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COMMUTATIVITY OF RINGS WITH POLYNOMIAL CONSTRAINTS

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Abstract. Let p, q and r be fixed non-negative integers. In this note, it is shown that if R is left (right) s-unital ring satisfying $[f(x^py^q)-x^ry,x]=0$ ($[f(x^py^q)-yx^r,x]=0$, respectively) where $f(\lambda)\in\lambda^2\mathbb{Z}[\lambda]$, then R is commutative. Moreover, commutativity of R is also obtained under different sets of constraints on integral exponents. Also, we provide some counterexamples which show that the hypotheses are not altogether superfluous. Thus, many well-known commutativity theorems become corollaries of our results.

Keywords: automorphism, commutativity, local ring, polynomial identity, s-unital ring

MSC 2000: 16U80, 16U99

1. Introduction

Throughout the paper, R will denote an associative ring, $N(R)$ the set of nilpotent elements of R, $U(R)$ the group of units of R, $\mathbb{Z}[X,Y]$ the ring of polynomials in two commuting indeterminates, $\mathbb{Z}\langle X,Y\rangle$ the ring of polynomials in two non-commuting indeterminates over the ring $\mathbb{Z}$ of integers and $\mathbb{Z}[X]$ the totality of all polynomials in X over $\mathbb{Z}$, the ring of integers. For any $x,y\in R$, $[x,y]=xy-yx$.

A ring R is said to be a left (right) s-unital ring if $x\in Rx$ for each x in R ($x\in xR$, respectively) and R is called s-unital in case it is a left as well as a right s-unital.

Now, we consider the following ring properties:

(P) For each x in R, there exist polynomials $f(\lambda)\in\lambda^2\mathbb{Z}[\lambda]$ and $g(\lambda),h(\lambda)\in\mathbb{Z}[\lambda]$ depending on x such that

$$g(x)[f(x),y]h(x)=\pm y^n[x,y^m]$$

for all y in R and fixed integers $n\geqslant 0$, $m>1$.

(P_1) For each $x \in R$, there exist polynomials $f(\lambda)$ in $\lambda^2\mathbb{Z}[\lambda]$ and $g(\lambda)$, $h(\lambda)$ in $\mathbb{Z}[\lambda]$ depending on x such that

$$g(x)[f(x), y]h(x) = \pm[x, y^m]y^n$$

for all $y \in R$ and fixed integers $n \geqslant 0$, $m > 1$.

(P_2) Let p, q and r be fixed non-negative integers. For each $x, y \in R$ there exists a polynomial $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ such that

$$[f(x^p y^q) - x^r y, x] = 0.$$

(P_2^*) For each $x, y \in R$ there exist a polynomial $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ and non-negative integers p, q, r such that

$$[f(x^p y^q) - x^r y, x] = 0.$$

(P_3) Let p, q and r be fixed non-negative integers. For each $x, y \in R$ there exists a polynomial $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ such that

$$[f(x^p y^q) - y x^r, x] = 0.$$

(P_3^*) For each $x, y \in R$ there exist a polynomial $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ and non-negative integers p, q, r such that

$$[f(x^p y^q) - y x^r, x] = 0.$$

(P_4) For each $y \in R$ there exist $f(\lambda)$, $g(\lambda)$ in $\lambda^2\mathbb{Z}[\lambda]$ such that

$$x^t[x^n, y] = g(y)[x, f(y)]x^s \quad \& \quad x^t[x^m, y] = g(y)[x, f(y)]x^s$$

for all $x \in R$ where $m \geqslant 1$, $n \geqslant 1$ and s, t are fixed non-negative integers with $(m, n) = 1$ and at least one of s and t is non-zero.

