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ON THE STIELTJES MOMENT PROBLEM ON SEMIGROUPS

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Abstract. We characterize finitely generated abelian semigroups such that every completely positive definite function (a function all of whose shifts are positive definite) is an integral of nonnegative miltiplicative real-valued functions (called nonnegative characters).

Keywords: semigroup, abelian, commutative, finitely generated, positive definite, completely positive definite, character

MSC 2000: 44A60, 43A35

1. Introduction

Stieltjes [21] showed that a sequence $(s_n)_{n=0}^{\infty}$ of reals is the moment sequence of a measure on $\mathbb{R}_+$, in the sense that

$$s_n = \int_{\mathbb{R}_+} x^n \, \mathrm{d}\mu(x), \quad n \in \mathbb{N}_0$$

for some measure μ on $\mathbb{R}_+$, if and only if

$$\sum_{j,k=0}^{n} c_j c_k s_{j+k} \geqslant 0 \quad \text{and} \quad \sum_{j,k=0}^{n} c_j c_k s_{j+k+1} \geqslant 0$$

for every choice of $n \in \mathbb{N}_0$ and $c_0, \ldots, c_n \in \mathbb{R}$.

The moment problem thus solved by Stieltjes can be generalized to arbitrary abelian semigroups instead of $\mathbb{N}_0$. Suppose $(S, +)$ is an abelian semigroup. For arbitrary subsets H and K of S, define $H + K = \{x + y \mid x \in H,\ y \in K\}$. A *positive definite function on S* is a function $\varphi\colon S + S \to \mathbb{R}$ such that

$$\sum_{j,k=1}^{n} c_j c_k \varphi(s_j + s_k) \geqslant 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{R}$. Denote by $\mathcal{P}(S)$ the set of all positive definite functions on S. A *character* on S is a function $\sigma\colon S \to \mathbb{R}$, not identically zero, such that $\sigma(s+t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by S^* the set of all characters on S. Denote by $\mathcal{A}(S^*)$ the smallest σ-ring of subsets of S^* rendering $\sigma \mapsto \sigma(s)\colon S^* \to \mathbb{R}$ measurable for each $s \in S$, and by $F_+(S^*)$ the set of all measures defined on $\mathcal{A}(S^*)$ and integrating $\sigma \mapsto \sigma(s)$ for all $s \in S+S$. For $\mu \in F_+(S^*)$, define $\mathcal{L}\mu\colon S+S \to \mathbb{R}$ by

$$\mathcal{L}\mu(s) = \int_{S^*} \sigma(s)\,\mathrm{d}\mu(\sigma), \quad s \in S+S.$$

A *moment function on* S is a function $\varphi\colon S+S \to \mathbb{R}$ such that $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S^*)$, and a moment function φ is *determinate* if there is only one such μ. Denote by $\mathcal{H}(S)$ the set of all moment functions on S, and by $\mathcal{H}_D(S)$ the subset of determinate moment functions. We have $\mathcal{H}(S) \subset \mathcal{P}(S)$ since if $\mu \in F_+(S^*)$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{R}$ then

$$\sum_{j,k=1}^{n} c_j c_k \mathcal{L}\mu(s_j + s_k) = \int_{S^*} \Bigl(\sum_{j=1}^{n} c_j \sigma(s_j)\Bigr)^2 \mathrm{d}\mu(\sigma) \geqslant 0.$$

The semigroup S is *semiperfect* if $\mathcal{H}(S) = \mathcal{P}(S)$, and *perfect* if $\mathcal{H}_D(S) = \mathcal{P}(S)$.

The semigroup $\mathbb{Q}_+$ is perfect ([4], Section 6.5). The semigroup $\mathbb{N}_0$ is semiperfect by Hamburger's Theorem (see [1] or [4], 6.2.2). Likewise, the semigroup $\mathbb{Z}$ is semiperfect. This was first shown in [16]; see [4], 6.4.1 for a simple proof. For $k \geqslant 2$ the semigroups $\mathbb{N}_0^k$ and $\mathbb{Z}^k$ are not semiperfect. For $\mathbb{N}_0^k$, this was first shown in [3] and independently in [20]; see [4], 6.3.4. For $\mathbb{Z}^k$, see [4], 6.4.8.

These results are subsumed in the following result of Sakakibara [19]: *A subsemigroup of $\mathbb{Z}^k$ containing 0 is semiperfect if and only if it is $\{0\}$ or isomorphic to $\mathbb{Z}$ or $\mathbb{N}_0$.*

An even more general result was shown in [11]. The presentation requires some terminology. An abelian semigroup H is *archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbb{N}$ such that $nx = y + z$. An *archimedean component* of an abelian semigroup S is an archimedean subsemigroup of S which is maximal for the inclusion ordering. Every abelian semigroup is the disjoint union of its archimedean components ([14], Section 4.3). An abelian semigroup S is $\mathbb{R}$-*separative* if S^* separates points in S. If S is an abelian semigroup and V is a subset of S, let $E(V)$ denote the set of those $v \in V$ such that the conditions $s, t \in S$, $2s, 2t \in V$, and $s + t = v$ imply $s = t$. For every subset U of S, let $C(U)$ be the union of those finite subsets V of S such that $E(V) \subset U$. A *C-finite semigroup* is an $\mathbb{R}$-separative semigroup S such that $C(U)$ is a finite set for every finite subset U of S. Now the main result of [11] states: *A countable C-finite semigroup S satisfying $S = S + S$ is semiperfect* *if and only if each archimedean component of S is isomorphic to the product of a finite group of exponent* 1 *or* 2 *and one of the semigroups* $\{0\}$, $\mathbb{Z}$, $\mathbb{N}$. (The *exponent* of a finite abelian group F is the smallest $n \in \mathbb{N}$ such that $nx = 0$ for all $x \in F$.)

Semiperfect $\mathbb{R}$-separative finitely generated abelian semigroups can be characterized by showing that every semiperfect $\mathbb{R}$-separative finitely generated abelian semigroup S is C-finite and satisfies $S = S + S$, so that the result from [11] applies.

Semiperfect finitely generated abelian semigroups can be characterized by a slightly complicated criterion. This is so far unpublished.

Suppose S is an abelian semigroup. Define an abelian semigroup $\widetilde{S}$ by $\widetilde{S} = S \cup \{0\}$ where 0 is some element outside S which is a zero for the union. For $r \in S$ and $\varphi\colon S \to \mathbb{R}$, define $E_r\varphi\colon \widetilde{S} \to \mathbb{R}$ by $E_r\varphi(s) = \varphi(r + s)$ for $s \in \widetilde{S}$. A function $\varphi\colon S \to \mathbb{R}$ is *completely positive definite* if $E_r\varphi \in \mathcal{P}(\widetilde{S})$ for all $r \in S$. Denote by $\mathcal{P}_c(S)$ the set of all completely positive definite functions on S. Denote by S^*_+ the set of all nonnegative characters on S. Denote by $\mathcal{A}(S^*_+)$ the smallest σ-ring of subsets of S^*_+ rendering $\sigma \mapsto \sigma(s)\colon S^*_+ \to \mathbb{R}_+$ measurable for all $s \in S$ (so $\mathcal{A}(S^*_+) = \{A \cap S^*_+ \mid A \in \mathcal{A}(S^*)\}$), and by $F_+(S^*_+)$ the set of all measures defined on $\mathcal{A}(S^*_+)$ and integrating $\sigma \mapsto \sigma(s)$ for all $s \in S$. For $\mu \in F_+(S^*_+)$, define $\mathcal{L}\mu\colon S \to \mathbb{R}_+$ by

$$\mathcal{L}\mu(s) = \int_{S^*_+} \sigma(s)\,\mathrm{d}\mu(\sigma), \quad s \in S.$$

A *Stieltjes moment function on* S is a function $\varphi\colon S \to \mathbb{R}$ such that $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S^*_+)$, and a Stieltjes moment function φ on S is *Stieltjes determinate* if there is only one such μ. Denote by $\mathcal{H}_S(S)$ the set of all Stieltjes moment functions on S, and by $\mathcal{H}_{S,D}(S)$ the subset of Stieltjes determinate Stieltjes moment functions. We have $\mathcal{H}_{S,D} \subset \mathcal{H}_S(S) \subset \mathcal{P}_c(S)$ since if $\mu \in F_+(S^*_+)$, $r \in S$, $s_1, \dots, s_n \in \widetilde{S}$, and $c_1, \dots, c_n \in \mathbb{R}$ then

$$\sum_{j=1}^{n} c_j c_k E_r \mathcal{L}\mu(s_j + s_k) = \int_{S^*_+} \sigma(r) \Bigl(\sum_{j=1}^{n} c_j \sigma(s_j) \Bigr)^2 \mathrm{d}\mu(\sigma) \geqslant 0.$$

(If $\sigma \in S^*$, we extend σ to a character on $\widetilde{S}$, also denoted by σ, by $\sigma(0) = 1$.) The semigroup S is *Stieltjes semiperfect* if $\mathcal{H}_S(S) = \mathcal{P}_c(S)$, and *Stieltjes perfect* if $\mathcal{H}_{S,D}(S) = \mathcal{P}_c(S)$.

Every perfect semigroup is Stieltjes perfect ([8], Lemma 3.2). The semigroup $\mathbb{N}_0$ is Stieltjes semiperfect by the result of Stieltjes mentioned initially. Likewise, the semigroup $\mathbb{Z}$ is Stieltjes semiperfect ([4], 6.4.7). For $k \geqslant 2$, the semigroups $\mathbb{N}_0^k$ and $\mathbb{Z}^k$ are not Stieltjes semiperfect. For $\mathbb{N}_0^k$, this was shown in [4]. Indeed, for $k \geqslant 2$ there exists a function $\varphi \in \mathcal{P}_c(\mathbb{N}_0^k) \setminus \mathcal{H}(\mathbb{N}_0^k)$ ([4], 6.3.7), as well as a function $\varphi\colon \mathbb{N}_0^k \to \mathbb{R}$ such that $E_r\varphi \in \mathcal{H}(\mathbb{N}_0^k)$ for all $r \in \mathbb{N}_0^k$, yet $\varphi \notin \mathcal{H}_S(\mathbb{N}_0^k)$ ([4], 6.3.12). For $\mathbb{Z}^k$, see our characterization of Stieltjes semiperfect finitely generated abelian semigroups below.

The purpose of the present paper is to characterize Stieltjes semiperfect finitely generated abelian semigroups. We shall do this by defining "c-finite" semigroups in such a way that every $\mathbb{R}_+$-separative finitely generated abelian semigroup (that is, every finitely generated abelian semigroup S such that S_+^* separates points in S) is c-finite, characterizing Stieltjes semiperfect countable c-finite semigroups, and reducing the case of an arbitrary finitely generated abelian semigroup to the $\mathbb{R}_+$-separative case.

In Section 2 we show that in order for an abelian semigroup S to be Stieltjes semiperfect it is necessary that $S = S + S$ (Theorem 1). Fortunately, the hypothesis $S = S + S$ implies the validity of an indeterminate method of moments given in [10]. This allows one to show that $\mathcal{H}_S(S)$ is closed under pointwise convergence (Lemma 1), and one can then describe $\mathcal{H}_S(S)$ as the polar of a certain convex cone $\mathbb{R}[S]_{++}$ in $\mathbb{R}^{(S)}$ with respect to the natural duality between $\mathbb{R}^{(S)}$ and $\mathbb{R}^S$. It follows that S is Stieltjes semiperfect if and only if a certain convex cone, $\Sigma_c(S)$, is dense in $\mathbb{R}[S]_{++}$ with respect to the finest locally convex topology on $\mathbb{R}^{(S)}$ (Theorem 2). In Section 3 we characterize $\mathbb{R}_+$-separative abelian semigroups by three equivalent conditions (Proposition 1). We then define certain mappings e and c of the set of subsets of an abelian semigroup S into itself and note some of their properties (Proposition 2). A sufficient condition for an element v of a subset V of S to belong to $e(V)$ is given in Proposition 3. Proposition 4 describes, given a subset U of S, a subset of S containing $c(U)$. Proposition 5 establishes the quite important fact that $c(\emptyset) = \emptyset$ if S is $\mathbb{R}_+$-separative. Proposition 6 establishes a formula that permits one to show that if S is a countable c-finite semigroup then $\Sigma_c(S)$ is closed in the finest locally convex topology on $\mathbb{R}^{(S)}$, so that if S furthermore satisfies $S = S + S$ then S is Stieltjes semiperfect if and only if $\Sigma_c(S) = \mathbb{R}[S]_{++}$ (Theorem 3). In Section 4 we use the formula from Proposition 6 to prove a sequence of Lemmas that lead up to the establishment of three necessary conditions for the Stieltjes semiperfectness of a countable c-finite semigroup (Theorem 4). In Section 5 we describe certain faces of the convex cone $\mathbb{R}[S]_{++}$, for an $\mathbb{R}_+$-separative abelian semigroup S, in Proposition 7. This leads to the fact that for a c-finite semigroup S, the convex cone $\mathbb{R}[S]_{++}$ is generated by its extreme rays (Proposition 8). This is an important ingredient in the proof that the necessary conditions found in Theorem 4 are also sufficient for the semiperfectness of a c-finite semigroup, even if it is not countable (Theorem 5). In Section 6 we characterize Stieltjes semiperfect $\mathbb{R}_+$-separative finitely generated abelian semigroups by showing that every $\mathbb{R}_+$-separative finitely generated abelian semigroup is c-finite, so that Theorem 5 applies (Theorem 6). In Section 7 we characterize arbitrary Stieltjes semiperfect finitely generated abelian semigroups by reducing the general case to the $\mathbb{R}_+$-separative case (Theorem 7). We then give examples of a finitely generated abelian semigroup which is Stieltjes semiperfect but not semiperfect and a finitely generated abelian semigroup that is semiperfect but not Stieltjes semiperfect. In Section 8 we turn to Schur-increasing functions. The main theorem (Theorem 8) states that there is a function $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ such that $E_r\varphi$ is a moment function for all $r \in \mathbb{N}_0^2$, yet φ is not Schur-decreasing of order 3. In Section 9 we characterize functions $\psi\colon S \to \mathbb{R}$ with the property that $\mathrm{e}^{-t\psi}$ is a Stieltjes moment function for each $t > 0$ (Theorem 9).

2. Preliminaries

In this section, we characterize Stieltjes semiperfect abelian semigroups in terms of the density of a certain convex cone in the semigroup algebra in another convex cone with respect to the finest locally convex topology.

Theorem 1. *In order that an abelian semigroup S be Stieltjes semiperfect, it is necessary that $S = S + S$.*

Proof. Suppose $S \neq S + S$. Choose $a \in S \setminus (S + S)$. Define $\varphi = 1_{\{a\}}$ (the indicator function of the subset $\{a\}$ of S). Then $E_a\varphi = 1_{\{0\}}$. Indeed, the conditions $s \in \widetilde{S}$ and $a + s = a$ imply $s = 0$ since $a \notin S + S$. Now $1_{\{0\}} \in \mathcal{P}(\widetilde{S})$. To see this, suppose $s_1, \ldots, s_n \in \widetilde{S}$ and $c_1, \ldots, c_n \in \mathbb{R}$. We may assume $s_1 = 0$ and $s_j \in S$ for $j > 1$. Since the conditions $x, y \in \widetilde{S}$ and $x + y = 0$ imply $x = y = 0$ then

$$\sum_{j,k=1}^{n} c_j c_k 1_{\{0\}}(s_j + s_k) = c_1^2 \geqslant 0.$$

For $r \in S \setminus \{a\}$ we have $E_r\varphi = 0$. Indeed, for $s \in \widetilde{S}$ we have $E_r\varphi(s) = \varphi(r + s) = 0$ since if $s = 0$ then $r + s = r \neq a$ while if $s \in S$ then $r + s \neq a$ because of $a \notin S + S$. Thus $E_r\varphi \in \mathcal{P}(\widetilde{S})$ for all $r \in S$, that is, $\varphi \in \mathcal{P}_c(S)$. Now $\varphi \notin \mathcal{H}_S(S)$. To see this, suppose $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S_+^*)$. Then $1 = \varphi(a) = \int_{S_+^*} \sigma(a)\,\mathrm{d}\mu(\sigma)$, so $\mu(\{\sigma \in S_+^* \mid \sigma(a) > 0\}) > 0$, whence $0 < \int_{S_+^*} \sigma(a)^2\,\mathrm{d}\mu(\sigma) = \varphi(2a)$, contradicting the fact that $\varphi(2a) = 0$ because of $2a \neq a$, which is a consequence of the fact that $a \notin S + S$. $\square$

For an abelian semigroup S and for $s \in S$, define $\widehat{s}\colon S^* \to \mathbb{R}$ by $\widehat{s}(\sigma) = \sigma(s)$ for $\sigma \in S^*$.

Suppose U is a countable abelian semigroup. Then U^* is a Polish space (when considered with the topology of pointwise convergence), $\mathcal{A}(U^*) = \mathcal{B}(U^*)$ (the Borel σ-field), and every bounded measure μ on $\mathcal{B}(U^*)$ is a *Radon measure* in the sense that $\mu(B) = \sup\{\mu(C) \mid \mathcal{K}(U^*) \ni C \subset B\}$ for each $B \in \mathcal{B}(U^*)$ where $\mathcal{K}(U^*)$ is the set of all compact subsets of U^*. If $\mu \in F_+(U^*)$ then $\widehat{u}^2\mu$ (the measure with density $\widehat{u}^2$ with respect to μ) is a bounded measure for each $u \in U$ (since $\int \widehat{u}^2 \, d\mu = \mathcal{L}\mu(2u) < \infty$), and we define the *$\mathcal{L}$-topology* on $F_+(U^*)$ by the condition that a net (μ_i) in $F_+(U^*)$ converges to a measure $\mu \in F_+(U^*)$ if and only if for each $u \in U$ the net $(\widehat{u}^2\mu_i)$ converges weakly to $\widehat{u}^2\mu$ (the weak topology being defined in [4], Section 2.3).

