \frac{1}{2}\varrho_n f''(x) C_n \psi_x^2(x) + \varrho_n R_n(x),$$

where

$$R_n(x) = \sum_{k=0}^n \binom{n}{k}\Big(\frac{x}{b_n}\Big)^k\Big(1 - \frac{x}{b_n}\Big)^{n-k}\mu\Big(x, \frac{c_n k}{n}\Big)\Big(c_n\frac{k}{n} - x\Big)^2.$$

It follows that

$$\begin{aligned}\frac{1}{1+x^\alpha}&|\varrho_n(C_n f(x) - f(x)) - axf''(x) - bxf'(x)| \\ &\leqslant \frac{1}{1+x^\alpha}\Big|\varrho_n \frac{1}{2} f''(x) C_n \psi_x^2(x) - axf''(x)\Big| \\ &\quad + \frac{1}{1+x^\alpha}|\varrho_n f'(x) C_n \psi_x(x) - bxf'(x)| + \frac{\varrho_n}{1+x^\alpha}|R_n(x)|,\end{aligned}$$

where the first two members on the right-hand side tend to 0 uniformly: simply apply Lemma 2.3, taking also into account that f'' is bounded by assumption and that

$$\begin{aligned}\frac{1}{1+x^\alpha}|\varrho_n f'(x)C_n\psi_x(x)-bxf'(x)| &\leqslant N\frac{|f'(x)|}{1+x}\|\varrho_n C_n(\psi_x)-be_1\|_{\alpha-1}\\ &\leqslant NK\|\varrho_n C_n(\psi_x)-be_1\|_{\alpha-1}\end{aligned}$$

by virtue of (2) (here N is a suitable positive constant).

Therefore, in order to establish (2.11) in $[0,b_n]$, it is sufficient to show that $\lim_{n\to+\infty}\varrho_n(1+x^\alpha)^{-1}|R_n(x)| = 0$ uniformly. To this aim, note that the assumptions on f together with the definition (4) ensure that $|\mu(x,t)| \leqslant M$ for every $(x,t)\in[0,b_n]\times[0,c_n]$ and that $\lim_{t\to x}\mu(x,t)=0$ uniformly with respect to $x\in[0,b_n]$.

Now fix $\varepsilon>0$ and choose $\delta>0$ such that $|\mu(x,t)|<\varepsilon$ whenever $|x-t|<\delta$; then (2.9) and the second limit in (2.10) yield

$$\frac{\varrho_n}{1+x^\alpha}\left|\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(c_n\frac{k}{n}-x\right)^4\right| < \frac{a\varepsilon\delta^2}{2M}$$

and

$$\frac{|\varrho_n C_n\psi_x^2(x)-2ax|}{1+x^\alpha} < a/2$$

for every $x\in[0,b_n]$ if n is large enough, say $n\geqslant n_0$. It follows that for every $x\in[0,b_n]$ and $n\geqslant n_0$

$$\begin{aligned}&\frac{\varrho_n}{1+x^\alpha}|R_n(x)|\\ &\quad\leqslant \frac{\varepsilon\varrho_n}{1+x^\alpha}\left|\sum_{\substack{k=0\\|c_nk/n-x|<\delta}}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(c_n\frac{k}{n}-x\right)^2\right|\\ &\qquad+\frac{\varrho_n}{1+x^\alpha}\left|\sum_{\substack{k=0\\|c_nk/n-x|\geqslant\delta}}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\mu\left(x,\frac{c_nk}{n}\right)\left(c_n\frac{k}{n}-x\right)^2\right|\\ &\quad\leqslant\frac{\varepsilon\varrho_n}{1+x^\alpha}C_n\psi_x^2(x)+\frac{\varrho_n M}{\delta^2(1+x^\alpha)}\left|\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(c_n\frac{k}{n}-x\right)^4\right|\\ &\quad\leqslant\varepsilon\frac{|\varrho_n C_n\psi_x^2(x)-2ax|}{1+x^\alpha}+\frac{2a\varepsilon x}{1+x^\alpha}+\frac{a\varepsilon}{2}\leqslant 3a\varepsilon.\end{aligned}$$

Therefore $\limsup_{n\to+\infty}\varrho_n(1+x^\alpha)^{-1}|R_n(x)|\leqslant 3a\varepsilon$ and, consequently, since ε is arbitrary, the proof is complete in this first case.

Now, if $x \in\,]b_n, c_n]$, since $C_n f(x) = f(c_n)$ by the definition (2.1), applying Taylor's formula together with (2.4) and (2.5) gives

$$\begin{aligned}
&\frac{1}{1+x^\alpha}|\varrho_n(f(c_n) - f(x)) - axf''(x) - bxf'(x)| \\
&\quad = \frac{1}{1+x^\alpha}\Big|\varrho_n f'(c_n)(c_n - x) \\
&\qquad - \varrho_n \frac{1}{2} f''(d_{n,x})(c_n - x)^2 - axf''(x) - bxf'(x)\Big| \\
&\quad \leqslant \frac{1}{1+x^\alpha}|\varrho_n C_n \psi_x(x) f'(c_n) - bxf'(x)| \\
&\qquad + \frac{1}{1+x^\alpha}\left|\varrho_n C_n \psi_x^2(x)\frac{f''(d_{n,x})}{2} - axf''(x)\right| := I_1 + I_2,
\end{aligned}$$

$d_{n,x}$ being a suitable point between x and c_n. Next we show that each I_i tends to 0 uniformly; indeed, on account of (3), for a suitable $N > 0$ we have

$$\begin{aligned}
I_1 &\leqslant \frac{1}{1+x^\alpha}|\varrho_n C_n \psi_x(x) f'(c_n) - bxf'(c_n)| + \frac{1}{1+x^\alpha}|bxf'(c_n) - bxf'(x)| \\
&\leqslant \frac{N|f'(c_n)|}{1+b_n}\|\varrho_n C_n(\psi_x) - be_1\|_{\alpha-1} + \frac{Mbc_n(c_n - b_n)}{1+b_n^\alpha},
\end{aligned}$$

and the term on the right-hand side tends to 0 due to the first limit in (2.10) and to (2), because $|f'(c_n)|/(1+b_n) \approx |f'(c_n)|/(1+c_n)$ as $n \to +\infty$.

Similarly, since $|f''(x)| \leqslant M$ for every $x \geqslant 0$ by assumption, we get

$$\begin{aligned}
I_2 &\leqslant \frac{1}{1+x^\alpha}\left|\varrho_n C_n \psi_x^2(x)\frac{f''(d_{n,x})}{2} - axf''(d_{n,x})\right| + \frac{1}{1+x^\alpha}|axf''(d_{n,x}) - axf''(x)| \\
&\leqslant \frac{M}{2}\|\varrho_n C_n(\psi_x^2) - 2ae_1\|_\alpha + \frac{2Mac_n}{1+b_n^\alpha},
\end{aligned}$$

which easily yields $I_2 \to 0$, too, because of the second limit in (2.10).

At last, when $x > c_n$ and therefore $C_n f(x) = f(x)$ by definition, we have, for n large enough and a suitable $N > 0$ (see the last part of the proof of Lemma 2.3)

$$\frac{1}{1+x^\alpha}|axf''(x) + bxf'(x)| \leqslant \frac{Mac_n}{1+c_n^\alpha} + \frac{N|f'(x)|}{1+x} \cdot \frac{bc_n}{1+c_n^{\alpha-1}},$$

where again the term on the right-hand side tends to 0 because of (2).

The proof of the theorem is now complete. □

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