(P_4^*) For each $x, y \in R$ there exist polynomials $f(\lambda)$, $g(\lambda)$ in $\lambda^2\mathbb{Z}[\lambda]$ and non-negative integers s, t and $m \geqslant 1$, $n \geqslant 1$ with $(m, n) = 1$ such that

$$x^t[x^n, y] = g(y)[x, f(y)]x^s \quad \& \quad x^t[x^m, y] = g(y)[x, f(y)]x^s.$$

(P_5) For each $y \in R$ there exist $f(\lambda)$, $g(\lambda)$ in $\lambda^2\mathbb{Z}[\lambda]$ such that

$$[x^n, y]x^t = g(y)[x, f(y)]x^s \quad \& \quad [x^m, y]x^t = g(y)[x, f(y)]x^s$$

for all $x \in R$ where $m \geqslant 1$, $n \geqslant 1$ and s, t are fixed non-negative integers with $(m, n) = 1$ and at least one of s and t is non-zero.

(P_5^*) For each x and y in R there exist polynomials $f(\lambda), g(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ and non-negative integers t and $m \geqslant 1$, $n \geqslant 1$ with $(m, n) = 1$ such that

$$[x^n, y]x^t = g(y)[x, f(y)]x^s \quad \& \quad [x^m, y]x^t = g(y)[x, f(y)]x^s.$$

Q(m) For all x, y in R, $m[x, y] = 0$ implies that $[x, y] = 0$, where m is a positive integer.

(CH) For each $x, y \in R$ there exist $f(\lambda)$, $h(\lambda)$ in $\lambda^2\mathbb{Z}[\lambda]$ such that $[x - f(x), y - h(y)] = 0$.

There are several results dealing with the conditions under which R is commutative. Generally such conditions are imposed either on the ring itself or on its commutator. A nice theorem due to Herstein [6] asserts that rings satisfying the polynomial identity $(x + y)^k = x^k + y^k$ for some $k > 1$ must have a nil commutator ideal. Among other classes of rings in which $C(R)$ is known to be nil is the class of rings satisfying the polynomial identity $[x^k, y] = [x, y^k]$ for some $k > 1$ (see [5]). This class includes the rings satisfying the polynomial identity $(x + y)^k = x^k + y^k$. Motivated by this observation, Bell [4] proved that a ring R with unity 1 satisfying the polynomial identity $[x^k, y] = [x, y^k]$ is commutative if the additive group $(R, +)$ is k-torsion free. In attempts to generalize this result, several authors have considered various special cases of (P) and (P_1) (cf. [1], [2], [5], [6], [7], [11], [12], [14], [16]). In most of the cases the underlying polynomials are assumed to be monomials.

In an attempt to prove commutativity of rings satisfying such conditions, the author [11] has shown that a ring with unity 1 is commutative if, for all $x \in R$, there exist polynomials $f(\lambda), g(\lambda), h(\lambda) \in \mathbb{Z}[\lambda]$ such that $g(x)[f(x), y]h(x) = y^t[x, y^n]$ and $g(x)[f(x), y]h(x) = y^t[x, y^m]$ for all y in R, where t, m, n are fixed positive integers with $(m, n) = 1$. In the same paper it is conjectured that an m-torsion free ring with unity 1 satisfies the condition (P) is commutative. In Section 2, we shall prove this conjecture and, in Section 3, study commutativity theorems through a Streb's result: if R satisfies (P_2), (P_3), (P_4) or (P_5), then Q(m) is replaced by some other suitable constraints on the exponent m. On the other hand, in Section 4, commutativity of rings satisfying any one of the properties (P_2^*), $(P_3)^*$, (P_4^*), $(P_5)^*$, is investigated.

2. Commutativity theorems for rings with unity

Theorem 2.1. *Let R be a ring with unity 1 satisfying the property* (P). *If R also satisfies* Q(m), *then R is commutative (and conversely).*

We begin with the following known results.