For an arbitrary abelian semigroup S we have

$$\mathcal{A}(S^*) = \bigcup_{U \in \mathcal{D}(S)} p_{S,U}^{-1}\bigl(\mathcal{B}(U^*)\bigr)$$

where $\mathcal{D}(S)$ is the set of all countable subsemigroups of S and where $p_{S,U} \colon S^* \to U^* \cup \{0\}$, for $U \in \mathcal{D}(S)$, is defined by $p_{S,U}(\sigma) = \sigma\big|U$ for $\sigma \in S^*$. By abuse of notation, for $\mu \in F_+(S^*)$ and $U \in \mathcal{D}(S)$ we denote by $\mu^{p_{S,U}}$ the image measure of $\mu\big|p_{S,U}^{-1}(U^*)$ (that is, the restriction of μ to the σ-ring $p_{S,U}^{-1}\bigl(\mathcal{A}(U^*)\bigr)$) under the mapping $p_{S,U}\big|p_{S,U}^{-1}(U^*) \colon p_{S,U}^{-1}(U^*) \to U^*$. Then $\mu^{p_{S,U}} \in F_+(U^*)$ and $\mathcal{L}(\mu^{p_{S,U}}) = (\mathcal{L}\mu)\big|(U+U)$. We define the *$\mathcal{L}$-topology* on $F_+(S^*)$ by the condition that a net (μ_i) in $F_+(S^*)$ converges to a measure $\mu \in F_+(S^*)$ if and only if for each $U \in \mathcal{D}(S)$ the net $(\mu_i^{p_{S,U}})$ converges, in the $\mathcal{L}$-topology on $F_+(U^*)$, to $\mu^{p_{S,U}}$.

Let $\mathcal{D}_0(S)$ be the set of those $U \in \mathcal{D}(S)$ such that $p_{S,U}$ maps S^* onto U^*. Then each element of $\mathcal{D}(S)$ is contained in some element of $\mathcal{D}_0(S)$ ([5], Theorem 3). Hence

$$\mathcal{A}(S^*) = \bigcup_{U \in \mathcal{D}_0(S)} p_{S,U}^{-1}(\mathcal{B}(U^*)).$$

If $U \in \mathcal{D}_0(S)$ then $p_{S,U}$ maps S_+^* onto U_+^* ([8], Lemma 3.1).

Lemma 1. *If S is an abelian semigroup satisfying $S = S+S$ then $\mathcal{H}_S(S)$ is closed in $\mathbb{R}^S$ under pointwise convergence.*

Proof. Suppose φ is an element of the closure of $\mathcal{H}_S(S)$ under pointwise convergence. Choose a net (φ_i) in $\mathcal{H}_S(S)$ which converges pointwise to φ. For each i choose $\mu_i \in F_+(S_+^*)$ such that $\varphi_i = \mathcal{L}\mu_i$. We may assume that (μ_i) is a universal net. By the main result in [10], (μ_i) converges in the $\mathcal{L}$-topology to some $\mu \in F_+(S^*)$ such that $\mathcal{L}\mu = \varphi$. For $U \in \mathcal{D}(S)$ the net $(\mu_i^{p_{S,U}})$ converges in the $\mathcal{L}$-topology to $\mu^{p_{S,U}}$, so for $u \in U$ the net $(\widehat{u}^2\mu_i^{p_{S,U}})$ converges weakly to $\widehat{u}^2\mu^{p_{S,U}}$. Since for each i the measure $\widehat{u}^2\mu_i^{p_{S,U}}$ is supported by the closed set U_+^*, we have $(\widehat{u}^2\mu^{p_{S,U}})(U^* \setminus U_+^*) = 0$. This being so for all $u \in U$, it follows that $\mu^{p_{S,U}}(U^* \setminus U_+^*) = 0$. Now $\mu_*(S^* \setminus S_+^*) = 0$. To see this, we must show that if $A \in \mathcal{A}(S^*)$ and $A \cap S_+^* = \emptyset$ then $\mu(A) = 0$. Choose $U \in \mathcal{D}_0(S)$ and $B \in \mathcal{B}(U^*)$ such that $A = p_{S,U}^{-1}(B)$. Since $p_{S,U}$ maps S_+^* onto U_+^* then $B \cap U_+^* = \emptyset$. Hence $\mu(A) = \mu^{p_{S,U}}(B) = 0$. This shows that $\mu_*(S^* \setminus S_+^*) = 0$. If we now define $\lambda \in F_+(S_+^*)$ by the condition that $\lambda(A \cap S_+^*) = \mu(A)$ for all $A \in \mathcal{A}(S^*)$ then $\mathcal{L}\lambda = \varphi$, so $\varphi \in \mathcal{H}_S(S)$, as desired. $\square$

Suppose S is an abelian semigroup. Denote by $\mathbb{R}[S]$ the space $\mathbb{R}^{(S)}$ of finitely supported real-valued functions on S, equipped with the multiplication $*$ defined by

$$a * b(u) = \sum_{s,t \in S\colon\, s+t=u} a(s)b(t)$$

for $a, b \in \mathbb{R}[S]$ and $u \in S$. Then $\mathbb{R}[S]$ is a commutative real algebra. For $r \in S$, define $\delta_r \in \mathbb{R}[S]$ by $\delta_r(s) = \delta_{r,s}$ (the Kronecker delta) for $s \in S$. For $a \in \mathbb{R}[S]$, write $a^{*2} = a * a$. For $a \in \mathbb{R}[S]$, denote by $\operatorname{supp} a$ the *support* of a, that is, the set of those $s \in S$ such that $a(s) \neq 0$.

Define a bilinear form $\langle \cdot, \cdot \rangle \colon\ \mathbb{R}[S] \times \mathbb{R}^S \to \mathbb{R}$ by

$$\langle a, \varphi \rangle = \sum_{s \in S} a(s)\varphi(s)$$

for $a \in \mathbb{R}[S]$ and $\varphi \in \mathbb{R}^S$. Under this bilinear form, the spaces $\mathbb{R}[S]$ and $\mathbb{R}^S$ are in duality. The finest locally convex topology on $\mathbb{R}[S]$, and the topology of pointwise convergence on $\mathbb{R}^S$, are compatible with the duality.

Define a convex cone $\mathbb{R}[S]_{++}$ in $\mathbb{R}[S]$ by

$$\mathbb{R}[S]_{++} = \{a \in \mathbb{R}[S] \mid \langle a, \sigma \rangle \geqslant 0 \ \ \forall \sigma \in S_+^*\}.$$

Define another convex cone $\Sigma_c(S)$ in $\mathbb{R}[S]$ by

$$\Sigma_c(S) = \{\delta_{r_1} * a_1^{*2} + \ldots + \delta_{r_n} * a_n^{*2} \mid r_1, \ldots, r_n \in S,\ \ a_1, \ldots, a_n \in \mathbb{R}[\widetilde{S}]\}.$$

For $r \in S$, $a \in \mathbb{R}[\widetilde{S}]$, and $\sigma \in S_+^*$ we have $\langle \delta_r * a^{*2}, \sigma \rangle = \sigma(r)\langle a, \sigma \rangle^2 \geqslant 0$. It follows that $\Sigma_c(S) \subset \mathbb{R}[S]_{++}$.

For every subset A of $\mathbb{R}[S]$, define a convex cone $A^\perp$ in $\mathbb{R}^S$, closed under pointwise convergence, by

$$A^\perp = \{\varphi \in \mathbb{R}^S \mid \langle a, \varphi \rangle \geqslant 0 \ \ \forall a \in A\}.$$

Theorem 2. *Suppose S is an abelian semigroup satisfying $S = S + S$. Then $\mathcal{P}_c(S) = \Sigma_c(S)^\perp$ and $\mathcal{H}_S(S) = \mathbb{R}[S]_{++}^\perp$. Hence S is Stieltjes semiperfect if and only if $\Sigma_c(S)$ is dense in $\mathbb{R}[S]_{++}$ with respect to the finest locally convex topology.*

P r o o f. As the proof of [9], Proposition 3, using Lemma 1. $\square$

An abelian semigroup S is $\mathbb{R}_+$-*separative* if S_+^* separates points in S.

Suppose S is an abelian semigroup. Let $\mathcal{J}(S)$ denote the set of all archimedean components of S. For $H, K \in \mathcal{J}(S)$, the subsemigroup $H + K$ of S is archimedean, hence contained in a unique archimedean component of S, which we denote by $H \vee K$. Then $(\mathcal{J}(S), \vee)$ is a *semilattice*, that is, an abelian semigroup with all elements *idempotent* ($H \vee H = H$ for $H \in \mathcal{J}(S)$). Define a relation $\leqslant$ in $\mathcal{J}(S)$ by the condition that $H \leqslant K$ if and only if $H \vee K = K$. Then $\leqslant$ is a partial ordering in $\mathcal{J}(S)$, and for $H, K \in \mathcal{J}(S)$ the element $H \vee K$ is the least upper bound on $\{H, K\}$ in the partially ordered set $(\mathcal{J}(S), \leqslant)$.

For every abelian semigroup X, let G_X denote the abelian group having a presentation with generators $g_X(x)$, $x \in X$, and relations $g_X(x+y) = g_X(x)+g_X(y)$ for $x, y \in X$. The mapping $g_X \colon X \to G_X$ is a homomorphism and $G_X = g_X(X) - g_X(X)$. For $x, y \in X$ we have $g_X(x) = g_X(y)$ if and only if $x + a = y + a$ for some $a \in X$.

If X and Y are subsemigroups of an abelian semigroup S satisfying $X + Y \subset Y$, there is a unique homomorphism $g_{X,Y} \colon G_X \to G_Y$ such that $g_Y(x + y) = g_{X,Y}(g_X(x)) + g_Y(y)$ for all $x \in X$ and $y \in Y$. The mapping $g_{X,X}$ is the identity on G_X. If Z is a third subsemigroup of S such that $X + Z \subset Z$ and $Y + Z \subset Z$ then $g_{X,Z} = g_{Y,Z} \circ g_{X,Y}$. Hence, if X and Y are subsemigroups of S such that $X + Y \subset X \cap Y$ then $g_{X,Y}$ is an isomorphism between G_X and G_Y, and $g_{Y,X}$ is its inverse.

A *face* of an abelian semigroup S is a subsemigroup X of S such that the conditions $x, y \in S$ and $x + y \in X$ imply $x, y \in X$. Every face of S is the union of those archimedean components of S which it contains. If $\mathcal{K}$ is a subset of $\mathcal{J}(S)$ then the set $\bigcup_{K \in \mathcal{K}} K$ is a face of S if and only if $\mathcal{K}$ is a face of the semilattice $\mathcal{J}(S)$, that is, a subsemigroup of $\mathcal{J}(S)$ such that if $H \in \mathcal{J}(S)$ and $H \leqslant K \in \mathcal{K}$ then $H \in \mathcal{K}$. For $K \in \mathcal{J}(S)$, the set

$$\mathcal{K}_K = \{H \in \mathcal{J}(S) \mid H \leqslant K\}$$

is easily seen to be a face of $\mathcal{J}(S)$. It follows that the set

$$X_K = \bigcup_{H \in \mathcal{K}_K} H = \bigcup_{H \leqslant K} H$$

is a face of S. Note that X_K is the least face of S containing K. Since the condition $H \in \mathcal{K}_K$ implies $H + K \subset H \vee K = K$, we have $X_K + K \subset K$. As above, it follows that g_{K,X_K} is an isomorphism between G_K and G_{X_K} with $g_{X_K,K}$ as its inverse. If $K \in \mathcal{J}(S)$ then K is the greatest element of $\mathcal{J}(S)$ contained in X_K. Hence, if $K, L \in \mathcal{J}(S)$ and $K \neq L$ then $X_K \neq X_L$.

Proposition 1. *For an abelian semigroup S the following three conditions are equivalent:*

(i) *S is $\mathbb{R}_+$-separative;*

(ii) *for each $H \in \mathcal{J}(S)$, the semigroup H is cancellative and the group G_H is torsion-free;*

(iii) *the conditions $x, y \in S$, $k \in \mathbb{N}$, and $kx = ky$ imply $x = y$.*

P r o o f. The equivalence (i) $\Leftrightarrow$ (ii) is Theorem 0.1 on p. 135 in [17]. The equivalence (i) $\Leftrightarrow$ (iii) follows from [6], Theorem 1. $\square$

Suppose S is an abelian semigroup. For every subset V of S, denote by $e(V)$ the set of those $v \in V$ such that the conditions $r \in S$, $s, t \in \widetilde{S}$, $r + 2s, r + 2t \in V$, and $r + s + t = v$ imply $r + s = r + t$. For every subset U of S, denote by $c(U)$ the union of all finite subsets V of S such that $e(V) \subset U$.

Let us say that $v \in S$ is the *S-midpoint* of u and w if there exist $r \in S$ and $s, t \in \widetilde{S}$ such that $u = r + 2s$, $w = r + 2t$, and $v = r + s + t$. If S is $\mathbb{R}_+$-separative then $e(V)$ is precisely the set of those $v \in V$ such that v is not the S-midpoint of two distinct elements of V. Note that an S-midpoint is not the same as a midpoint in the usual sense. For example, if $S = \mathbb{N}_0 \setminus \{1\}$, it is easily seen that 3 is not the S-midpoint of 2 and 4.

Proposition 2. *If U and V are subsets of S then*

(i) *$e(V) \subset V$;*

(ii) *if $U \subset V$ then $U \cap e(V) \subset e(U)$;*

(iii) *$U \subset c(U)$;*

(iv) *if $U \subset V$ then $c(U) \subset c(V)$;*

(v) *every finite subset of $c(U)$ is contained in a finite set W such that $e(W) \subset U$;*

(vi) *$c(c(U) = c(U)$;*

(vii) *if U is finite then $c(U) = c(e(U))$;*

(viii) *$e(c(U)) \subset e(U)$;*

(ix) *if $r \in S$ and $s, t \in \widetilde{S}$ then $r + s + t \in c(\{r + 2s, r + 2t\})$.*

P r o o f. (i) through (viii): Analogous to [11], Theorem 2.

(ix): Define $V = \{r+2s, r+s+t, r+2t\}$. It suffices to show $e(V) \subset \{r+2s, r+2t\}$ since it then follows that $r + s + t \in V \subset c(\{r + 2s, r + 2t\})$. To show $e(V) \subset \{r+2s, r+2t\}$, it suffices to show that if $r+s+t \in e(V)$ then $r+s+t \in \{r+2s, r+2t\}$. Suppose $r + s + t \in e(V)$. Since $r + 2s, r + 2t \in V$, it follows that $r + s = r + t$. Hence $r + s + t = r + 2t \in \{r + 2s, r + 2t\}$, as desired. $\square$

If S is an $\mathbb{R}_+$-separative abelian semigroup, for every subset U of S we denote by $\mathrm{Conv}(U)$ the set of those $u \in S$ such that

$$(1) \qquad (n+1)u = u + u_1 + \ldots + u_n \quad \text{for some } n \in \mathbb{N} \text{ and } u_1, \ldots, u_n \in U$$

and by $\mathrm{Ex}(U)$ the set of those $u \in U$ such that (1) implies $u_1 = \ldots = u_n = u$. Note that if S is a subsemigroup of a torsion-free abelian group G then $\mathrm{Conv}(U)$ is the intersection of S with the convex hull $\mathrm{conv}(U)$ of U in the enveloping real vector space of G while $\mathrm{Ex}(U)$ is the set of all extreme points of the convex set $\mathrm{conv}(U)$.

Proposition 3. *Suppose S is an $\mathbb{R}_+$-separative abelian semigroup and V is a subset of S. Then* $\mathrm{Ex}(V) \subset e(V)$.

Proof. Assume $v \in \mathrm{Ex}(V)$; we have to show $v \in e(V)$. Suppose $r \in S$, $s, t \in \widetilde{S}$, $r + 2s, r + 2t \in V$, and $r + s + t = v$; we have to show $r + s = r + t$. We have $2v = 2(r + s + t) = (r + 2s) + (r + 2t)$. Since $r + 2s, r + 2t \in V$, by the definition of $\mathrm{Ex}(V)$ it follows that $r + 2s = r + 2t$. Hence $2(r + s) = 2(r + t)$, and since S is $\mathbb{R}_+$-separative, by Proposition 1 it follows that $r + s = r + t$, as desired. □

Lemma 2. *Suppose S is an $\mathbb{R}$-separative abelian semigroup, $a, b, x \in S$, and $n \in \mathbb{N}$. If $a + nx = b + nx$ then $a + x = b + x$.*

Proof. Suppose $\sigma \in S^*$. From $a + nx = b + nx$ we get $\sigma(a)\sigma(x)^n = \sigma(b)\sigma(x)^n$. If $\sigma(x) \neq 0$, we may divide by $\sigma(x)^{n-1}$ to get $\sigma(a)\sigma(x) = \sigma(b)\sigma(x)$. If $\sigma(x) = 0$, it is trivial that $\sigma(a)\sigma(x) = \sigma(b)\sigma(x)$. Thus $\sigma(a)\sigma(x) = \sigma(b)\sigma(x)$ in every case. This being so for each $\sigma \in S^*$, since S is $\mathbb{R}$-separative it follows that $a + x = b + x$. □

Lemma 3. *Suppose S is an $\mathbb{R}$-separative abelian semigroup and V is a finite subset of S. Then* $\mathrm{Conv}(V) = \mathrm{Conv}(\mathrm{Ex}(V))$.

Proof. The inclusion $\mathrm{Conv}(\mathrm{Ex}(V)) \subset \mathrm{Conv}(V)$ is trivial. For the converse inclusion, it suffices to show $V \subset \mathrm{Conv}\,\mathrm{Ex}(V))$. Let $\mathcal{U}$ be the set of those subsets U of V such that for all $v \in V \setminus \mathrm{Ex}(V)$ there exist $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U \setminus \{v\}$ such that $(n + 1)v = v + u_1 + \ldots + u_n$.

Then $V \in \mathcal{U}$. To see this, suppose $v \in V \setminus \mathrm{Ex}(V)$. By the definition of $\mathrm{Ex}(V)$, there exist $p \in \mathbb{N}$ and $v_1, \ldots, v_p \in V$ such that $(p+1)v = v + v_1 + \ldots + v_p$ and such that it is not the case that $v_1 = \ldots = v_p = v$. We may assume that for some $n \in \{1, \ldots, p\}$ we have $v_j \neq v$ for $j \leqslant n$ and $v_j = v$ for $j > n$. Then $(p+1)v = (p-n+1)v + v_1 + \ldots + v_n$. By Lemma 2 it follows that $(n+1)v = v + v_1 + \ldots + v_n$. Moreover, $v_1, \ldots, v_n \in V \setminus \{v\}$, as desired.

Since $V \in \mathcal{U}$ then U is a nonempty set of subsets of the finite set V. We can therefore choose a set $U \in \mathcal{U}$ which is minimal with respect to the inclusion ordering. If $U \subset \mathrm{Ex}(V)$, we are done. Thus we may assume $U \not\subset \mathrm{Ex}(V)$. Choose $u \in U \setminus \mathrm{Ex}(V)$ and define $U' = U \setminus \{u\}$. We shall derive a contradiction by showing $U' \in \mathcal{U}$.