Lemma 2.1 [9, p. 221]**.** *If x, y are elements of a ring R with $[x,[x,y]]=0$, then $[x^n,y]=nx^{n-1}[x,y]$ for any positive integer n.*

Lemma 2.2 [10, Theorem]**.** *Let f be a polynomial in n noncommuting indeterminates $x_1,x_2,\dots,x_n$ with relatively prime integral coefficients. Then the following assertions are equivalent:*
(i) *For any ring satisfying the polynomial identity $f=0$, $C(R)$ is a nil ideal.*
(ii) *For every prime p, $(GF(p))_2$ fails to satisfy $f=0$.*

Lemma 2.3 [19, Hauptsatz 3]**.** *Let R satisfy a polynomial identity of the form $[x,y]=p(x,y)$, where $p(X,Y)$ in $\mathbb{Z}\langle X,Y\rangle$ has the following properties:*
(a) *$p(X,Y)$ is in the kernel of the natural homomorphism from $\mathbb{Z}\langle X,Y\rangle$ to $\mathbb{Z}[X,Y]$;*
(b) *each monomial of $p(X,Y)$ has total degree at least* 3;
(c) *each monomial of $p(X,Y)$ has X-degree at least* 2, *or each monomial of $p(X,Y)$ has Y-degree at least* 2.
Then R is commutative.

Here we shall also prove the following lemma which will be repeatedly referred to in [15, Lemma] for a fixed exponent n, but with a slight modification in the proof it can be established for a variable exponent n.

Lemma 2.4. *Let R be a ring with unity 1 and let f be any polynomial function of two variables with the property $f(x+1,y)=f(x,y)$ for all x, y in R. If for all x, y in R there exists a positive integer $n=n(x,y)$ such that $x^nf(x,y)=0$ (or $f(x,y)x^n=0$), then necessarily $f(x,y)=0$.*

P r o o f. It is given that $x^nf(x,y)=0$, $n=n(x,y)\geqslant 1$. Choose an integer $n_1=n(1+x,y)$ such that $(1+x)^{n_1}f(x,y)=0$. If $k=\max\{n,n_1\}$, then $x^kf(x,y)=0$ and

$$(1+x)^kf(x,y)=0.$$

We have

$$f(x,y)=\{(1+x)-x\}^{2k+1}f(x,y).$$

Expanding the expression on the right-hand side by the binomial theorem, we get $f(x,y)=0$.

A similar proof is valid in the case that R satisfies $f(x,y)x^n=0$. □

Lemma 2.5. *Let R be a ring with unity 1 satisfying either* (P) *or* (P_1). *Then*

$$C(R) \subseteq N(R).$$

P r o o f. Let R satisfy the condition (P). By our hypothesis we have

$$g(x)[f(x), y]h(x) = \pm y^n[x, y^m]. \tag{1}$$

Replacing y by $x+y$ in (1) and using (1), we get

$$y^n[x, y^m] = (x+y)^n[x, (x+y)^m] \tag{2}$$

for all x, y in R. Equation (2) is a polynomial identity and one can observe that $x = e_{11}$, $y = -e_{11} + e_{12}$ fail to satisfy this equality in $(GF(p))_2$, p a prime, and hence by Lemma 2.2, $C(R) \subseteq N(R)$.

On the other hand, if R satisfies the condition (P_1), then, using the same argument with $x = e_{11}$, $y = -e_{11} + e_{21}$, we get the required result. □

P r o o f of Theorem 2.1. Suppose R satisfies the condition (P). Now, we shall show that nilpotents are central. Let $u \in N(R)$. Then there exists a minimal positive integer t such that

$$u^k \in Z(R) \tag{1}$$

for all integers $k \geqslant t$. If $t = 1$, each such u is central. Therefore, assume now that $t > 1$. Replacing y by u^{t-1} in (P), we get

$$g(x)[f(x), u^{t-1}]h(x) = \pm u^{n(t-1)}[x, u^{m(t-1)}]$$

for all $x \in R$. Now in view of (1) and the fact that $m(t-1) \geqslant t$ for $m > 1$, we get