To see that $U' \in \mathcal{U}$, suppose $v \in V \setminus \mathrm{Ex}(V)$. Choose $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U \setminus \{v\}$ such that $(n + 1)v = v + u_1 + \ldots + u_n$. We may assume that for some $m \in \{0, \ldots, n\}$

we have $u_i = u$ if and only if $i > m$. Then

$$(*) \qquad (n+1)v = v + u_1 + \ldots + u_m + (n-m)u.$$

If $m = n$, we are done. Suppose $m < n$. Since $u \notin \mathrm{Ex}(V)$, there exist $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in U \setminus \{u\} = U'$ such that $(k+1)u = u + t_1 + \ldots + t_k$. Now

$$\begin{aligned} k(n+1)v + (n-m)u &= k(v + u_1 + \ldots + u_m) + (n-m)(k+1)u \\ &= k(v + u_1 + \ldots + u_m) + (n-m)(u + t_1 + \ldots + t_k). \end{aligned}$$

Hence $\sigma(v)^{k(n+1)}\sigma(u)^{n-m} = \sigma(v)^k\sigma(u_1)^k \ldots \sigma(u_m)^k\sigma(u)^{n-m}\sigma(t_1)^{n-m} \ldots \sigma(t_k)^{n-m}$ for each $\sigma \in S^*$. The equation $(*)$ shows that if $\sigma(u) = 0$ then $\sigma(v) = 0$. Hence we may infer $\sigma(v)^{k(n+1)} = \sigma(v)^k\sigma(u_1)^k \ldots \sigma(u_m)^k\sigma(t_1)^{n-m} \ldots \sigma(t_k)^{n-m}$. This being so for all $\sigma \in S^*$, since S is $\mathbb{R}$-separative we have $k(n+1)v = kv + k(u_1 + \ldots + u_m) + (n-m)(t_1 + \ldots + t_k)$. We may assume that for some $j \in \{0, \ldots, k\}$ we have $t_i = v$ if and only if $i > j$. Then $k(n+1)v = (k + (n-m)(k-j))v + k(u_1 + \ldots + u_m) + (n-m)(t_1 + \ldots + t_j)$. By Lemma 2 it follows that $(km + (n-m)j + 1)v = v + k(u_1 + \ldots + u_m) + (n-m)(t_1 + \ldots + t_j)$, which shows that $U' \in \mathcal{U}$, the desired contradiction. □

Proposition 4. *Suppose S is an $\mathbb{R}$-separative abelian semigroup and U is a subset of S. Then $c(U) \subset \mathrm{Conv}(U)$.*

Proof. Suppose V is a finite subset of S such that $e(V) \subset U$; we have to show $V \subset \mathrm{Conv}(U)$. By Proposition 3 we have $\mathrm{Ex}(V) \subset e(V) \subset U$, so by Lemma 3, $V \subset \mathrm{Conv}(V) = \mathrm{Conv}(\mathrm{Ex}(V)) \subset \mathrm{Conv}(U)$. □

Proposition 5. *If S is an $\mathbb{R}$-separative abelian semigroup then $c(\emptyset) = \emptyset$.*

Proof. By Proposition 4, $c(\emptyset) \subset \mathrm{Conv}(\emptyset) = \emptyset$. □

A pair (r, U), where $r \in S$ and where U is a subset of $\widetilde{S}$, is *proper* if the conditions $s, t \in U$ and $r + s = r + t$ imply $s = t$. A pair $(r, a) \in S \times \mathbb{R}[\widetilde{S}]$ is *proper* if the pair $(r, \operatorname{supp} a)$ is proper. For every subset T of S, define $2T = \{2t \mid t \in T\}$. For $r \in S$, write $r + 2T = \{r + 2t \mid t \in T\}$.

Lemma 4. *If $(r_1, a_1), \ldots, (r_n, a_n) \in S \times \mathbb{R}[\widetilde{S}]$ are proper and if*

$$v \in e\Big(\bigcup_{j=1}^{n}(r_j + 2\operatorname{supp} a_j)\Big)$$

then $\sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}(v) > 0$.

P r o o f. We have

$$\sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}(v) = \sum_{j=1}^{n} \sum_{s,t \in \operatorname{supp} a\colon r_j+s+t=v} a_j(s)a_j(t).$$

If $j \in \{1,\dots,n\}$, $s,t \in \operatorname{supp} a_j$, and $r_j+s+t=v$ then, since $r_j+2s, r_j+2t \in \bigcup_{i=1}^{n}(r_i+2\operatorname{supp} a_i)$ and since $v \in e\Big(\bigcup_{i=1}^{n}(r_i+2\operatorname{supp} a_i)\Big)$, we have $r_j+s=r_j+t$. Since (r_j,a_j) is proper, it follows that $s=t$. Thus

$$\sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}(v) = \sum_{j=1}^{n} \sum_{s \in \operatorname{supp} a_j\colon r_j+2s=v} a_j(s)^2 > 0.$$

(At least one term is positive since $v \in \bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)$.) □

Proposition 6. *If $(r_1,a_1),\dots,(r_n,a_n) \in S \times \mathbb{R}[\widetilde{S}]$ are proper then*

$$c\Big(\operatorname{supp}\sum_{j=1}^{n}\delta_{r_j} * a_j^{*2}\Big) = c\Big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\Big).$$

P r o o f. We have $\operatorname{supp}\sum_{j=1}^{n}\delta_{r_j} * a_j^{*2} \subset \bigcup_{j=1}^{n}(r_j+\operatorname{supp} a_j+\operatorname{supp} a_j)$. If $j \in \{1,\dots,n\}$ and $s,t \in \operatorname{supp} a_j$ then $r_j+s+t \in c(\{r_j+2s, r_j+2t\}) \subset c(r_j+2\operatorname{supp} a_j)$ (Proposition 2 (ix) and (iv)). Hence

$$\operatorname{supp}\sum_{j=1}^{n}\delta_{r_j} * a_j^{*2} \subset \bigcup_{j=1}^{n} c(r_j+2\operatorname{supp} a_j) \subset c\big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\big)$$

(Proposition 2 (iv)), so

$$c\Big(\operatorname{supp}\sum_{j=1}^{n}\delta_{r_j} * a_j^{*2}\Big) \subset c\Big[c\Big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\Big)\Big] = c\Big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\Big)$$

(Proposition 2 (iv) and (vi)). Conversely, by Lemma 4 we have

$$e\Big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\Big) \subset \operatorname{supp}\sum_{j=1}^{n}\delta_{r_j} * a_j^{*2},$$

so

$$c\Big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\Big) = c\Big[e\Big(\bigcup_{j=1}^{n}(r_j+2\operatorname{supp} a_j)\Big)\Big] \subset c\Big(\operatorname{supp}\sum_{j=1}^{n}\delta_{r_j} * a_j^{*2}\Big)$$

(Proposition 2 (vii) and (iv)). □

Corollary 1. *If S is $\mathbb{R}_+$-separative and if $(r_j, a_j) \in S \times \mathbb{R}[\widetilde{S}]$ is proper and $a_j \neq 0$ for $j = 1, \dots, n$ then $\sum_{j=1}^{n} \delta_{r_j} * a_j^{*2} \neq 0$.*

P r o o f. By Proposition 6, $c\Big(\operatorname{supp} \sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}\Big) = c\Big(\bigcup_{j=1}^{n} (r_j + 2 \operatorname{supp} a_j)\Big)$. The latter set contains the nonempty set $\bigcup_{j=1}^{n} (r_j + 2 \operatorname{supp} a_j)$ (Proposition 2 (iii)) and so is nonempty. By Proposition 5 it follows that $\operatorname{supp} \sum_{j=1}^{n} \delta_{r_j} * a_j^{*2} \neq \emptyset$, that is, $\sum_{j=1}^{n} \delta_{r_j} * a_j^{*2} \neq 0$, as desired. □

Lemma 5. *If $(r, a) \in S \times \mathbb{R}[\widetilde{S}]$ then there is some $b \in \mathbb{R}[\widetilde{S}]$ such that* $\operatorname{supp} b \subset \operatorname{supp} a$, *the pair (r, b) is proper, and $\delta_r * a^{*2} = \delta_r * b^{*2}$.*

P r o o f. Define an equivalence relation $\sim$ in $\operatorname{supp} a$ by the condition that $s \sim t$ if and only if $r + s = r + t$. For $s \in \operatorname{supp} a$, denote by $[s]$ the equivalence class containing s. Let X be a set that contains exactly one element from each equivalence class. Define

$$b = \sum_{x \in X} \Big(\sum_{s \in [x]} a(s) \Big) \delta_x.$$

Clearly $\operatorname{supp} b \subset \operatorname{supp} a$ and the pair (r, b) is proper. Now

$$\delta_r * b^{*2} = \sum_{x,y \in X} \sum_{s \in [x]} \sum_{t \in [y]} a(s)a(t)\delta_{r+x+y} = \sum_{x,y \in X} \sum_{s \in [x]} \sum_{t \in [y]} a(s)a(t)\delta_{r+s+t}$$

since for $x, y \in X$, $s \in [x]$, and $t \in [y]$ we have $r + x + y = (r + x) + y = (r + s) + y = s + (r + y) = s + (r + t) = r + s + t$. The last sum reduces to $\delta_r * a^{*2}$. □

Theorem 3. *Suppose S is a countable c-finite semigroup. Then the convex cone $\Sigma_c(S)$ is closed in $\mathbb{R}[S]$ with respect to the finest locally convex topology. Hence, if S furthermore satisfies $S = S + S$ then S is Stieltjes semiperfect if and only if $\Sigma_c(S) = \mathbb{R}[S]_{++}$.*

P r o o f. Since $\mathbb{R}[S]$ is countable-dimensional, by [4], 6.3.3 it suffices to show that $\Sigma_c(S) \cap \mathbb{R}^{(U)}$ is closed, in the canonical topology on the finite-dimensional space $\mathbb{R}^{(U)} = \{a \in \mathbb{R}[S] \mid \operatorname{supp} a \subset U\}$, for every finite subset U of S.

It even suffices to show that $\Sigma_c(S) \cap \mathbb{R}^{(V)}$ is closed in $\mathbb{R}^{(V)}$ for every finite subset V of S satisfying $V = c(V)$. Indeed, every finite subset U of S is contained in such a set V, namely, the set $V = c(U)$. (Use Proposition 2 (iii) and (vi).)

Let Ω be the set of all pairs (r, U) such that $r \in S$, U is a finite subset of $\widetilde{S}$, the pair (r, U) is proper, and $r + 2U \subset V$. For every subset Ω' of Ω, let $\Sigma_c(\Omega')$ be the subcone of $\Sigma_c(S)$ generated by elements of the form $\delta_r * a^{*2}$ where $(r,a) \in S \times \mathbb{R}[\widetilde{S}]$ is such that $\operatorname{supp} a$ is contained in a set U such that $(r,U) \in \Omega'$. Then

$$\Sigma_c(S) \cap \mathbb{R}^{(V)} = \Sigma_c(\Omega). \tag{2}$$

To see this, first suppose $(r,U) \in \Omega$, $a \in \mathbb{R}[\widetilde{S}]$, and $\operatorname{supp} a \subset U$; we have to show $\delta_r * a^{*2} \in \mathbb{R}^{(V)}$, that is, $\operatorname{supp}(\delta_r * a^{*2}) \subset V$. We have $\operatorname{supp}(\delta_r * a^{*2}) \subset c\big(\operatorname{supp}(\delta_r * a^{*2})\big) = c(r + 2\operatorname{supp} a) \subset c(r+2U) \subset c(V) = V$ by Proposition 2 (iii), Proposition 6, and Proposition 2 (iv). This shows $\Sigma_c(\Omega) \subset \Sigma_c(S) \cap \mathbb{R}^{(V)}$. For the converse inclusion, suppose $b \in \Sigma_c(S) \cap \mathbb{R}^{(V)}$. Choose $(r_1,a_1),\dots,(r_n,a_n) \in S \times \mathbb{R}[\widetilde{S}]$ such that $b = \sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}$. By Lemma 5 we may suppose that (r_j,a_j) is proper for $j = 1,\dots,n$. Define $U_j = \operatorname{supp} a_j$ for $j = 1,\dots,n$. By Proposition 6,

$$\bigcup_{j=1}^{n} (r_j + 2U_j) \subset c\Bigg(\bigcup_{j=1}^{n} (r_j + 2U_j)\Bigg) = c(\operatorname{supp} b) \subset c(V) = V.$$

(We used Proposition 2 (iii) and (iv).) Thus $(r_j,U_j) \in \Omega$ for $j = 1,\dots,n$, so $b \in \Sigma_c(\Omega)$. This shows $\Sigma_c(S) \cap \mathbb{R}^{(V)} \subset \Sigma_c(\Omega)$, and completes the proof of (2).

For each $(r,U) \in \Omega$, the set $r + 2U$ is a subset of the finite set V. Since V has only finitely many subsets, we may choose a finite subset Ω' of Ω such that $\{r' + 2U' \mid (r',U') \in \Omega'\} = \{r + 2U \mid (r,U) \in \Omega\}$. Now

$$\Sigma_c(S) \cap \mathbb{R}^{(V)} = \Sigma_c(\Omega'). \tag{3}$$

The inclusion $\Sigma_c(\Omega') \subset \Sigma_c(S) \cap \mathbb{R}^{(V)}$ follows from (2) since $\Sigma_c(\Omega')$ is obviously a subset of $\Sigma_c(\Omega)$. For the converse inclusion, suppose $b \in \Sigma_c(S) \cap \mathbb{R}^{(V)}$. By (2) we have $b \in \Sigma_c(\Omega)$. Thus there exist $(r_1,U_1),\dots,(r_n,U_n) \in \Omega$ and $a_1,\dots,a_n \in \mathbb{R}[\widetilde{S}]$ such that $\operatorname{supp} a_j \subset U_j$ for each j and $b = \sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}$. Suppose $j \in \{1,\dots,n\}$. Choose $(r_j',U_j') \in \Omega'$ such that $r_j' + 2U_j' = r_j + 2U_j$. For each $u \in U_j$ choose $u' \in U_j'$ such that $r_j' + 2u' = r_j + 2u$. Define

$$a_j' = \sum_{u \in U_j} a(u)\delta_{u'}.$$

Then

$$\delta_{r_j'} * {a_j'}^{*2} = \sum_{u,v \in U_j} a_j(u)a_j(v)\delta_{r_j'+u'+v'} = \sum_{u,v \in U_j} a_j(u)a_j(v)\delta_{r_j+u+v}$$

since for $u,v \in U_j$ we have $2(r_j'+u'+v') = (r_j'+2u')+(r_j'+2v') = (r_j+2u)+(r_j+2v) = 2(r_j+u+v)$, hence $r_j'+u'+v' = r_j+u+v$ by Proposition 1 (iii). The last sum reduces to $\delta_{r_j} * a_j^{*2}$. Thus $b = \sum_{j=1}^{n} \delta_{r'_j} * {a'_j}^{*2}$, which shows $b \in \Sigma_c(\Omega')$. This completes the proof of (3).

It now suffices to show that $\Sigma_c(\Omega')$ is closed. For each $(r, U) \in \Omega'$, choose a compact subset $B_{r,U}$ of $\mathbb{R}^{(U)} \setminus \{0\}$ which intersects every ray from the origin. The set

$$C = \operatorname{conv}\{\delta_r * a^{*2} \mid (r, U) \in \Omega',\ \ a \in B_{r,U}\}$$

is again compact ([13], 2.8). Moreover,

$$\Sigma_c(\Omega') = \{\lambda c \mid c \in C,\ \ \lambda \geqslant 0\}.$$

It therefore suffices to show that $0 \notin C$. Suppose $b \in C$. Then there exist $(r_1, U_1), \ldots, (r_n, U_n) \in \Omega'$, $a_j \in B_{r_j,U_j}$ $(j = 1, \ldots, n)$, and $\alpha_1, \ldots, \alpha_n \geqslant 0$ such that $\sum_{j=1}^{n} \alpha_j = 1$ and

$$b = \sum_{j=1}^{n} \alpha_j \delta_{r_j} * a_j^{*2} = \sum_{j=1}^{n} \delta_{r_j} * (\sqrt{\alpha_j} a_j)^{*2}.$$

By Corollary 1 it follows that $b \neq 0$, as desired. $\square$

We have shown that $\Sigma_c(S)$ is closed in $\mathbb{R}[S]$ with respect to the finest locally convex topology. The remaining claim follows from Theorem 2.

4. Necessity

In this section, we derive some conditions that are necessary in order for a countable c-finite semigroup to be Stieltjes semiperfect. In the next section, these conditions will turn out to be sufficient, even if the semigroup is not countable.

Lemma 6. *Suppose S is a Stieltjes semiperfect countable c-finite semigroup and K is an archimedean component of S. Then $c(U) \cap K = \operatorname{Conv}(U) \cap K$ for every finite subset U of K.*

Proof. Suppose U is a finite subset of K. By Proposition 4, $c(U) \subset \operatorname{Conv}(U)$. Hence $c(U) \cap K \subset \operatorname{Conv}(U) \cap K$. For the converse inclusion, suppose $v \in \operatorname{Conv}(U) \cap K$. Choose $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U$ such that $(n+1)v = v + u_1 + \ldots + u_n$. Define

$$b = \delta_{u_1} + \ldots + \delta_{u_n} - n\delta_v.$$

For $\sigma \in S_+^*$, either $\sigma(v) = 0$ or

$$\sigma(v) = \sigma(u_1)^{1/n} \dots \sigma(u_n)^{1/n} \leqslant \frac{1}{n}[\sigma(u_1) + \dots + \sigma(u_n)],$$

that is, $\langle b, \sigma \rangle \geqslant 0$. This being so for all $\sigma \in S_+^*$, we have $b \in \mathbb{R}[S]_{++}$. By Theorem 3 it follows that $b \in \Sigma_c(S)$. Thus we may choose $(r_1, a_1), \dots, (r_n, a_n) \in S \times \mathbb{R}[\widetilde{S}]$ such that $b = \sum\limits_{j=1}^{n} \delta_{r_j} * a_j^{*2}$. By Lemma 5 we may assume that (r_j, a_j) is proper for $j = 1, \dots, n$. By Lemma 4, if $w \in e\Big(\bigcup\limits_{j=1}^{n} (r_j + 2 \operatorname{supp} a_j)\Big)$ then $b(w) > 0$, so $w \in \{u_1, \dots, u_n\} \subset U$. Thus $e\Big(\bigcup\limits_{j=1}^{n} (r_j + 2 \operatorname{supp} a_j)\Big) \subset U$. It follows that

$$v \in \operatorname{supp} b \subset c(\operatorname{supp} b) = c\Big(\bigcup_{j=1}^{n} (r_j + 2 \operatorname{supp} a_j)\Big) = c\Big[e\Big(\bigcup_{j=1}^{n} (r_j + 2 \operatorname{supp} a_j)\Big)\Big] \subset c(U).$$

(We used Proposition 2 (iii), Proposition 6, and Proposition 2 (vii) and (iv).) This shows $\operatorname{Conv}(U) \cap K \subset c(U) \cap K$ and completes the proof. $\square$

For every cancellative abelian semigroup K such that the group G_K is torsion-free, let Q_K be the enveloping rational vector space of G_K. If A is a subset of Q_K, say that A consists of *equidistant points* if there exist $u, w \in Q_k$ and $p, q \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ such that $A = \{u + jw \mid j \in \mathbb{Z},\ p \leqslant j \leqslant q\}$.

Lemma 7. *Suppose S is a Stieltjes semiperfect countable c-finite semigroup and K is an archimedean component of S. If P is a 1-dimensional affine subspace of Q_K then $P \cap K$ consists of equidistant points.*

Proof. As the proof of [11], Lemma 3. $\square$

Lemma 8. *If X is a face of an abelian semigroup S then $\Sigma_c(S) \cap \mathbb{R}[X] = \Sigma_c(X)$.*

Proof. The inclusion $\Sigma_c(X) \subset \Sigma_c(S) \cap \mathbb{R}[X]$ is trivial. For the converse inclusion, suppose $b \in \Sigma_c(S) \cap \mathbb{R}[X]$. Choose $r_1, \dots, r_n \in S$ and $a_1, \dots, a_n \in \mathbb{R}[\widetilde{S}]$ such that $b = \sum\limits_{j=1}^{n} \delta_{r_j} * a_j^{*2}$. By a permutation of $\{1, \dots, n\}$, we may assume that there is some $m \in \{0, \dots, n\}$ such that $r_j \in X$ if and only if $j \leqslant m$. Now the mapping $c \mapsto c\big|X \colon\ \mathbb{R}[S] \to \mathbb{R}[X]$ is an algebra homomorphism ([11], Lemma 10). Hence $b = b\big|X = \sum\limits_{j=1}^{m} \delta_{r_j} * (a_j\big|X)^{*2}$, which shows $b \in \Sigma_c(X)$. This shows $\Sigma_c(S) \cap \mathbb{R}[X] \subset \Sigma_c(X)$ and completes the proof. $\square$

Lemma 9. *Suppose S is an $\mathbb{R}_+$-separative abelian semigroup and U is a subset of S. Then*

$$\operatorname{Conv}(U) \subset \bigcup_{K\in\mathcal{J}(S)} \operatorname{conv}\Big(\bigcup_{H\leqslant K} g_{H,K}(U\cap H)\Big)$$

where conv *denotes convex hull in the enveloping real vector space of K.*

Proof. Suppose $v \in \operatorname{Conv}(U)$. Choose $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U$ such that $(n+1)v = v + u_1 + \ldots + u_n$. Let $K, H_1, \ldots, H_n$ be the archimedean components of S containing $v, u_1, \ldots, u_n$ respectively. The preceding equation shows that $K = K \vee H_1 \vee \ldots \vee H_n$, so $H_j \leqslant K$ for $j = 1, \ldots, n$. Moreover, $(n+1)v = v + g_{H_1,K}(u_1) + \ldots + g_{H_n,K}(u_n)$. Since G_K is cancellative, we may subtract v from both sides. Dividing the resulting equation by n, we get an expression of v as a convex combination of $g_1(u_1), \ldots, g_n(u_n)$. Since the latter elements belong to $\bigcup_{H\leqslant K}(U\cap H)$, this shows $v \in \operatorname{conv}\Big(\bigcup_{H\leqslant K}(U\cap H)\Big)$, as desired. □

Lemma 10. *Suppose S is an $\mathbb{R}_+$-separative abelian semigroup and K is an archimedean component of S. Let P be a 1-dimensional linear subspace of Q_K and suppose $h \in G_K$ is such that $P \cap K \subset \{nh \mid n \in \mathbb{N}\}$. Assume $k \in \mathbb{N}$ and $kh, (k+1)h, (k+2)h \in K$. Define*

$$b = \delta_{kh} + \delta_{(k+2)h} - 2\delta_{(k+1)h}.$$

Then $b \in \mathbb{R}[S]_{++}$. Now assume $b \in \Sigma_c(S)$. Then there exist $r, s, t \in \widetilde{S}$ such that $r + 2s = kh$ and $r + 2t = (k+2)h$.

Proof. For $\sigma \in S_+^*$ we have $\sigma((k+1)h)^k = \sigma(k(k+1)h) = \sigma(kh)^{k+1}$, so $\sigma((k+1)h) = \sigma(kh)^{(k+1)/k}$. Similarly, $\sigma((k+1)h) = \sigma((k+2)h)^{(k+1)/(k+2)}$. It follows that

$$\sigma((k+1)h) = \sqrt{\sigma(kh)\sigma((k+2)h)} \leqslant \frac{1}{2}[\sigma(kh) + \sigma((k+2)h)].$$

This being so for all $\sigma \in S_+^*$, we have shown $b \in \mathbb{R}[S]_{++}$.

Now suppose $b \in \Sigma_c(S)$. Recall that X_K denotes the least face of S containing K, which is equal to the union of those $H \in \mathcal{J}(S)$ such that $H \leqslant K$. Since $b \in \Sigma_c(S) \cap \mathbb{R}[X_K]$, by Lemma 8 it follows that $b \in \Sigma_c(X_K)$. Thus we may choose $r_1, \ldots, r_n \in X_K$ and $a_1, \ldots, a_n \in \mathbb{R}[\widetilde{X}_K]$ such that $b = \sum_{j=1}^{n} \delta_{r_j} * a_j^{*2}$. By Lemma 5 we may assume that (r_j, a_j) is proper for $j = 1, \ldots, n$. Since $b((k+1)h) < 0$, we may choose j such that $\delta_{r_j} * a_j^{*2}((k+1)h) < 0$. Now

$$0 > \delta_{r_j} * a_j^{*2}((k+1)h) = \sum_{s,t\in\operatorname{supp}a_j:\ r_j+s+t=(k+1)h} a_j(s)a_j(t).$$

Thus we may choose $s, t \in \operatorname{supp} a_j$ such that $r_j + s + t = (k+1)h$ and such that $a_j(s)$ and $a_j(t)$ are of opposite signs. In particular, $s \neq t$. Now

$$\begin{aligned} r_j + 2s, r_j + 2t \in \bigcup_{i=1}^{n}(r_i + 2\operatorname{supp} a_i) \subset X_K \cap c\Big(\bigcup_{i=1}^{n}(r_i + 2\operatorname{supp} a_i)\Big) \\ = X_K \cap c(\operatorname{supp} b) \subset X_K \cap \bigcup_{L \geqslant K} \operatorname{conv}(g_{K,L}(\operatorname{supp} b)) = K \cap \operatorname{conv}(\operatorname{supp} b) \\ = \{kh, (k+1)h, (k+2)h\}, \end{aligned}$$

by Proposition 2 (iii), Proposition 6, Proposition 4, and Lemma 9. We cannot have $r_j + 2s = r_j + 2t$, which would imply $2(r_j + s) = 2(r_j + t)$, hence $r_j + s = r_j + t$ since S is $\mathbb{R}_+$-separative (Proposition 1), hence $s = t$ since (r_j, a_j) is proper. Thus $r_j + 2s$ and $r_j + 2t$ are two distinct elements of $\{kh, (k+1)h, (k+2)h\}$ with midpoint $(k+1)h$. It follows that, interchanging s and t, if necessary, we may assume $r_j + 2s = kh$ and $r_j + 2t = (k+2)h$, as desired. $\square$

Lemma 11. *Suppose S is a c-finite semigroup and $J, K \in \mathcal{J}(S)$ are such that $J \leqslant K$. Suppose $t \in J$ and $k \in \mathbb{N}$. Let $\mathcal{M}$ be the set of those $L \in \mathcal{J}(S)$ such that $J \leqslant L \leqslant K$ and such that $\{g_{J,K}(nt) \mid n = 1, \ldots, k\} \cap L \neq \emptyset$. Then $\mathcal{M}$ is finite.*

P r o o f. For $L \in \mathcal{M}$ we have $g_{J,L}(kt) \in L$. Indeed, we can choose $n \in \{1, \ldots, k\}$ such that $g_{J,L}(nt) \in L$. If $n = k$, we have the desired conclusion. Otherwise, $L \ni g_{J,L}(nt) + (k-n)t = g_{J,L}(nt) + (k-n)g_{J,L}(t) = g_{J,L}(kt)$. This proves that $g_{J,L}(kt) \in L$ for $L \in \mathcal{M}$. Given $L \in \mathcal{M}$, define $V = \{3kt, g_{J,L}(3kt)\}$. With $r = x = kt$ and $y = g_{J,L}(kt)$ we have $r + 2x = 3kt \in V$, $r + 2y = g_{J,L}(3kt) \in V$, and $r + x + y = g_{J,L}(3kt)$. If $L \neq J$ then $r + x \neq r + y$, so the preceding shows $g_{J,L}(3kt) \notin e(V)$. Thus $e(V) \subset \{3kt\}$. On the other hand, if $L = J$ then it is trivial that $e(V) \subset \{3kt\}$. Thus $e(V) \subset \{3kt\}$ in every case. Since V is a finite set, it follows that $V \subset c(\{3kt\})$. In particular, $g_{J,L}(3kt) \in c(\{3kt\})$. This proves $\{g_{J,L}(3kt) \mid L \in \mathcal{M}\} \subset c(\{3kt\})$. The latter set is finite since S is c-finite. Thus the set $\{g_{J,L}(3kt) \mid L \in \mathcal{M}\}$ is finite. Since $g_{J,L}(3kt) \in L$ for each $L \in \mathcal{M}$, the mapping $L \mapsto g_{J,L}(3kt)$ is one-to-one, so $\mathcal{M}$ is finite. $\square$

Lemma 12. *Suppose S is an $\mathbb{R}_+$-separative abelian semigroup and $J, K \in \mathcal{J}(S)$ are such that the conditions $H \in \mathcal{J}(S)$ and $J \leqslant H \leqslant K$ imply $H = J$ or $H = K$. Suppose $u \in J$, $v = g_{J,K}(u) \in K$, and define*

$$c = \delta_u - \delta_v.$$

Then $c \in \mathbb{R}[S]_{++}$. Now suppose $c \in \Sigma_c(S)$. Then there exist $r \in S$ and $x, y \in \widetilde{S}$ such that $r + 2x = u$ and $r + 2y = v$.

P r o o f. Suppose $\sigma \in S_+^*$. By the analogue of [5], Proposition 3, for nonnegative characters instead of arbitrary characters there exist a face Y of S and a character $\gamma \in (G_Y)_+^*$ such that

$$\sigma(y) = \begin{cases} \gamma \circ g_Y(y) & \text{if } y \in Y, \\ 0 & \text{if } y \notin Y. \end{cases}$$

First suppose $K \subset Y$. Then also $J \subset Y$, and we have $\langle c, \sigma \rangle = \gamma \circ g_Y(u) - \gamma \circ g_Y(v)$. But $g_Y(v) = g_Y \circ g_{J,K}(u) = g_Y(u)$, so $\langle c, \sigma \rangle = 0$. Now suppose $K \not\subset Y$. Then $K \cap Y = \emptyset$, so $\langle c, \sigma \rangle = \sigma(u) \geqslant 0$. Thus $\langle c, \sigma \rangle \geqslant 0$ in every case. This being so for all $\sigma \in S_+^*$, we have shown $c \in \mathbb{R}[S]_{++}$.

Now suppose $c \in \Sigma_c(S)$. Recall that X_K denotes the least face of S containing K, which is the union of those $H \in \mathcal{J}(S)$ such that $H \leqslant K$. Since $c \in \Sigma_c(S) \cap \mathbb{R}[X_K]$, by Lemma 8 we have $c \in \Sigma_c(X_K)$. Choose $r_1, \dots, r_n \in X_K$ and $a_1, \dots, a_n \in \mathbb{R}[\widetilde{X}_K]$ such that $c = \sum\limits_{j=1}^{n} \delta_{r_j} * a_j^{*2}$. By Lemma 5 we may assume that (r_j, a_j) is proper for $j = 1, \dots, n$. Since $c(v) < 0$, we may choose j such that $\delta_{r_j} * a_j^{*2}(v) < 0$. Now

$$0 > \delta_{r_j} * a_j^{*2}(v) = \sum_{x,y \in \operatorname{supp} a_j \,:\, r_j + x + y = v} a_j(x) a_j(y).$$

Thus we may choose $x, y \in \operatorname{supp} a_j$ such that $r_j + x + y = v$ and such that $a_j(x)$ and $a_j(y)$ are of opposite signs. In particular, $x \neq y$. Now

$$\begin{aligned} r_j + 2x, r_j + 2y \in \bigcup_{i=1}^{n} (r_i + 2 \operatorname{supp} a_i) &\subset X_K \cap c\Big(\bigcup_{i=1}^{n} (r_i + 2 \operatorname{supp} a_i)\Big) \\ &= X_K \cap c(\operatorname{supp} c) \\ &\subset X_K \cap \Big(\bigcup_{L \geqslant J} \operatorname{conv}(g_{J,L}(u)) \cup \bigcup_{L \geqslant K} \operatorname{conv}(g_{K,L}(v)) \Big) \\ &= \{u, v\}. \end{aligned}$$

(We used Proposition 2 (iii), Proposition 6, Proposition 4, and Lemma 9.) We cannot have $r_j + 2x = r_j + 2y$, which would imply $2(r_j + x) = 2(r_j + y)$, hence $r_j + x = r_j + y$ since S is $\mathbb{R}_+$-separative (Proposition 1), hence $x = y$ since (r_j, a_j) is proper. Thus $r_j + 2x$ and $r_j + 2y$ are two distinct elements of $\{u, v\}$, so by interchanging x and y, if necessary, we may assume $r_j + 2x = u$ and $r_j + 2y = v$, as desired. $\square$

Lemma 13. *Suppose S is a Stieltjes semiperfect countable c-finite semigroup and K is an archimedean component of S. If P is a 1-dimensional linear subspace of Q_K which intersects K then the semigroup $P \cap K$ is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$.*

P r o o f. The semigroup $P \cap K$ consists of equidistant points by Lemma 7. It follows that $P \cap K$ is isomorphic to a subsemigroup of $\mathbb{Z}$. If this semigroup intersects both $\mathbb{N}$ and $-\mathbb{N}$, it is a group, hence isomorphic to $\mathbb{Z}$. Thus we may assume that $P \cap K$ is isomorphic to a subsemigroup of $\mathbb{N}_0$. Now $P \cap K$ is archimedean. To see this, suppose $x, y \in P \cap K$. Since K is archimedean, we can choose $z \in K$ and $n \in \mathbb{N}$ such that $nx = y + z$. Since $nx \in P$ and $y \in P$, it follows that $z \in P \cap K$. This shows that $P \cap K$ is archimedean. Since $P \cap K$ is isomorphic to a subsemigroup of $\mathbb{N}_0$, that semigroup must be contained in one of the archimedean components of $\mathbb{N}_0$, which are $\{0\}$ and $\mathbb{N}$. Thus we may assume that $P \cap K$ is isomorphic to a subsemigroup of $\mathbb{N}$. Since $P \cap K$ consists of equidistant points, it follows that $P \cap K$ is isomorphic to $\{n \in \mathbb{N} \mid n \geqslant k\}$ for some $k \in \mathbb{N}$. It remains to be shown that $k = 1$. Suppose $k \geqslant 2$. Choose $h \in g_K(K)$ such that $P \cap K = \{nh \mid n \in \mathbb{N},\ \ n \geqslant k\}$. Define

$$b = \delta_{kh} + \delta_{(k+2)h} - 2\delta_{(k+1)h}.$$

By Lemma 10, $b \in \mathbb{R}[S]_{++}$. By Theorem 3 it follows that $b \in \Sigma_c(S)$. By Lemma 10 there exist $r \in S$ and $s, t \in \widetilde{S}$ such that $r + 2s = kh$ and $r + 2t = (k+2)h$. Let G, I, and J be the archimedean components of $\widetilde{S}$ containing r, s, and t, respectively. If $H \in \mathcal{J}(S)$ and $H \leqslant K$ then $g_{H,K}(H) + K = H + K \subset K$, so $g_{H,K}(H) \subset \{nh \mid n \in \mathbb{N}_0\}$. Now $g_{G,K}(r) = ph$, $g_{I,K}(s) = qh$, and $g_{J,K}(t) = uh$ for some $p, q, u \in \mathbb{N}_0$. Since $r + 2s, r + 2t \in K$ then $G \vee I = G \vee J = K$. If we had $q > 0$, it would follow that $r + s = (p+q)h$ and $kh = r + 2s = (p+2q)h$, so $(p+q)h \in K$ and $p + q < p + 2q = k$, contradicting the fact that $K = \{nh \mid n \geqslant k\}$. Hence $q = 0$, so $p = k$. Since $(k+2)h = r + 2t = (p+2u)h$, it follows that $u = 1$, that is, $g_{J,K}(t) = h$.

Let $\mathcal{M}$ be the set of those $L \in \mathcal{J}(S)$ such that $J \leqslant L \leqslant K$ and $\{g_{J,L}(nt) \mid n = 1, \ldots, k\} \cap L \neq \emptyset$. By Lemma 11, $\mathcal{M}$ is finite. Suppose $L \in \mathcal{M}$. Let P_L be the linear subspace of Q_L spanned by $g_{J,L}(t)$. By the argument applied to K, the semigroup $P_L \cap L$ is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\{n \in \mathbb{N} \mid n \geqslant l\}$ for some $l \in \mathbb{N}$. Since $g_{L,K}(L) + K = L + K \subset K$, we have $g_{L,K}(L) \subset \{nh \mid n \in \mathbb{N}_0\}$. If L were a group, it would follow that $g_{L,K} = 0$, hence $g_{J,K} = g_{L,K} \circ g_{J,L} = 0$, contradicting the fact that $g_{J,K}(t) = h \neq 0$. Thus $P_L \cap L = \{ne_L \mid n \geqslant l_L\}$ for a unique $e_L \in G_L$ and a unique $l_L \in \mathbb{N}$. Since $g_{J,L}(t) \in G_L$, we have $g_{J,L}(t) = pe_L$ for some $p \in \mathbb{Z}$. Necessarily, $p \geqslant 1$. Since $g_{L,K}(e_L) \in G_K$, we have $g_{L,K}(e_L) = qh$ for some $q \in \mathbb{Z}$. Now $h = g_{J,K}(t) = g_{L,K}(g_{J,L}(t)) = pqh$, so $p = q = 1$, that is, $g_{J,L}(t) = e_L$ and $g_{L,K}(e_L) = h$. Since $\{g_{J,L}(nt) \mid n = 1, \ldots, k\} \cap L \neq \emptyset$, it follows that $l_L \leqslant k$. Thus $ke_L \in L$.