$$g(x)[f(x), u^{t-1}]h(x) = 0. \tag{2}$$

Replacing y by $1 + u^{t-1}$ in (P), we get

$$g(x)[f(x), 1+u^{t-1}]h(x) = \pm(1+u^{t-1})^n[x, (1+u^{t-1})^m].$$

This, in view of (2), yields that

$$(1+u^{t-1})^n[x, (1+u^{t-1})^m] = 0.$$

However, since $1+u^{t-1}$ is invertible, the last equation reduces to

$$[x,(1+u^{t-1})^m]=0.$$

That is,

$$0=[x,1+mu^{t-1}]=[x,(1+u^{t-1})^m].$$

This yields that

$$m[x,u^{t-1}]=0$$

for all $x\in R$, and the application of Q(m) gives that $u^{t-1}\in Z(R)$. This is a contradiction, and hence $t=1$. Thus we obtain $N(R)\subseteq Z(R)$. Combining this fact with Lemma 2.5, we have

$$C(R)\subseteq N(R)\subseteq Z(R). \tag{3}$$

Note that the left hand side of the equality involved in (P) remains unchanged if y is replaced by $1+y$; therefore

$$(1+y)^n[x,(1+y)^m]-y^n[x,y^m]=0.$$

But, in view of (3), Lemma 2.1 is applicable in the present case, and the last identity implies that

$$m[x,y]\{(1+y)^{m+n-1}-y^{m+n-1}\}=0 \tag{4}$$

or

$$m[x\{(1+y)^{m+n-1}-y^{m+n-1}\},y]=0$$

for all $x,y\in R$. Applying the property Q(m) to (4), we get

$$[x\{(1+y)^{m+n-1}-y^{m+n-1}\},y]=0. \tag{5}$$

Equation (5) is a polynomial identity and can be rewritten in the form

$$[x,y]=[x,y]yh(y)$$

for some $h(X)\in\mathbb{Z}[X]$. Hence, by Lemma 2.3, R is commutative. □

Corollary 2.1. *Let $m > 1$ and n be fixed non-negative integers, and R a ring with unity 1 in which for every $x \in R$ there exist integers $p = p(x) \geqslant 0$, $k = k(x) \geqslant 0$, $r = r(x) \geqslant 0$, depending on x, such that*

$$x^p[x^k, y]x^r = \pm y^n[x, y^m]$$

for all $y \in R$. If R satisfies Q(m), *then R is commutative (and conversely).*

Using similar arguments with the necessary variations, one can prove

Theorem 2.2. *Let R be a ring with unity* 1 *possessing the property* (P_1). *If R satisfies* Q(m), *then R is commutative (and conversely).*

Remark 2.1. The ring of 3×3 strictly upper triangular matrices over a field provides an example showing that the above theorems are not valid for arbitrary rings. Moreover, the following ring shows that the property Q(m) in the hypotheses of the above theorems cannot be deleted.

Example 2.1. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha, \beta, \gamma \in GF(4) \right\}$ be the set of matrices. It is readily verified that R (with the usual matrix addition and multiplication) is a non-commutative local ring with unity I, the identity matrix. Further, R satisfies

$$x^{48} \in Z(R) \quad \text{for all} \quad x \in R. \tag{1}$$

Since $N'(R)$ consists of all matrices x in R with zero diagonal elements, it contains exactly 16 elements. For any $x \in N'(R)$, $x^2 = 0$ and hence $x^{48} = 0 \in Z(R)$. The set $R/N'(R)$ is a multiplicative group of order 48 and hence $x^{48} = I \in Z(R)$ for all $x \in R/N'(R)$. In view of (1), it follows that R satisfies the conditions (P), (P_1) and the hypothesis of Corollary 2.1 for the same k and m and for arbitrary non-negative integers p, r, n. This shows that the assumption that R has the property Q(m) in Theorems 2.1, 2.2 and Corollary 2.1 cannot be eliminated.