Define $\mathcal{M}_1 = \{L \in \mathcal{M} \mid l_L = 1\}$ and $\mathcal{M}_2 = \{L \in \mathcal{M} \mid l_L \geqslant 2\}$. Then $\mathcal{M}$ is the disjoint union $\mathcal{M}_1 \cup \mathcal{M}_2$. Since $J \in \mathcal{M}_1$ and $K \in \mathcal{M}_2$, it is easily seen that there exist $L_1 \in \mathcal{M}_1$ and $L_2 \in \mathcal{M}_2$ such that $L_1 < L_2$ and such that the conditions $L \in \mathcal{M}$ and $L_1 \leqslant L \leqslant L_2$ imply $L \in \{L_1, L_2\}$. There is no essential loss of generality in assuming $L_1 = J$ and $L_2 = K$. Now define

$$c = \delta_{kt} - \delta_{kh}.$$

By Lemma 12 we have $c \in \mathbb{R}[S]_{++}$. By Theorem 3 it follows that $c \in \Sigma_c(S)$. By Lemma 12 again there exist $r \in S$ and $x, y \in \widetilde{S}$ such that $r + 2x = kt$ and $r + 2y = kh$. Let A, B, and C be the archimedean components of $\widetilde{S}$ containing r, x, and y, respectively. Since $r + 2x = kt \in (A \vee B) \cap J$ then $A \vee B = J$ (since distinct archimedean components are disjoint). Similarly, $A \vee C = K$. For $H \leqslant K$ we have $g_{H,K}(H) + K = H + K \subset K$. It follows that $g_{H,K}(H) \subset \{nh \mid n \in \mathbb{N}_0\}$. Now $g_{A,K}(r) = ph$, $g_{B,K}(x) = qh$, and $g_{C,K}(y) = uh$ for some $p, q, u \in \mathbb{N}_0$. If we had $u > 0$, it would follow that $K \ni r + y = (p+u)h$ and $p + u < p + 2u = k$ (because of $kh = r + 2y = (p + 2u)h$), a contradiction. Thus $u = 0$, that is, $g_{C,K}(y) = 0$. Now $K \ni t + y = g_{J,K}(t) + g_{C,K}(y) = h$, contradicting the fact that $k \geqslant 2$. This contradiction completes the proof. $\square$

The *dimension* of a cancellative abelian semigroup K such that the group G_K is torsion-free is the dimension of the rational vector space Q_K.

Lemma 14. *If S is a Stieltjes semiperfect countable c-finite semigroup and K is an archimedean component of S then the dimension of K is at most* 1.

P r o o f. As the proof of [11], Lemma 6. $\square$

Theorem 4. *In order that a countable c-finite semigroup S be Stieltjes semiperfect, it is necessary that the following three conditions be satisfied:*

(i) *Each archimedean component of S is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$;*

(ii) *if K and L are archimedean components of S, isomorphic to $\mathbb{N}$, such that $K < L$ and $g_{K,L} \neq 0$, there is an archimedean component H of S such that $H \leqslant L$, $g_{H,L} = 0$, and $H \nleqslant K$.*

(iii) *if K is an archimedean component of S isomorphic to $\mathbb{N}$, there is an archimedean component H of S such that $H \leqslant K$ and $g_{H,K} = 0$.*

P r o o f. (i): Suppose K is an archimedean component of S. By Lemma 14, the dimension of K is at most 1. If the dimension is 0 then K is isomorphic to $\{0\}$. Thus we may assume that the dimension is 1. By Lemma 13 applied to $P = Q_K$, K is isomorphic to $\mathbb{Z}$ or $\mathbb{N}$.

(ii): Let e and f be the generators of K and L, respectively. Since $g_{K,L}(K) \subset G_L$ then $g_{K,L}(e) = pf$ for some $p \in \mathbb{Z}$. Since $pf + f = e + f \in L = \{nf \mid n \in \mathbb{N}\}$ then $p \geqslant 0$. We cannot have $p = 0$, which would imply $g_{K,L} = 0$, contradicting the hypothesis. Thus $p \in \mathbb{N}$.

Let $\mathcal{M}$ be the set of those archimedean components M of S such that $K \leqslant M \leqslant L$. Suppose $M \in \mathcal{M}$. By (i), M is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$. Since $g_{M,L}(M) \subset G_L = \{nf \mid n \in \mathbb{Z}\}$ and $g_{M,L}(M) + f = M + f \subset M + L \subset L = \{nf \mid n \in \mathbb{N}\}$ then $g_{M,L}(M) \subset \{nf \mid n \in \mathbb{N}_0\}$. If M were a group, it would follow that $g_{M,L} = 0$, hence $g_{K,L} = g_{M,L} \circ g_{K,M} = 0$, contradicting the hypothesis. Thus M must be isomorphic to $\mathbb{N}$. Let h_M be the generator of M. Then $g_{K,M}(e) \in G_M = \{nh_M \mid n \in \mathbb{Z}\}$, so $g_{K,M}(e) = p_M h_M$ for some $p_M \in \mathbb{Z}$. Since $p_M h_m + h_M = e + h_M \in K + M \subset M = \{nh_M \mid n \in \mathbb{N}\}$, we must have $p_M \geqslant 0$. We cannot have $p_M = 0$, which would imply $g_{K,M} = 0$, hence $g_{K,L} = g_{M,L} \circ g_{K,M} = 0$, contradicting the hypothesis. Thus $p_M \in \mathbb{N}$.

The set $\mathcal{M}$ is finite by Lemma 11. Since $\mathcal{M}$ is finite, we may choose a maximal element M of $\{N \in \mathcal{M} \mid N < L\}$. If we find some $H \in \mathcal{J}(S)$ such that $H \leqslant L$, $g_{H,L} = 0$, and $H \nleqslant L$, it follows that $H \nleqslant K$. Thus we may as well assume $M = K$, i.e., that the conditions $N \in \mathcal{J}(S)$ and $K \leqslant N \leqslant L$ imply $N \in \{K, L\}$.

Define

$$c = \delta_e - \delta_{pf}.$$

Then $c \in \mathbb{R}[S]_{++}$ by Lemma 12. By Theorem 3 it follows that $c \in \Sigma_c(S)$. By Lemma 12 it follows that there exist $r \in S$ and $s, t \in \widetilde{S}$ such that $r + 2s = e$ and $r + 2t = pf$. Let F, G, and H be the archimedean components of S containing r, s, and t, respectively. Then $g_{F,K}(r) = qe$ and $g_{G,K}(s) = ue$ for some $q, u \in \mathbb{N}_0$. Now $(q + 2u)e = r + 2s = e$, so $q + 2u = 1$ and therefore $q = 1$ and $u = 0$. It follows that $pf = r + 2t = pf + 2g_{H,L}(t)$, so $g_{H,L}(t) = 0$. If H is isomorphic to $\mathbb{N}$, it easily follows that $g_{H,L} = 0$. Otherwise, H is isomorphic to $\{0\}$ or $\mathbb{Z}$, and (as we have seen) it follows that $g_{H,L} = 0$. Thus $g_{H,L} = 0$ in every case. We cannot have $H \leqslant K$, which would imply $r + 2t \in F \vee H$ and $F \vee H \leqslant K$, contradicting the fact that $r + 2t \in L$. Thus $H \nleqslant K$. This completes the proof of the necessity of condition (ii).

(iii): Let e be the generator of K. By Theorem 1, in order that S be Stieltjes semiperfect, it is necessary that $S = S + S$. Thus it is necessary that there exist $s, t \in S$ such that $e = s + t$. Let H and I be the archimedean components of S containing s and t, respectively. Then $g_{H,K}(s) \in G_K = \{ne \mid n \in \mathbb{Z}\}$, so $g_{H,K}(s) = pe$ for some $p \in \mathbb{Z}$. Since $K \ni e + s = e + g_{H,K}(s) = (p+1)e$ then $p \in \mathbb{N}_0$. Similarly, $g_{I,K}(t) = qe$ for some $q \in \mathbb{N}_0$. Now $e = s + t = g_{H,K}(s) + g_{I,K}(t) = (p + q)e$, so $p + q = 1$ and therefore either $p = 0$ and $q = 1$ or vice versa. By symmetry, we may assume $p = 0$. Thus $g_{H,K}(s) = 0$. Since H is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$, it follows that $g_{H,K}$ is identically zero. $\square$

5. Sufficiency

In this section, we show that the necessary conditions from Theorem 4 for the Stieltjes semiperfectness of a countable c-finite semigroup S are also sufficient, even if S is not countable.

Lemma 15. *Suppose S is an $\mathbb{R}_+$-separative abelian semigroup and $b \in \mathbb{R}[S]_{++}$. Suppose $v \in \operatorname{Ex}(\operatorname{supp} b)$. Then $b(v) > 0$.*

P r o o f. Let K be the archimedean component of S containing v. From the fact that $v \in \operatorname{Ex}(\operatorname{supp} b)$ it follows that v is a vertex of the convex polytope $\operatorname{conv}\Bigl(\bigcup\limits_{H\leqslant K} g_{H,K}(\operatorname{supp} b\cap H)\Bigr)$. To see this, assume $H_1,\dots,H_n \in \mathcal{J}(S)$ with $H_j \leqslant K$ for each j, $u_j \in \operatorname{supp} b \cap H_j$ for each j, $\alpha_j > 0$ for each j, $\sum\limits_{j=1}^{n} \alpha_j = 1$, and $v = \sum\limits_{j=1}^{n} \alpha_j g_{H_j,K}(u_j)$. Since $\mathbb{Q}$-linearly independent families in Q_K are $\mathbb{R}$-linearly independent in the enveloping real vector space then we may assume that the α_j are rational. Multiplying by a common denominator, we get a relation of the form $kv = \sum\limits_{j=1}^{n} g_{H_j,K}(k_j u_j)$ where the k_j are in $\mathbb{N}$ and $k = \sum\limits_{j=1}^{n} k_j$. Now $(k+1)v = v + \sum\limits_{j=1}^{n} g_{H_j,K}(k_j u_j) = v + k_1u_1 + \ldots + k_nu_n$. Since $v \in \operatorname{Ex}(\operatorname{supp} b)$, it follows that $u_1 = \ldots = u_n = v$. This shows that v is a vertex of $\operatorname{conv}\Bigl(\bigcup\limits_{H\leqslant K} g_{H,K}(\operatorname{supp} b\cap H)\Bigr)$. Moreover, the conditions $H \in \mathcal{J}(S)$, $H \leqslant K$, $w \in \operatorname{supp} b \cap H$, and $g_{H,K}(w) = v$ imply $H = K$. To see this, note that these conditions imply $2v = v + g_{H,K}(w) = v + w$, so $v = w$ follows from the fact that $v \in \operatorname{Ex}(\operatorname{supp} b)$. Since v is a vertex of the convex polytope $\operatorname{conv}\Bigl(\bigcup\limits_{H\leqslant K} g_{H,K}(\operatorname{supp} b\cap H)\Bigr)$, by [13], 7.5, there is a homomorphism ξ of G_K into the group $(\mathbb{R},+)$ such that $\xi(w) < \xi(v)$ for all $w \in \bigcup\limits_{H\leqslant K} g_{H,K}(\operatorname{supp} b\cap H)\setminus\{v\}$. For $t > 0$, define $\sigma_t \in S_+^*$ by $\sigma_t\big|H = \mathrm{e}^{\xi\circ g_{H,K}}$ for $H \leqslant K$ and $\sigma_t\big|H = 0$ for $H \not\leqslant K$. Then

$$0 \leqslant \lim_{t\to\infty} \frac{\langle b, \sigma_t\rangle}{\sigma_t(v)} = b(v).$$

Since $v \in \operatorname{supp} b$, it follows that $b(v) > 0$. □

Corollary 2. $\mathbb{R}[S]_{++} \cap (-\mathbb{R}[S]_{++}) = \{0\}$.

P r o o f. Suppose $b \in \mathbb{R}[S]_{++} \cap (-\mathbb{R}[S]_{++})$ but $b \neq 0$. Then $\emptyset \neq \operatorname{supp} b \subset \operatorname{Conv}(\operatorname{supp} b) = \operatorname{Conv}(\operatorname{Ex}(\operatorname{supp} b))$ (Lemma 3), so $\operatorname{Ex}(\operatorname{supp} b) \neq \emptyset$. Choose $v \in \operatorname{Ex}(\operatorname{supp} b)$. Then $b(v) > 0$ by Lemma 15. But Lemma 15 also applies to $-b$ instead of b, so $-b(v) > 0$, a contradiction. □

Proposition 7. *Suppose S is an $\mathbb{R}_+$-separative abelian semigroup and W is a subset of S satisfying $W = \operatorname{Conv}(W)$. Then $\mathbb{R}[S]_{++} \cap \mathbb{R}^{(W)}$ is a face of $\mathbb{R}[S]_{++}$. That is, if $b, c \in \mathbb{R}[S]_{++}$ and $b + c \in \mathbb{R}^{(W)}$ then $b, c \in \mathbb{R}^{(W)}$.*

P r o o f. Suppose $b, c \in \mathbb{R}[S]_{++}$ and $b + c \in \mathbb{R}^{(W)}$, that is, $\operatorname{supp}(b+c) \subset W$; we have to show $\operatorname{supp} b \cup \operatorname{supp} c \subset W$. Since $\operatorname{supp} b \cup \operatorname{supp} c \subset \operatorname{Conv}(\operatorname{supp} b \cup \operatorname{supp} c) =$

$\mathrm{Conv}(\mathrm{Ex}(\operatorname{supp} b \cup \operatorname{supp} c))$ (Lemma 3), and since $W = \mathrm{Conv}(W)$, it suffices to show $\mathrm{Ex}(\operatorname{supp} b \cup \operatorname{supp} c) \subset W$. Suppose $v \in \mathrm{Ex}(\operatorname{supp} b \cup \operatorname{supp} c)$. In particular, $v \in \operatorname{supp} b \cup \operatorname{supp} c$, so by interchanging b and c, if necessary, we may assume $v \in \operatorname{supp} b$. From $v \in \operatorname{supp} b \cap \mathrm{Ex}(\operatorname{supp} b \cup \operatorname{supp} c)$ it follows that $v \in \mathrm{Ex}(\operatorname{supp} b)$. By Lemma 15 it follows that $b(v) > 0$. Now either $v \in \operatorname{supp} c$, in which case $c(v) > 0$ (similarly), or $v \notin \operatorname{supp} c$, that is, $c(v) = 0$. In either case, $c(v) \geqslant 0$. Thus $b(v) + c(v) > 0$, so $v \in \operatorname{supp}(b + c) \subset W$, as desired. □

Proposition 8. *Suppose S is a c-finite semigroup, each of whose archimedean components is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$. If U is a finite subset of S then the set $W = \mathrm{Conv}(U)$ in is finite and satisfies $W = \mathrm{Conv}(W)$. It follows that the convex cone $\mathbb{R}[S]_{++} \cap \mathbb{R}^{(W)}$ is generated by its extreme rays, and that these are even extreme rays in $\mathbb{R}[S]_{++}$. Thus $\mathbb{R}[S]_{++}$ is generated by its extreme rays, so in order to show $\mathbb{R}[S]_{++} \subset \Sigma_c(S)$ it suffices to show that each element which generates an extreme ray in $\mathbb{R}[S]_{++}$ has the form $\delta_r * a^{*2}$ for some $r \in S$ and some $a \in \mathbb{R}[\widetilde{S}]$.*

P r o o f. By Lemma 9, $W \subset \bigcup_{K \in \mathcal{J}(S)} \mathrm{conv}\Big(\bigcup_{H \leqslant K} g_{H,K}(U \cap H)\Big)$. For $K \in \mathcal{J}(S)$ the set $\bigcup_{H \leqslant K} g_{H,K}(U \cap H)$ is finite since the finite set U intersects only finitely many $H \in \mathcal{J}(S)$ and has a finite intersection with each of them. It therefore suffices to show that there are only finitely many $K \in \mathcal{J}(S)$ such that $\Big(\bigcup_{H \leqslant K} g_{H,K}(U \cap H)\Big) \cap K \neq \emptyset$. For this, it suffices to show that for each $H \in \mathcal{J}(S)$ there are only finitely many $K \in \mathcal{J}(S)$ such that $H \leqslant K$ and $g_{H,K}(H) \cap K \neq \emptyset$.

Suppose $H \in \mathcal{J}(S)$ and let $\mathcal{K}$ be the set of those $K \in \mathcal{J}(S)$ such that $H \leqslant K$ and $g_{H,K}(H) \cap K \neq \emptyset$. Choose an element e of H which generates H either as a semigroup or as a group. If $K \in \mathcal{K}$ then $g_{H,K}(e) \in K$. To see this, first suppose H is a group. If K is isomorphic to $\mathbb{N}$, let f be the generator of K. Then $g_{H,K}(H) \subset G_K = \{nf \mid n \in \mathbb{Z}\}$ and $g_{H,K}(H) + f \subset H + K \subset K = \{nf \mid n \in \mathbb{N}\}$, so $g_{H,K}(H) \subset \{nf \mid n \in \mathbb{N}_0\}$. Since $g_{H,K}(H)$ is a group, it follows that $g_{H,K}(H) = \{0\}$, hence $g_{H,K}(H) \cap K = \emptyset$, a contradiction. Thus K is also a group. Now $g_{H,K}(e) \in G_K = K$, as claimed. Next, suppose H is isomorphic to $\mathbb{N}$. Since $g_{H,K}(H) \cap K \neq \emptyset$, there is some $p \in \mathbb{N}$ such that $g_{H,K}(pe) \in K$. Now $g_{H,K}(e) \in G_K$ and $p g_{H,K}(e) \in K$. It easily follows that $g_{H,K}(e) \in K$.

Thus $g_{H,K}(e) \in K$ for each $K \in \mathcal{K}$. By Lemma 11 it follows that $\mathcal{K}$ is finite.

We have shown that for every finite subset U of S the set $W = \mathrm{Conv}(U)$ is finite. We leave it as an exercise to show that $W = \mathrm{Conv}(W)$. Now the convex cone $\Gamma_W = \mathbb{R}[S]_{++} \cap \mathbb{R}^{(W)}$ is finite-dimensional, closed, and satisfies $\Gamma_W \cap (-\Gamma_W) = \{0\}$ (Corollary 2). As is well known, it follows that Γ_W is generated by its extreme rays. These are also extreme in $\mathbb{R}[S]_{++}$ since Γ_W is a face of $\mathbb{R}[S]_{++}$ (Proposition 7).

If $b \in \mathbb{R}[S]_{++}$, define $U = \operatorname{supp} b$ and $W = \operatorname{Conv}(U)$. Then $U \subset W$, so $b \in \Gamma_W$. Hence b is the sum of certain generators of extreme rays in Γ_W. These also generate extreme rays in $\mathbb{R}[S]_{++}$. Thus $\mathbb{R}[S]_{++}$ is generated by its extreme rays. $\square$

Theorem 5. *Suppose S is a c-finite semigroup. Then S is Stieltjes semiperfect if the following three conditions are satisfied:*

(i) *Each archimedean component of S is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$;*

(ii) *if K and L are archimedean components of S, both isomorphic to $\mathbb{N}$, such that $K < L$ and $g_{K,L} \neq 0$ then there is an archimedean component H of S such that $H \leqslant L$, $g_{H,L} = 0$, and $H \nleqslant K$.*

(iii) *if K is an archimedean component of S isomorphic to $\mathbb{N}$, there is an archimedean component H of S such that $H \leqslant K$ and $g_{H,K} = 0$.*

If S is countable, the above three conditions are also necessary for the Stieltjes semiperfectness of S.

Proof. To show the sufficiency of the conditions, note that (iii) implies that $S = S + S$, so by Theorem 2 it suffices to show that $\Sigma_c(S)$ is dense in $\mathbb{R}[S]_{++}$. (Perhaps we ought to indicate why (iii) implies $S = S + S$. Suppose $u \in S$; we have to show that there exist $s, t \in S$ such that $s + t = u$. Let K be the archimedean component of S containing u. If K is a group then, denoting by 0 the zero of K, we have $u = u + 0$, as desired. This takes care of the cases $K = \{0\}$ and $K = \mathbb{Z}$. By (i), the case $K = \mathbb{N}$ remains. By (iii) there is an archimedean component H of S such that $H \leqslant K$ and $g_{H,K} = 0$. Choosing any $x \in H$, we have $u = u + 0 = u + g_{H,K}(x) = u + x$, as desired.) We can even show $\Sigma_c(S) = \mathbb{R}[S]_{++}$. The inclusion $\Sigma_c(S) \subset \mathbb{R}[S]_{++}$ being automatical, by Proposition 8 it suffices to show that if b is an element that generates an extreme ray in $\mathbb{R}[S]_{++}$ then $b = \delta_r * a^{*2}$ for some $r \in S$ and some $a \in \mathbb{R}[\widetilde{S}]$. Choose $K \in \mathcal{J}(S)$ minimal with the property that $b\big|K \neq 0$.

First suppose K is a group. In this case, let $\mathcal{H}$ be the subsemilattice of $\mathcal{J}(S)$ generated by those $H \in \mathcal{J}(S)$ such that $b\big|H \neq 0$.

Next, suppose K is isomorphic to $\mathbb{N}$. In this case, let $\mathcal{G}$ be the set of those $L \in \mathcal{J}(S)$ such that $K < L$ and $g_{K,L} \neq 0$. Then $\mathcal{G}$ is finite. Indeed, let e be the generator of K. As in the proof of Proposition 8, one can show that $g_{K,L}(e) \in L$ for all $L \in \mathcal{G}$. By Lemma 11 it follows that $\mathcal{G}$ is finite. For each $L \in \mathcal{G}$ which is isomorphic to $\mathbb{N}$, choose $H_L \in \mathcal{J}(S)$ such that $H_L \leqslant L$, $g_{H_L,L} = 0$, and $H_L \nleqslant K$, which is possible by (ii). Now let $\mathcal{H}$ be the subsemilattice of $\mathcal{J}(S)$ generated by the union of $\{H_L \mid L \in \mathcal{G}\}$, the set of those $H \in \mathcal{J}(S)$ such that $b\big|H \neq 0$, and a set of the form $\{H\}$ where $H \in \mathcal{J}(S)$ is so chosen that $H \leqslant K$ and $g_{H,K} = 0$ (which is possible by (iii)).

In both cases, $\mathcal{H}$ is a finitely generated semilattice, hence finite, and $b \in \mathbb{R}[S_{\mathcal{H}}]$ where $S_{\mathcal{H}}$ is the subsemigroup of S defined by

$$S_{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} H.$$

Define an abelian semigroup $G_{\mathcal{H}}$, containing $S_{\mathcal{H}}$ as a subsemigroup, by

$$G_{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} G_H \quad \text{(disjoint union)}$$

and the addition law

$$x + y = g_{I,I\vee J}(x) + g_{J,I\vee J}(y) \quad \text{(sum in the group } G_{I\vee J})$$

for $I, J \in \mathcal{H}$, $x \in I$, and $y \in J$. Define a subsemigroup $T_{\mathcal{H}}$ of $G_{\mathcal{H}}$, containing $S_{\mathcal{H}}$, by

$$T_{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} T_H$$

where

$$T_H = \begin{cases} H & \text{if } H = \{0\} \text{ or } H = \mathbb{Z}, \\ \mathbb{N}_0 & \text{if } H = \mathbb{N}. \end{cases}$$

For $I, J \in \mathcal{H}$ such that $I \leqslant J$, define an algebra homomorphism $\Phi_{I,J}\colon\ \mathbb{R}[T_I] \to \mathbb{R}[T_J]$ by

$$\Phi_{I,J}a(t) = \sum_{s \in g_{I,J}^{-1}(t)} a(s)$$

for $a \in \mathbb{R}[T_I]$ and $t \in T_J$ (cf. the proof of [9], Proposition 6). Define a linear mapping $\Lambda\colon\ \mathbb{R}[T_{\mathcal{H}}] \to \prod_{H \in \mathcal{H}} \mathbb{R}[T_H]$ by

$$\Lambda a(J) = \sum_{I \leqslant J} \Phi_{I,J}(a\big|I)$$

for $a \in \mathbb{R}[T_{\mathcal{H}}]$ and $J \in \mathcal{H}$. If $\prod_{H \in \mathcal{H}} \mathbb{R}[T_H]$ is considered with the multiplication $\times$ defined by

$$f \times g(H) = f(H) * g(H)$$

for $f, g \in \prod_{H \in \mathcal{H}} \mathbb{R}[T_H]$ and $H \in \mathcal{H}$ then Λ is an algebra isomorphism ([9], Proposition 6). Moreover, $\Lambda\mathbb{R}[T_{\mathcal{H}}]_{++} = \prod_{H \in \mathcal{H}} \mathbb{R}[T_H]_{++}$ (cf. the proof of the analogous equality in [9], Proposition 6).

Define $\mathcal{H}^* = \{H \in \mathcal{H} \mid T_H \neq H\}$. Let $\mathcal{H}_1^*$ be the set of those $L \in \mathcal{H}^*$ for which there is some $H \in \mathcal{H}$ such that $H \leqslant L$ and $g_{H,L} = 0$, and let $\mathcal{H}_0^*$ be the complementary subset of $\mathcal{H}^*$. For $L \in \mathcal{H}_1^*$, let $\mathcal{H}_L$ be the set of those $H \in \mathcal{H}$ such that $H \leqslant L$ and $g_{H,L} = 0$. Then $\mathcal{H}_L$ is a semilattice (cf. [9], proof of Proposition 10) which, being finite, has a greatest element, which we denote by L'. Now for $f \in \prod_{H \in \mathcal{H}} \mathbb{R}[T_H]$ we have $f \in \Lambda\mathbb{R}[S_{\mathcal{H}}]$ if and only if $f(L)(0) = \Phi_{L',L}(f(L'))(0)$ for all $L \in \mathcal{H}_1^*$ and $f(L)(0) = 0$ for all $L \in \mathcal{H}_0^*$ (cf. [9], Proposition 10, and [11], proof of Theorem 4). Note that for $L \in \mathcal{H}_1^*$ we have $\Phi_{L',L}(f(L'))(0) = \sum_{s \in L'} f(L')(s)$.

Define $f = \Lambda b$. From the fact that K is minimal in $\mathcal{J}(S)$ with the property that $b\big|K \neq 0$, it follows that K is minimal in $\mathcal{H}$ with the property that $f(K) \neq 0$. Since b generates an extreme ray in $\mathbb{R}[S]_{++}$, b also generates an extreme ray in $\mathbb{R}[S_{\mathcal{H}}]_{++}$, so f generates an extreme ray in $\Lambda\mathbb{R}[S_{\mathcal{H}}]_{++}$.

Let $\sim$ be the smallest equivalence relation in $\mathcal{H}$ such that if $L \in \mathcal{H}_1^*$ then $L \sim L'$. Let $\mathcal{K}$ be the equivalence class containing K. Then $f\big|(\mathcal{H} \setminus \mathcal{K}) = 0$. To see this, define $f_1, f_2 \in \prod_{H \in \mathcal{H}} \mathbb{R}[T_H]$ by $f_1\big|\mathcal{K} = f\big|\mathcal{K}$, $f_1\big|(\mathcal{H} \setminus \mathcal{K}) = 0$, $f_2\big|\mathcal{K} = 0$, and $f_2\big|(\mathcal{H} \setminus \mathcal{K}) = f\big|(\mathcal{H} \setminus \mathcal{K})$. Then $f = f_1 + f_2$. From the facts that $f \in \Lambda\mathbb{R}[S_{\mathcal{H}}]_{++}$ and that each of the sets $\mathcal{K}$ and $\mathcal{H} \setminus \mathcal{K}$ is a union of equivalence classes with respect to $\sim$, it easily follows that $f_1, f_2 \in \Lambda\mathbb{R}[S_{\mathcal{H}}]_{++}$. Since f generates an extreme ray in $\Lambda\mathbb{R}[S_{\mathcal{H}}]_{++}$, it follows that there exist $\alpha_1, \alpha_2 \geqslant 0$ such that $f_i = \alpha_i f$ for $i = 1, 2$. Now $0 = f_2\big|\mathcal{K} = \alpha_2 f\big|\mathcal{K}$. Since $f\big|\mathcal{K} \neq 0$ (because of $K \in \mathcal{K}$), it follows that $\alpha_2 = 0$, so $f_2 = 0$ and therefore $f = f_1$. This proves $f\big|(\mathcal{H} \setminus \mathcal{K}) = 0$.

Define $\mathcal{D} = \{(L, L') \mid L \in \mathcal{H}_1^*\}$ and $\mathcal{E} = \{(L', L) \mid L \in \mathcal{H}_1^*\}$. A *path* is a sequence $(L_0, \ldots, L_n)$ of pairwise distinct elements $L_j \in \mathcal{H}$ such that $(L_{j-1}, L_j) \in \mathcal{D} \cup \mathcal{E}$ for $j = 1, \ldots, n$. We admit the case $n = 0$. Such a path is a path *from* L_0 *to* L_n. The *signature* of this path is the sequence $(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ where $\mathcal{F}_j = \mathcal{D}$ (resp. $\mathcal{E}$) if $(L_{j-1}, L_j) \in \mathcal{D}$ (resp. $\mathcal{E}$). The signature of a path cannot have the form $(\ldots, \mathcal{E}, \mathcal{D}, \ldots)$. Indeed, this would imply that for some j we had $(L_{j-1}, L_j) \in \mathcal{E}$ and $(L_j, L_{j+1}) \in \mathcal{D}$, so $L_{j-1} = L_j' = L_{j+1}$, contradicting the hypothesis that the L_i are pairwise distinct. From the definition of the equivalence relation $\sim$ it follows that two elements of $\mathcal{H}$ are equivalent if and only if there is a path from one to the other. In particular, $\mathcal{K}$ is the set of those $H \in \mathcal{H}$ such that there is a path from K to H.

For each $L \in \mathcal{K}$ there is a unique path from K to L. Indeed, if there were two distinct paths, there would be a *cycle*, that is, a sequence $(L_0, \ldots, L_n)$, with $n \geqslant 1$, which has all the properties of a path except that $L_0 = L_n$. The signature of the cycle cannot contain only $\mathcal{D}$'s, which would imply $L_0 > L_1 > \ldots > L_n = L_0$, a contradiction. Similarly, it cannot contain only $\mathcal{E}$'s. Hence, after a cyclic permutation, if necessary, it contains $(\ldots, \mathcal{E}, \mathcal{D}, \ldots)$, which is impossible.

For $L \in \mathcal{K}$, let $\mathcal{K}_L$ be the set of those $M \in \mathcal{K}$ for which there is a path from L to M with signature $(\mathcal{E}, \dots, \mathcal{E})$. If $L \in \mathcal{K}$ is isomorphic to $\mathbb{N}$ and $f(L)(0) = 0$ then either $f\big|\mathcal{K}_L = 0$ or $f\big|(\mathcal{K} \setminus \mathcal{K}_L) = 0$. To see this, suppose $f\big|\mathcal{K}_L \neq 0$. Define $f_1, f_2 \in \prod_{H\in\mathcal{H}} \mathbb{R}[T_H]$ by $f_1\big|\mathcal{K}_L = f\big|\mathcal{K}_L$, $f_1\big|(\mathcal{H}\setminus\mathcal{K}_L) = 0$, $f_2\big|\mathcal{K}_L = 0$, and $f_2\big|(\mathcal{H}\setminus\mathcal{K}_L) = f\big|(\mathcal{H} \setminus \mathcal{K}_L)$. It is easily seen that $f_1, f_2 \in \Lambda\mathbb{R}[S_\mathcal{H}]_{++}$. Since $f = f_1 + f_2$ and since f generates an extreme ray in $\Lambda\mathbb{R}[S_\mathcal{H}]_{++}$, there exist $\alpha_1, \alpha_2 \geqslant 0$ such that $f_i = \alpha_i f$ for $i = 1, 2$. In particular, $0 = f_2\big|\mathcal{K}_L = \alpha_2 f\big|\mathcal{K}_L$. Since $f\big|\mathcal{K}_L \neq 0$, it follows that $\alpha_2 = 0$, so $f_2 = 0$, that is, $f\big|(\mathcal{H} \setminus \mathcal{K}_L) = 0$ and in particular $f\big|(\mathcal{K} \setminus \mathcal{K}_L) = 0$.

For each $L \in \mathcal{K}$ the element $f(L)$ either is zero or generates an extreme ray in $\mathbb{R}[T_L]_{++}$. To see this, suppose $f(L) \neq 0$. Assume $a, b \in \mathbb{R}[T_L]_{++}$ and $f(L) = a + b$; we have to show that a and b are nonnegative multiples of $f(L)$. Choose an ordering $(L_0, \dots, L_p)$ of $\mathcal{K}$ such that if $j \in \{0, \dots, p\}$ and if $(M_0, \dots, M_n)$ is the unique path from L to L_j then $\{M_0, \dots, M_{n-1}\} \subset \{L_0, \dots, L_{j-1}\}$. We define $f_1, f_2 \in \Lambda\mathbb{R}[S_\mathcal{H}]_{++}$, with $f = f_1 + f_2$. We do this by defining $f_1(L_j)$ and $f_2(L_j)$ by induction on j. First suppose $j = 0$. Then $L_0 = L$. Define $f_1(L) = a$ and $f_2(L) = b$. Now suppose $j \geqslant 1$ and that $f_1(L_i)$ and $f_2(L_i)$ have been chosen for $i < j$. Let $(M_0, \dots, M_n)$ be the unique path from L to L_j. Then $f_1(M_{n-1})$ and $f_2(M_{n-1})$ have already been chosen. Now either $(M_{n-1}, L_j) \in \mathcal{D}$ or $(M_{n-1}, L_j) \in \mathcal{E}$. First suppose $(M_{n-1}, L_j) \in \mathcal{D}$. This means that $M_{n-1} \in \mathcal{H}_1^*$ and $L_j = M'_{n-1}$. We have to choose $f_1(L_j)$ and $f_2(L_j)$ in such a way that $f_1(L_j) + f_2(L_j) = f(L_j)$ and such that

$$\sum_{s\in L_j} f_1(L_j)(s) = f_1(M_{n-1})(0) \quad \text{and} \quad \sum_{s\in L_j} f_2(L_j)(s) = f_2(M_{n-1})(0).$$

By the induction hypothesis we have $f_1(M_{n-1}) + f_2(M_{n-1}) = f(M_{n-1})$ and in particular $f_1(M_{n-1})(0) + f_2(M_{n-1})(0) = f(M_{n-1})(0)$. If $f(M_{n-1})(0) > 0$, there is the unique solution

$$f_1(L_j) = \frac{f_1(M_{n-1})(0)}{f(M_{n-1})(0)} f(L_j), \quad f_2(L_j) = \frac{f_2(M_{n-1})(0)}{f(M_{n-1})(0)} f(L_j).$$

If $f(M_{n-1})(0) = 0$, there are in general many solutions. This covers all cases since $f(M_{n-1})(0) \geqslant 0$ because of $f(M_{n-1}) \in \mathbb{R}[T_{M_{n-1}}]_{++}$. Similar reasoning covers the case $(M_{n-1}, L_j) \in \mathcal{E}$. Now from $f = f_1 + f_2$ and from the fact that f generates an extreme ray in $\Lambda\mathbb{R}[S_\mathcal{H}]_{++}$ it follows that f_1 and f_2 are nonnegative multiples of f. In particular, a and b are nonnegative multiples of $f(L)$, as desired.