3. Commutativity theorems through a Streb's result

In an attempt to generalize famous Jacobson's "$x^n = x$" theorem it was proved by Herstein [7] that if for each $x, y \in R$ there exists a polynomial $f(t) \in t^2\mathbb{Z}[t]$ such that $[x - f(x), y] = 0$, then R is commutative. In their paper [17], Putcha and Yaqub established that if for each $x, y \in R$ there exists a polynomial $f(t) \in t^2\mathbb{Z}[t]$ such that $xy - f(xy)$ is central, then R^2 is central. Further, the author jointly with

Bell and Quadri [3] established the commutativity of R with unity 1 satisfying the polynomial identity $[xy-f(xy),x]=0$, where $f(t)\in t^2\mathbb{Z}[t]$. The aim of this section is to generalize the above results to the rings possessing the above properties; also other commutativity theorems for one-sided s-unital rings are obtained under different sets of conditions.

In view of Example 2.1, it is natural to ask under what additional conditions, R turns out to be commutative if the property Q(m) is dropped from the hypotheses of Theorems 2.1 and 2.2. The following theorem yields an answer to this question.

Theorem 3.1. *Let R be a left (right) s-unital ring with the property* (P_2), *(*(P_3)*, respectively). Then R is commutative (and conversely).*

Theorem 3.2. *Let R be a left (right) s-unital ring with the property* (P_4) *(*(P_5)*, respectively). Then R is commutative (and conversely).*

In order to develop the proof of the above theorems, we consider the following types of rings.

$(1)_l$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(1)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(1) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(2) $M_\sigma(F)=\left\{\begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a,b\in F\right\}$, where F is a finite field with a non-trivial automorphism σ.

(3) A non-commutative ring with no non-zero divisors of zero.

(4) $S=\langle 1\rangle+T$, T being a non-commutative subring of S such that

$$T[T,T]=[T,T]T=0.$$

In [18], Streb classified non-commutative rings, which has been used effectively to establish several commutativity theorems (cf. [12], [13], [14]). It can be observed from the proof of [13, Corollary 1] that if R is a non-commutative left s-unital ring, then there exists a factorsubring B of R which is of the type $(1)_l$, (2), (3) or (4). This gives a result which plays a vital role in our subsequent discussion (cf. [14, Meta theorem]).

Lemma 3.1. *Let P be a ring property which is inherited by factor subrings. If no rings of type* $(1)_l$, (2), (3) *or* (4) *satisfy* (P)*, then every left s-unital ring satisfying* (P) *is commutative.*

We pause to remark that the dual of the above lemma holds; if P is a ring property which is inherited by factorsubrings, and if no rings of type $(1)_r$, (2), (3) or (4) satisfy (P), then every right s-unital ring satisfying (P) is commutative.

We begin with the following known results.

Lemma 3.2 [12, Lemma 1]. *Let R be a left (right) s-unital ring and not a right (left, respectively) s-unital one. Then R has a factorsubring of type $(1)_l$ ($(1)_r$, respectively).*

Lemma 3.3 [13, Corollary 1]. *Let R be a non-commutative ring satisfying* (CH). *Then there exists a factorsubring of R which is of type* (1) *or* (2).

Now, we establish the following results called steps.

Step 3.1. Let B be a ring of type $(1)_l$ or (2). Then B does not satisfy $(\mathrm{P}_2)^*$.

Proof. Let B be of type $(1)_l$. Then in $(\mathrm{GF(P)})_2$, p a prime, putting $x = e_{11}$ and $y = e_{12}$ in the hypothesis, we get

$$[f(x^p y^q) - x^r y, x] = e_{12} \neq 0.$$

Suppose that B is a ring of type (2). Taking $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$ ($\sigma(a) \neq a$), $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in $B = M_\sigma(F)$, one observes that

$$[f(x^p y^q) - x^r y, x] = a^r(a - \sigma(a))e_{12} \neq 0,$$

and this shows that B is not of type (2).

Similar arguments maybe used if R has the property $(\mathrm{P}_3)^*$; then one can prove

Step 3.2. If a ring B is of type $(1)_r$ or (2), then B does not satisfy $(\mathrm{P}_3)^*$.

Proof of Theorem 3.1. It is enough to show that no rings of type $(1)_l$, (2), (3) or (4) satisfy (P_2). From Step 3.1, one can observe that no rings of type $(1)_l$ and (2) satisfy (P_2). Hence by Lemma 3.2, R is also right s-unital and hence it is s-unital. Thus in view of Proposition 1 of [8], we can assume that R has unity 1. Since $x = e_{22}$ and $y = e_{21}$ do not satisfy (P_2), by Lemma 3.3 we see that the commutator ideal of R is nil and hence no rings of type (3) satisfy (P_2).

Finally, suppose R is a ring of type (4). Let $t_1, t_2 \in T$ be such that $[t_1, t_2] \neq 0$. Then by hypothesis, we have

$$(1 + t_1)^r [t_1, t_2] = [1 + t_1, f((1 + t_1)^p\, t_2^q)] = 0.$$

This implies that $[t_1, t_2] = 0$. This leads to a contradiction and hence R is not of type (4).

Hence we have seen that no rings of type $(1)_l$, (2), (3) or (4) satisfy (P_2) and by Lemma 3.1, R is commutative.

Similar arguments maybe used if R possesses the property (P_3). □

Proof of Theorem 3.2. In $(GF(p))_2$, put $x = e_{11} + e_{12}$, $y = e_{12}$ in (P_4) to get

$$x^t[x^m, y] = g(y)[x, f(y)]x^s = e_{12} \neq 0.$$

Hence, R is not of type $(1)_l$; by Lemma 3.2, R is also right s-unital and hence it is s-unital. In view of Proposition 1 of [8], we may assume that the ring R has unity 1.

Consider the ring $M_\sigma(F)$, a ring of type (2). Notice that $N = Fe_{12}$. Hence for $b \in N$ and an arbitrary unit $u \in U(R)$ we obtain that there exists a polynomial $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$ such that

$$u^t[u^m, b] = g(b)[u, f(b)]u^s = 0$$

and

$$u^t[u^n, b] = g(b)[u, f(b)]u^s = 0.$$

Since $b^2 = 0$ and u is a unit of R, the last two equations yield that $[u^m, b] = 0$ and $[u^n, b] = 0$. Now for a non-central element $b = e_{12}$, $[u, e_{12}] = 0$ gives that e_{12} is central, which is a contradiction. Hence R cannot be of type (2).

By hypothesis, we have

$$x^t[x^m, y] = g(y)[x, f(y)]x^s. \tag{1}$$

Replacing x by $x + 1$ in (1) and then multiplying it by x^s, we get

$$(x + 1)^t[(x + 1)^m, y]x^s = g(y)[x, f(y)](x + 1)^s x^s. \tag{2}$$

Multiply (1) by $(x + 1)^s$ to get

$$x^t[x^m, y](x + 1)^s = g(y)[x, f(y)]x^s(x + 1)^s. \tag{3}$$

Now, compare (2) and (3) to get

$$(x + 1)^t[(x + 1)^m, y]x^s = x^t[x^m, y](x + 1)^s. \tag{4}$$

Equation (4) is a polynomial identity and $x = e_{11} + e_{12}$ and $y = e_{12} \in (GF(p))_2$ fail to satisfy (4). By Lemma 3.3, the commutator ideal of R is nil and hence no rings of type (3) satisfy (P_4).

Finally, let R be a ring of type (4). Suppose $[a, b] \neq 0$, where $a, b \in T$. There exists $f(\lambda)$ in $\lambda^2\mathbb{Z}[\lambda]$ such that

$$m[a, b] = (1 + a)^t[(1 + a)^m, b] = g(b)[a, f(b)](1 + a)^s = 0$$

and

$$n[a, b] = (1 + a)^t[(1 + a)^n, b] = g(b)[a, f(b)](1 + a)^s = 0.$$

Since $(m, n) = 1$, we get $[a, b] = 0$, and this gives a contradiction. Hence there is no ring of type (4) satisfying (P_4).

No rings of types $(1)_l$, (2), (3) or (4) satisfy (P_4). Thus by Lemma 3.1, R is commutative.