For $L \in \mathcal{K}$, since $f(L)$ is zero or generates an extreme ray in $\mathbb{R}[T_L]_{++}$, there exist $s_L \in T_L$ and $a_L \in \mathbb{R}[T_H]$ such that

$$f(L) = \delta_{s_L} * a_L^{*2}. \tag{4}$$

We are going to show that there exist $r \in S_{\mathcal{H}}$ and $c \in \mathbb{R}[S_{\mathcal{H}}]$ such that $b = \delta_r * c^{*2}$, which is equivalent to showing that there exist $r \in S_{\mathcal{H}}$ and $g \in \Lambda\mathbb{R}[S_{\mathcal{H}}]$ such that

$$f = \Lambda\delta_r \times g \times g.$$

Note that

$$\Lambda\delta_r(L) = \begin{cases} \delta_{g_{K,L}(r)} & \text{if } K \leqslant L, \\ 0 & \text{if } K \nleqslant L. \end{cases}$$

First suppose K is a group. We then put $r = s_K$ and $g(K) = a_K$. Then $f(K) = \delta_r * g(K)^{*2}$, as desired. Next, suppose K is isomorphic to $\mathbb{N}$. By the definition of $\mathcal{H}$ there is some $H \in \mathcal{H}$ such that $H \leqslant K$ and $g_{H,K} = 0$. Thus $K \in \mathcal{H}_1^*$. Now $K' < K$, so by the minimality of K we have $f(K') = 0$, hence (using the fact that $f \in \Lambda\mathbb{R}[S_{\mathcal{H}}]$) $0 = \sum_{s \in K'} f(K')(s) = f(K)(0)$. It follows that in (4) we may assume $s_K \geqslant 1$. We then take $r = s_K$ and $g(K) = a_K$.

We have now defined r and $g(K)$ in every case. The next step is to define $g(L)$ for $L \in \mathcal{K}_K \setminus \{K\}$, and we do this by induction on the length of the unique path from K to L. Suppose $L \in \mathcal{K}_K \setminus \{K\}$ and that $g(M)$ has been defined for every M for which the path from K to M is shorter than the path from K to L. From the fact that $L \in \mathcal{K}_K \setminus \{K\}$ it follows that $L \in \mathcal{H}_1^*$. Now L' is on the path from K to L, so the path from K to L' is shorter than the path from K to L. By the induction hypothesis it follows that $g(L')$ has already been chosen. First suppose $f(L)(0) = 0$. As we have seen, it follows that either $f\big|\mathcal{K}_L = 0$ or $f\big|(\mathcal{K} \setminus \mathcal{K}_L) = 0$. Since $K \notin \mathcal{K}_L$ and $f(K) \neq 0$, it follows that $f\big|\mathcal{K}_L = 0$. In particular, $f(L) = 0$. In this case, we may take $g(L) = 0$. Now suppose $f(L)(0) \neq 0$. Then in (4) we must have $s_L = 0$. Now (since f is in $\Lambda\mathbb{R}[S_{\mathcal{H}}]$) we have $a_L(0)^2 = a_L^{*2}(0) = f(L)(0) = \sum_{s \in L'} f(L')(s) = \big(\sum_{s \in L'} g(L')(s)\big)^2$, so we may take $g(L) = \pm a_L$ with the sign so chosen that $g(L)(0) = \sum_{s \in L'} g(L')(s)$. This completes the definition of $g\big|\mathcal{K}_K$.

To define $g\big|(\mathcal{K} \setminus \mathcal{K}_K)$, we again proceed by induction on the length of the path from K to the element of $\mathcal{K} \setminus \mathcal{K}_K$ in question. Suppose $L \in \mathcal{K} \setminus \mathcal{K}_K$ and that $g(M)$ has been defined for all $M \in \mathcal{K} \setminus \mathcal{K}_K$ such that the length of the path from K to M is less than the length of the path from K to L. First assume that the length of the path from K to L is 1. Since $L \notin \mathcal{K}_K$, the signature of the path must be $(\mathcal{D})$, i.e., we must have $L = K'$. Choose $g(K')$ such that $\sum_{s \in K'} g(K')(s) = g(K)(0)$ and furthermore such that if K' is isomorphic to $\mathbb{N}$ then $g(K')(0) = 0$. Now assume that the length of the path from K to L is 2. If the signature of the path begins with $\mathcal{E}$ then, as we have seen, it consists entirely of $\mathcal{E}$'s, so $L \in \mathcal{K}_K$, a contradiction. Thus the signature begins with $\mathcal{D}$ and therefore is either $(\mathcal{D}, \mathcal{D})$ or $(\mathcal{D}, \mathcal{E})$. First assume that the signature is $(\mathcal{D}, \mathcal{D})$. Then $L = (K')'$, so K' is isomorphic to $\mathbb{N}$.

Therefore, we have chosen $g(K')$ in such a way that $g(K')(0) = 0$. It follows that the requirement $g(K')(0) = \sum_{s \in L} g(L)(s)$ is satisfied if we take $g(L) = 0$ (which we do). Now suppose the signature is $(\mathcal{D}, \mathcal{E})$. Then $L' = K'$. We then take $g(L)$ to be any element of $\mathbb{R}[T_L]$ satisfying $g(L)(0) = \sum_{s \in K'} g(K')(s)$ and $\sum_{t \in L} g(L)(t) = 0$. This completes the definition of $g(L)$ for those $L \in \mathcal{K} \setminus \mathcal{K}_K$ such that the length of the path from K to L is at most 2. Note that for all $L \in \mathcal{K} \setminus \mathcal{K}_L$ we have chosen $g(L)$ in such a way that $\sum_{s \in L} g(L)(s) = 0$. Now suppose the length of the path from K to L is at least 3. We then define $g(L) = 0$. If the signature of the path from K to L ends with $\mathcal{D}$, we have $L = M'$ for some M such that the length of the path from K to M is 1 less than the length of the path from K to L. We then have to verify

$$g(M)(0) = \sum_{s \in L} g(L)(s). \tag{5}$$

This is trivial if the length of the path from K to M is at least 3, since in that case we have defined $g(M) = 0$. So suppose the length of the path from K to M is 2. As before, since $M \notin \mathcal{K}_K$, the signature of the path from K to M cannot begin with $\mathcal{E}$. Thus it begins with $\mathcal{D}$ and therefore has one of the forms $(\mathcal{D}, \mathcal{D})$ or $(\mathcal{D}, \mathcal{E})$. First consider the case $(\mathcal{D}, \mathcal{D})$. Then $M = (K')'$, so we have taken $g(M) = 0$, whence (5) is satisfied. Now consider the case $(\mathcal{D}, \mathcal{E})$. Then the signature of the path from K to L is $(\mathcal{D}, \mathcal{E}, \mathcal{D})$, which contains $\mathcal{E}$ and $\mathcal{D}$ immediately after each other in that order, which is impossible, as we have seen. Thus we may assume that the signature of the path from K to L ends with $\mathcal{E}$. Then we have to verify

$$g(L)(0) = \sum_{s \in L'} g(L')(s). \tag{6}$$

This is trivial if the length of the path from K to L' is at least 3 since in that case we have taken $g(L') = 0$. So suppose the length of the path from K to L' is 2. The signature of that path must be $(\mathcal{D}, \mathcal{D})$ or $(\mathcal{D}, \mathcal{E})$. In the first case, we have $L' = (K')'$, so we have taken $g((K')') = 0$, whence (6) is satisfied. In the latter case, we have chosen $g(L')$ so as to satisfy $\sum_{s \in L'} g(L')(s) = 0$, so that (6) is again satisfied. This completes the definition of $g\big|\mathcal{K}$. Finally, put $g\big|(\mathcal{H} \setminus \mathcal{K}) = 0$.

We have defined an element g of $\prod_{H \in \mathcal{H}} \mathbb{R}[T_H]$, and we claim that this g is in $\Lambda\mathbb{R}[S_{\mathcal{H}}]$. To see this, we have to verify that

$$g(L)(0) = \begin{cases} \sum_{s \in L'} g(L')(s) & \text{if } L \in \mathcal{H}_1^*, \\ 0 & \text{if } L \in \mathcal{H}_0^*. \end{cases}$$

The requirement in the case $L \in \mathcal{H}_0^*$ is automatically satisfied if $L \notin \mathcal{K}$ since in that case we took $g(L) = 0$. The requirement in the case $L \in \mathcal{H}_1^*$ is also automatically satisfied if $L \notin \mathcal{K}$ since in that case L' belongs to the equivalence class with respect to $\sim$ containing L, which class is disjoint with $\mathcal{K}$, so that $g(L)$ and $g(L')$ are both zero. Thus, in both cases it suffices to consider the case $L \in \mathcal{K}$. If $L \in \mathcal{H}_1^*$ then we took care of the requirement either when we chose $g(L)$ or when we chose $g(L')$, depending on which of the points L and L' is connected to K via the shortest path. It remains to consider the case $L \in \mathcal{H}_0^* \cap \mathcal{K}$. The path from K to L must have a signature ending with $\mathcal{D}$ since the assumption that it ended with $\mathcal{E}$ would imply $L \in \mathcal{H}_1^*$, a contradiction. Now, since the signature of a path cannot contain $(\ldots, \mathcal{E}, \mathcal{D}, \ldots)$, it follows that the signature of the path from K to L consists of $\mathcal{D}$'s alone. If the length of the path is at least 3 then we took $g(L) = 0$, so the requirement is met. Suppose the length of the path is 2, so the signature is $(\mathcal{D}, \mathcal{D})$, that is, $L = (K')'$. Then we took $g(L) = 0$, so the requirement is met. Now suppose the length of the path is 1, so the signature is $(\mathcal{D})$, that is, $L = K'$. Then, since L is isomorphic to $\mathbb{N}$ (being an element of $\mathcal{H}^*$), we took $g(L)$ such that $g(L)(0) = 0$, so the requirement is met. Finally, suppose the length of the path is zero, i.e., $L = K$. Note that by definition of $\mathcal{H}$, since $K = L$ is isomorphic to $\mathbb{N}$, there is some $H \in \mathcal{H}$ such that $H \leqslant K$ and $g_{H,K} = 0$. This, however, means that $K \in \mathcal{H}_1^*$, contradicting the hypothesis $L \in \mathcal{H}_0^*$. This completes the proof that $g \in \Lambda\mathbb{R}[S_{\mathcal{H}}]$.

We now have to verify

$$f(L) = \Lambda\delta_r(L) * g(L)^{*2} \tag{7}$$

for $L \in \mathcal{H}$. First suppose $L \in \mathcal{K}_K$. If $L = K$, we took $r = s_K$ and $g(K) = a_K$, so (7) is satisfied because of (4). Suppose $L \neq K$. If $f(L)(0) = 0$ then, as we saw, $f(L) = 0$, and we took $g(L) = 0$, so (7) is satisfied. Suppose $f(L)(0) \neq 0$. Then we took $g(L) = \pm a_K$, so (7) is satisfed if $\Lambda\delta_r = \delta_0$, which is true since (as we have noted) $\Lambda\delta_r = \delta_{g_{K,L}(r)}$ and $g_{K,L}(r) = 0$ because of $g_{K,L} = g_{L',L} \circ g_{K,L'} = 0$. (We used the fact that by definition, L' is in the set $\mathcal{H}_L$ of those $H \in \mathcal{H}$ such that $H \leqslant L$ and $g_{H,L} = 0$.) Next, we must verify (7) for $L \in \mathcal{H} \setminus \mathcal{K}_K$. If $L \notin \mathcal{K}$ then $f(L)$ and $g(L)$ are both zero, so (7) is trivially satisfied. Thus we may assume $L \in \mathcal{K} \setminus \mathcal{K}_K$. Then $f(L) = 0$, so we have to show

$$\Lambda\delta_r(L) * g(L)^{*2} = 0. \tag{8}$$

This is trivial if the length of the path from K to L is at least 3 since in that case we took $g(L) = 0$. Suppose the length of that path is 1 or 2. The signature of the path begins with $\mathcal{D}$. If it consists entirely of $\mathcal{D}$'s then $L < K$, whence $\Lambda\delta_r(L) = 0$, so (8) is satisfied. Thus we may assume that the signature is $(\mathcal{D}, \mathcal{E})$, that is, $L' = K'$.

If $K \not\leqslant L$ then $\Lambda\delta_r(L) = 0$, so (8) is satisfied. Thus we may assume $K \leqslant L$. Since $L \notin \mathcal{K}_K$ then $K \neq L$, so $K < L$. Since L', which is the greatest element of $\mathcal{H}_L$, is less than K then $K \notin \mathcal{H}_L$, that is, $g_{K,L} \neq 0$. It follows that $L \in \mathcal{G}$. Now for such L (isomorphic to $\mathbb{N}$, which the present L is) we chose $H_L \in \mathcal{J}(S)$ such that $H_L \leqslant L$, $g_{H_L,L} = 0$, and $H_L \not\leqslant K$. Moreover, H_L was one of the generators of the semilattice $\mathcal{H}$, so $H_L \in \mathcal{H}$. It follows that $H_L \in \mathcal{H}_L$, so $H \leqslant L' = K' < K$, contradicting the fact that $H \not\leqslant K$. This completes the proof of the sufficiency of the conditions. If S is countable, the necessity follows from Theorem 4. This completes the proof. $\square$

6. Stieltjes semiperfect $\mathbb{R}_+$-separative finitely generated semigroups

In this section, we characterize Stieltjes semiperfect $\mathbb{R}_+$-separative finitely generated abelian semigroups by an application of Theorem 5. In the next section, we shall do the same without the hypothesis of $\mathbb{R}_+$-separativity.

A *minimal face* of an abelian semigroup S is a face of S which is minimal with respect to the inclusion ordering. A minimal face of S is the same as a minimal element of $\mathcal{J}(S)$.

Theorem 6. *Suppose S is an $\mathbb{R}_+$-separative finitely generated abelian semigroup. Then $\mathcal{J}(S)$ is finite, and for $H \in \mathcal{J}(S)$ the group G_H is a free abelian group of finite rank. It follows that S is c-finite. Hence S is Stieltjes semiperfect if and only if the following three conditions are satisfied:*

(i) *Each archimedean component of S is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$;*
(ii) *if K and L are archimedean components of S, isomorphic to $\mathbb{N}$, such that $K < L$ and $g_{K,L} \neq 0$ then there is some archimedean component H of S such that $H \leqslant L$, $g_{H,L} \neq 0$, and $H \not\leqslant K$;*
(iii) *every minimal face of S is a group.*

Proof. The semilattice $\mathcal{J}(S)$ is finite. If $H \in \mathcal{J}(S)$ then X_H is a face of S, hence finitely generated. It follows that G_{X_H} is finitely generated. Now G_H is isomorphic to G_{X_H}, hence finitely generated. Being also torsion-free (Proposition 1 (ii)), G_H is a free abelian group of finite rank.

To see that S is c-finite, since S is $\mathbb{R}_+$-separative by hypothesis it suffices to show that if U is a finite subset of S then the set $c(U)$ is finite. By Proposition 4 and Lemma 9 the set $c(U)$ is contained in the set

$$W = \bigcup_{K \in \mathcal{J}(S)} K \cap \operatorname{conv}\Bigl(\bigcup_{H \leqslant K} g_{H,K}(U \cap H)\Bigr).$$

Thus it suffices to show that W is finite. Since U is finite, U intersects only finitely many $H \in \mathcal{J}(S)$ and intersects each of these in a finite set. Thus for each $K \in \mathcal{J}(S)$

the set $W_K = \bigcup_{H \leqslant K} g_{H,K}(U \cap H)$ is finite. Since G_K is a free abelian group, it follows that $K \cap \operatorname{conv}(W_K)$ is finite. Finally, since $\mathcal{J}(S)$ is finite then W is finite. This shows that S is c-finite.

In order that S be Stieltjes semiperfect, by Theorem 1 it is necessary that $S = S + S$. Hence Theorem 5 applies, so the conditions (i) through (iii) of that Theorem are together necessary and sufficient for the Stieltjes semiperfectness of S. Conditions (i) and (ii) of Theorem 5 are the same as conditions (i) and (ii) of the present Theorem. To see that condition (iii) of Theorem 5 is equivalent to condition (iii) of the present Theorem when the other two conditions are satisfied, first suppose condition (iii) of Theorem 5 is satisfied. Suppose K is a minimal face of S. Then K is a minimal element of $\mathcal{J}(S)$. If K is not a group then by (i), K is isomorphic to $\mathbb{N}$. By condition (iii) of Theorem 5 it follows that there is some $H \in \mathcal{J}(S)$ such that $H \leqslant K$ and $g_{H,K} = 0$. Since K is a minimal element of $\mathcal{J}(S)$ it follows that $H = K$. But then $g_{H,K}$ is $g_{K,K}$, which is the identity on G_K, hence nonzero, a contradiction. This shows that condition (iii) of Theorem 5 implies condition (iii) of the present Theorem, provided that the other two conditions are satisfied. Conversely, suppose condition (iii) of the present Theorem is satisfied. Assume that $K \in \mathcal{J}(S)$ is isomorphic to $\mathbb{N}$. Since $\mathcal{J}(S)$ is finite, there is a minimal element H of $\mathcal{J}(S)$ such that $H \leqslant K$. By hypothesis, H is a group. Let e be the generator of K. Then $g_{H,K}(H) + K = H + K \subset K = \{ne \mid n \in \mathbb{N}\}$, so $g_{H,K}(H) \subset \{ne \mid n \in \mathbb{N}_0\}$. Since $g_{H,K}(H)$ is a group, it follows that $g_{H,K} = 0$. Thus condition (iii) of Theorem 5 is satisfied. This completes the proof. □

7. Stieltjes semiperfect finitely generated semigroups

In this section we characterize Stieltjes semiperfect finitely generated abelian semigroups by an application of Theorem 6.

The *greatest* $\mathbb{R}_+$*-separative homomorphic image* of an abelian semigroup S is the quotient semigroup $U_S = S/{\sim}$ where $\sim$ is the congruence relation in S defined by the condition that $s \sim t$ if and only if $\sigma(s) = \sigma(t)$ for all $\sigma \in S_+^*$. Denote by $h_S \colon S \to U_S$ the quotient mapping.

Proposition 9. *Suppose S is an abelian semigroup. Then S is Stieltjes semiperfect if and only if the following two conditions are satisfied:*

(i) *U_S is Stieltjes semiperfect;*

(ii) *every completely positive definite function on S factors via h_S.*

Proof. First suppose S is Stieltjes semiperfect. Since every homomorphic image of a Stieltjes semiperfect semigroup is Stieltjes semiperfect (cf. [12], Proposition 1 or

[8], Lemma 3.5), it follows that (i) holds. If $\varphi \in \mathcal{P}_c(S)$ then $\varphi \in \mathcal{H}_S(S)$, and by the definition of $\mathcal{H}_S(S)$ it is obvious that (ii) holds.