Similar arguments maybe used if R satisfies the condition (P_5). □

Corollary 3.1 [4, Theorem 6]**.** *Let R be a ring with unity* 1*, and let $n > 1$ be a fixed integer. If R^+ is n-torsion free and R satisfies the identity $x^n y - xy^n = xy^n - y^n x$ for all $x, y \in R$ then R is commutative.*

A careful scrutiny of the proof of Steps 3.1 and 3.2 shows that if R is a left (right) s-unital ring with the property (P_2) (or (P_3)), then no rings of type $(1)_l$, (or $(1)_r$) satisfy (P_4) (or (P_5), respectively). Hence by Lemma 3.2, R is right (left) s-unital, and hence s-unital. Thus, by Proposition 1 of [8], we can assume that R has unity 1. Now, the application of Theorems 2.1 and 2.2 yields the following result.

Theorem 3.3. *Let R be a left (right) s-unital ring satisfying the property* (P) *((*P_1*), respectively). If R satisfies* Q(m)*, then R is commutative.*

Remark 3.1. The following example demonstrates that there are non-commutative left (right) s-unital rings with the property (P_1) (or (P)), (P_3) (or (P_2)) or (P_5) (or (P_4), respectively).

Example 3.1. Let

$$R_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$\Bigg($or

$$R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}\Bigg)$$

be a subring of $(GF(2))_2$. Then the non-commutative left (or right) s-unital ring (R_1) (or R_2) possesses the property (P_1) (or (P)), (P_3) (or (P_2)) or (P_5) (or (P_4), respectively).

As a corollary to the above Theorem 3.1, we get the following result improving the earlier results (for reference see [2], [3]).

Corollary 3.2. *Let R be a left (or right) s-unital ring in which for each $x, y \in R$ there exists an integer $n = n(x,y) > 1$ such that $[xy - (xy)^n, x] = 0$ (or $[yx - (xy)^n, x] = 0$, respectively). Then R is commutative (and conversely).*

4. Extensions to variable exponents

If the integral exponents p, q and r in the conditions (P_2), (P_3), (P_4) and (P_5) are also allowed to vary with the pair of ring's elements x, y then the weaker versions of the above conditions are (P_2^*), (P_3^*), (P_4^*) and (P_5^*).

From Theorems 3.1 and 3.2 it can be easily shown that no rings of type $(1)_l$ (or $(1)_r$) or (2) satisfy (P_2^*) (or (P_3^*)), (P_4^*) (or (P_5^*), respectively). We omit the details of the proof just to avoid repetition.

Combining this fact with Lemma 3.3, we obtain the following results.

Theorem 4.1. *Suppose that R is a left (or right) s-unital ring with the properties* (P_2^*) *and* (P_3^*). *If R satisfies* (CH), *then R is commutative.*

Theorem 4.2. *Suppose that R is a left (or right) s-unital ring with the properties* (P_4^*) *and* (P_5^*). *If R satisfies* (CH) *then R is commutative.*

Remark 4.1. The following example shows that in the hypothesis of Theorems 3.2 and 4.2, the presence of both the conditions in (P_4), (P_4^*), (P_5) and (P_5^*) is not superfluous (even if R has unity 1).

Example 4.1. Consider $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}$. Then R is a non-commutative ring with unity satisfying the condition $y^t[x, y^4] = [x^4, y]x^s$ where s and t maybe any non-negative integers.

We close our discussion with the following

Question. Let R be one-sided s-unital ring in which for each $y \in R$ there exist polynomials $f(t)$, $g(t)$, $h(t)$ in $\mathbb{Z}[t]$ such that

$$g(y)[x, f(y)]h(y) = \pm x^p[x^n, y]y^q$$

or

$$g(y)[x, f(y)]h(y) = \pm y^p[x, y^n]x^q$$

for all $x \in R$ and fixed integers $p \geqslant 0$, $q \geqslant 0$, $n > 1$. Moreover, if R satisfies $\mathrm{Q}(n)$, then R is commutative.

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