Conversely, suppose (i) and (ii) hold. Let $\varphi \in \mathcal{P}_c(S)$ be given. By (ii) there is a function $\Phi\colon U_S \to \mathbb{R}$ such that $\varphi = \Phi \circ h_S$. Since $h_S(S) = U_S$, one easily sees that $\Phi \in \mathcal{P}_c(U_S)$. Since U_S is Stieltjes semiperfect, it follows that $\Phi \in \mathcal{H}_S(U_S)$. Choose $\mu \in F_+((U_S)^*_+)$ such that $\Phi = \mathcal{L}\mu$. If $\mu^{h^*_S}$ is the image of μ under the mapping $h^*_S\colon (U_S)^*_+ \to S^*_+$ given by $h^*_S(\omega) = \omega \circ h_S$ for $\omega \in (U_S)^*_+$, it is easily seen that $\mu^{h^*_S} \in F_+(S^*_+)$ and $\mathcal{L}(\mu^{h^*_S}) = \varphi$, which shows $\varphi \in \mathcal{H}_S(S)$, as desired. □

We see that in order to characterize Stieltjes semiperfect finitely generated abelian semigroups, it suffices to answer the following question: *If S is a finitely generated abelian semigroup such that U_S is Stieltjes semiperfect, under what conditions does every completely positive definite function on S factor via h_S?*

Suppose U_S is Stieltjes semiperfect. Since U_S is finitely generated (being a homomorphic image of the finitely generated semigroup S) and $\mathbb{R}_+$-separative, by Theorem 6 it follows, in particular, that every minimal face of U_S is a group. It follows that for each $K \in \mathcal{J}(S)$ there is an idempotent ω (an element satisfying $\omega + \omega = \omega$) such that $\omega + K \subset K$. Define a semigroup G by

$$G = \bigcup_{H \in \mathcal{J}(S)} G_H \qquad \text{(disjoint union)}$$

and

$$x + y = g_{H,H\vee K}(x) + g_{K,H\vee K}(y) \qquad \text{(sum in the group } G_{H\vee K}\text{)}$$

for $H, K \in \mathcal{J}(S)$, $x \in G_H$, and $y \in G_K$. Define $g\colon S \to G$ by $g\big|H = g_H$ for $H \in \mathcal{J}(S)$. Every completely positive definite function on S factors via h_S if and only if every completely positive definite function on S factors via g. For $K \in \mathcal{J}(S)$, for each $\varphi \in \mathcal{P}_c(S)$ the function $\varphi\big|K$ factors via g if and only if for each $a \in K$ there is an idempotent ω such that $\omega + K \subset K$ and such that $\varphi(a) = \varphi(\omega + a)$ for each $\varphi \in \mathcal{P}_c(S)$. For $K \in \mathcal{J}(S)$ we denote by Ω_K the set of those $x \in K$ such that there is an idempotent ω such that $\omega + K \subset K$ and such that $\varphi(x) = \varphi(\omega + x)$ for each $\varphi \in \mathcal{P}_c(S)$. Define $\Omega = \bigcup_{K \in \mathcal{J}(S)} \Omega_K$.

Proposition 10. *Ω is an ideal of S, that is, $\Omega + S \subset S$.*

P r o o f. Suppose $a \in \Omega$ and $b \in S$; we have to show $a + b \in \Omega$. Let A and B be the archimedean components of S containing a and b, respectively; then $a + b$ belongs to the archimedean component $K = A \vee B$. Choose an idempotent ω such that $\omega + A \subset A$ and such that $\varphi(a) = \varphi(\omega + a)$ for each $\varphi \in \mathcal{P}_c(S)$. If H_ω is the archimedean component of S containing ω then $\omega + A \subset A \cap (H_\omega \vee A)$, so

$H_\omega \vee A = A$ (since distinct archimedean components are disjoint), that is, $H_\omega \leqslant A$. Since $A \vee B = K$, we have $A \leqslant K$, so $H_\omega \leqslant K$, hence $\omega + K \subset H_\omega \vee K = K$. Moreover, for $\varphi \in \mathcal{P}_c(S)$ we have by positive definiteness of $E_a\varphi$

$$\begin{aligned}|\varphi(a+b) - \varphi(\omega+a+b)|^2 &\leqslant \varphi(a+2b)[\varphi(a) + \varphi(2\omega + A) - 2\varphi(\omega+a)] \\ &= \varphi(a+2b)[\varphi(a) - \varphi(\omega+a)] = 0,\end{aligned}$$

hence $\varphi(a+b) = \varphi(\omega+a+b)$, as desired. □

Define $\frac{1}{2}\Omega = \{x \in S \mid 2x \in \Omega\}$.

Proposition 11. $\frac{1}{2}\Omega + S \subset \Omega$.

Proof. Suppose $a \in \frac{1}{2}\Omega$ and $b \in S$; we have to show $a + b \in \Omega$. Choose an idempotent ω such that $\omega + A \subset A$ where A is the archimedean component of S containing a (or equivalently, $2a$). As in the proof of Proposition 10, if K is the archimedean component of S containing $a+b$ then $\omega + K \subset K$. Moreover, for $\varphi \in \mathcal{P}_c(S)$ we have by positive definiteness of $E_b\varphi$

$$\begin{aligned}|\varphi(a+b) - \varphi(\omega+a+b)|^2 &\leqslant \varphi(b)[\varphi(2a+b) + \varphi(2\omega+2a+b) - 2\varphi(\omega+2a+b)] \\ &= \varphi(b)[\varphi(2a+b) - \varphi(\omega+2a+b)] = 0\end{aligned}$$

since $2a + b \in \Omega + S \subset \Omega$ (Proposition 10). Thus $\varphi(a+b) = \varphi(\omega+a+b)$, as desired. □

We have $S + S + S \subset \Omega$. By Theorem 1, in order that S be Stieltjes semiperfect it is necessary that $S = S + S$, which implies $S = S + S + S$, hence $S \subset \Omega$, that is, $S = \Omega$. Thus, under this hypothesis, every completely positive definite function on S factors via g, hence via h_S.

Theorem 7. *Suppose S is a finitely generated abelian semigroup. Then S is Stieltjes semiperfect if and only if the following three conditions are satisfied:*

(i) *Each archimedean component of U_S is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{N}$;*

(ii) *if K and L are archimedean components of U_S, isomorphic to $\mathbb{N}$, such that $K < L$ and $g_{K,L} \neq 0$ then there is an archimedean component H of U_S such that $H \leqslant L$, $g_{H,L} = 0$, and $H \nleqslant K$;*

(iii) *$S = S + S$.*

Proof. By Proposition 9 it is necessary that U_S be Stieltjes semiperfect. Since U_S is finitely generated and $\mathbb{R}_+$-separative, the necessity of (i) and (ii) follows by Theorem 6. Condition (iii) is necessary by Theorem 1.

Conversely, suppose the conditions are satisfied. From (iii) it follows that $U_S = U_S + U_S$, and as in the proof of Theorem 6 it follows that every minimal face of U_S

is a group. Thus the conditions of Theorem 6 are satisfied (for U_S instead of S), so U_S is Stieltjes semiperfect. By Proposition 9 it only remains to be shown that every completely positive definite function on S factors via h_S. But we have seen that this follows from $S = S + S$. This completes the proof. $\square$

Example 1. There is a finitely generated abelian semigroup which is Stieltjes semiperfect but not semiperfect. To see this, let $A = \{0, a\}$ be the 2-element group, let $A \times \mathbb{N}_0$ be the product semigroup, and let S be the subsemigroup $(A \times \mathbb{N}_0) \setminus \{(a, 0), (a, 1)\}$. Then S has a zero, so $S = S + S$ and therefore S is Stieltjes semiperfect if and only if the greatest $\mathbb{R}_+$-separative homomorphic image of S is Stieltjes semiperfect. That image is $\mathbb{N}_0$, the quotient mapping being the composite of the inclusion mapping of S into $A \times \mathbb{N}_0$ and the projection of $A \times \mathbb{N}_0$ onto the second factor. Since $\mathbb{N}_0$ is Stieltjes semiperfect, so is S. However, S is not semiperfect. Indeed, S has the archimedean component $(A \times \mathbb{N}) \setminus \{a, 1\}$, which is not isomorphic to the product of a finite group of exponent 1 or 2 and one of the semigroups $\{0\}$, $\mathbb{Z}$, $\mathbb{N}$. Since S is $\mathbb{R}$-separative, by the main theorem in [9] it follows that S is not semiperfect.

Example 2. There is a finitely generated abelian semigroup which is semiperfect but not Stieltjes semiperfect. To see this, let $E = \{0, e\}$ be the 2-element semigroup with zero 0 and $e + e = e$, and let S be the subsemigroup $(E \times \mathbb{N}_0) \setminus \{(e, 0)\}$ of the product semigroup $E \times \mathbb{N}_0$. Then S has the archimedean components O, K, and L where $O = \{(0, 0)\}$, $K = \{0\} \times \mathbb{N}$, and $L = \{e\} \times \mathbb{N}$. Since each of these is isomorphic to $\{0\}$ or $\mathbb{N}$, by the main theorem in [9] it follows that S is semiperfect. However, the archimedean components K and L are isomorphic to $\mathbb{N}$ and satisfy $K < L$ and $g_{K,L} \neq 0$, and there is no archimedean component H of S such that $H \leqslant L$, $g_{H,L} = 0$, and $H \nleqslant K$. Hence S is not Stieltjes semiperfect.

8. Schur-increasing functions

Suppose E is a real vector space. If $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are elements of E^n, one says that p is *majorized* by q, written $p \prec q$, if there is a doubly stochastic $n \times n$ matrix Ω such that $p = q\Omega$. Information on the majorization ordering can be found in [18].

Now suppose S is an abelian semigroup. Define

$$\Pi(S) = \{a \in \mathbb{R}[S] \mid \langle a, 1\rangle = 1,\ \ a(s) \geqslant 0\ \ \forall s \in S\}$$

where 1 is the constant character. A function $\psi\colon\ S \to \mathbb{R}$ is *Schur-increasing of order* $n \in \mathbb{N}$ if the conditions $p, q \in \Pi(S)$ and $p \prec q$ imply

$$\langle p_1 * \ldots * p_n, \psi\rangle \leqslant \langle q_1 * \ldots * q_n, \psi\rangle.$$

The set of functions on S that are Schur-increasing of order n is denoted by $\mathcal{S}_n(S)$. A function is *Schur-increasing* if it is Schur-increasing of every order $n \in \mathbb{N}$. The set of all Schur-increasing functions on S is denoted by $\mathcal{S}(S)$. A function $\varphi\colon S \to \mathbb{R}$ is *Schur-decreasing* (of order n) if $-\varphi$ is Schur-increasing (of order n).

Proposition 12. *For every abelian semigroup S, $\mathcal{H}_S(S) \subset -\mathcal{S}(S)$.*

Proof. See [4], proof of 7.3.7. □

Again, suppose S is an abelian semigroup. A function $\psi\colon S+S \to \mathbb{R}$ is *negative definite* if

$$\sum_{j,k=1}^{n} c_j c_k \varphi(s_j + s_k) \leqslant 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{R}$ such that $\sum\limits_{j=1}^{n} c_j = 0$. Denote by $\mathcal{N}(S)$ the set of all negative definite functions on S. For an arbitrary function $\psi\colon S \to \mathbb{R}$ we have $\psi \in \mathcal{N}(S)$ if and only if $\mathrm{e}^{-t\psi} \in \mathcal{P}(S)$ for all $t > 0$, by a theorem that goes back to Schoenberg, cf. [4], 3.2.2. A function $\psi\colon S \to \mathbb{R}$ is *completely negative definite* if $E_r\psi \in \mathcal{N}(\widetilde{S})$ for all $r \in S$. Denote by $\mathcal{N}_c(S)$ the set of all completely negative definite functions on S. Then $\mathcal{S}_2(S) = \mathcal{N}(S)$ and $\mathcal{S}(S) \subset \mathcal{N}_c(S)$ ([4], p. 243 and 7.1.7). For an abelian semigroup S, the following two conditions are equivalent:

(i) Every completely positive definite function on S is Schur-decreasing;

(ii) every completely negative definite function on S is Schur-increasing

(see [4], 7.3.9).

Proposition 13. *If S is a Stieltjes semiperfect semigroup then $\mathcal{S}(S) = \mathcal{N}_c(S)$.*

Proof. If $\varphi \in \mathcal{P}_c(S)$ then $\varphi \in \mathcal{H}_S(S)$, so $\varphi \in -\mathcal{S}(S)$ by Proposition 12. Thus $\mathcal{P}_c(S) \subset -\mathcal{S}(S)$. By the above equivalent conditions, $\mathcal{N}_c(S) \subset \mathcal{S}(S)$. The converse inclusion being automatical, we have $\mathcal{N}_c(S) = \mathcal{S}(S)$. □

Berg ([2], p. 274) states: "For $S = \mathbb{N}_0$ or $S = \mathbb{Z}$ with the identity involution we have $\mathcal{S}(S) = \mathcal{CN}(S)$, cf. Theorem 7.3.9 in B-C-R, which can be extended from Radon perfect semigroups to semigroups, and probably to all semiperfect semigroups." ("$\mathcal{CN}(S)$" denotes $\mathcal{N}_c(S)$.)

So consider the following question: *In Proposition* 13*, can "Stieltjes semiperfect" be replaced with "semiperfect"?* Suppose S is a semiperfect semigroup with zero and $\varphi \in \mathcal{P}_c(S)$. For $r \in S$ we have $E_r\varphi \in \mathcal{P}(S) = \mathcal{H}(S)$. For an arbitrary abelian semigroup S, let $\mathcal{H}_c(S)$ denote the set of those functions $\varphi\colon S \to \mathbb{R}$ such that $E_r\varphi \in \mathcal{H}(S)$ for all $r \in S$. We have just seen that if S is a semiperfect semigroup with zero then $\mathcal{P}_c(S) \subset \mathcal{H}_c(S)$. The converse inclusion being automatical, we have

$\mathcal{P}_c(S) = \mathcal{H}_c(S)$. So the question is: *Is it true that* $\mathcal{H}_c(S) \subset -\mathcal{S}(S)$*?* We shall see that if this question is to be answered in the affirmative, the semiperfectness of S must be employed in some more subtle way. Indeed, the inclusion $\mathcal{H}_c(S) \subset -\mathcal{S}(S)$ is false for $S = \mathbb{N}_0^2$.

Theorem 8. *There is a function* $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ *such that* $E_r\varphi \in \mathcal{H}(\mathbb{N}_0^2)$ *for all* $r \in \mathbb{N}_0^2$*, yet* φ *is not Schur-decreasing of order* 3.

Proof. Identify $\mathbb{R}[\mathbb{N}_0^2]$ with the algebra $\mathbb{R}[x, y]$ of polynomials in two variables by identifying $\delta_{(m,n)}$ with the monomial $x^m y^n$ for $(m, n) \in \mathbb{N}_0^2$. Denote by $\mathbb{R}[x, y]_+$ the convex cone of nonnegative polynomials. Define a convex cone D in $\mathbb{R}[x, y]$ by

$$D = \{a + xb + yc + xyd \mid a, b, c, d \in \mathbb{R}[x, y]_+\}.$$

(In [4], 6.3.12, D is denoted by $D^{(2)}$.) Then

$$\mathcal{H}_c(\mathbb{N}_0^2) = D^\perp,$$

cf. the proof of [4], 6.3.12. As in [4], 7.3.13, denote by B the set of polynomials $p \in \mathbb{R}[x, y]$ with nonnegative coefficients and $p(1, 1) = 1$, and let $\widetilde{B}$ be the set of all polynomials of the form $q_1q_2q_3 - p_1p_2p_3$ where $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ are triples of polynomials in B such that $q \prec p$ (that is, $q = p\Omega$ for some doubly stochastic matrix Ω). Then

$$-\mathcal{S}(\mathbb{N}_0^2) = \widetilde{B}^\perp,$$

cf. the proof of [4], 7.3.13. Suppose it were true that $\mathcal{H}_c(\mathbb{N}_0^2) \subset -\mathcal{S}_3(\mathbb{N}_0^2)$. By the Bipolar Theorem (or the Hahn-Banach Theorem) it would follow that $\widetilde{B}$ were contained in the closure of D with respect to the finest locally convex topology on $\mathbb{R}[x, y]$. But D is already closed ([4], 6.3.12), so $\widetilde{B}$ would be contained in D. In particular, taking $p = (1, x, y)$,

$$\Omega = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix},$$

and $q = p\Omega$, we would have

$$r := q_1q_2q_3 - p_1p_2p_3 = a + xb + yc + xyd \tag{9}$$

for some $a, b, c, d \in \mathbb{R}[x, y]_+$. It is easy to see that if $f = \sum\limits_{i,j\in\mathbb{N}_0} f_{i,j}x^iy^j \in D$ and if (m, n) is a vertex of the convex polytope $\operatorname{conv}(\{(i, j) \mid f_{i,j} \neq 0\})$ in $\mathbb{R}^2$ then $f_{m,n} > 0$. It follows that if $f^{(k)} = \sum f^{(k)}_{i,j} x^iy^j \in D$ for $k = 1, \ldots, n$ then $\operatorname{conv}(\{(i, j) \mid f^{(k)}_{i,j} \neq$

$0\}) \subset \operatorname{conv}\Bigl(\Bigl\{(i,j) \Bigm| \sum_{m=1}^{n} f_{i,j}^{(m)} \neq 0\Bigr\}\Bigr)$ for $k = 1, \dots, n$. Since the polynomial r is of degree 3, from (9) it would therefore follow that a, b, c, and d were of degree at most 2 (since their degrees must be even, these polynomials being nonnegative). As shown by Hilbert [15] it would follow that these four polynomials were sums of squares of polynomials. But that is impossible, as shown in the proof of [4], 7.3.14. □

Note that Theorem 8 is simultaneously stronger than [4], 6.3.12, and [4], 7.3.13.

9. Semigroups of Stieltjes moment functions

Suppose S is an abelian semigroup with zero. Denote by $m\colon S_+^* \times S_+^* \to S_+^*$ pointwise multiplication, i.e., $m(\sigma, \tau) = \sigma \cdot \tau$ where $\sigma \cdot \tau \in S_+^*$ is defined by $\sigma \cdot \tau(s) = \sigma(s)\tau(s)$ for $s \in S$. Then m is measurable with respect to the σ-rings $\mathcal{A}(S_+^*) \otimes \mathcal{A}(S_+^*)$ and $\mathcal{A}(S_+^*)$, so if μ and ν are measures defined on $\mathcal{A}(S_+^*)$, we may define their *convolution* $\mu * \nu$ by $\mu * \nu = (\mu \otimes \nu)^m$, the image measure of $\mu \otimes \nu$ under the mapping m. If $\mu, \nu \in F_+(S_+^*)$ then $\mu * \nu \in F_+(S_+^*)$ and $\mathcal{L}(\mu * \nu) = \mathcal{L}\mu \cdot \mathcal{L}\nu$. We see from this that $\mathcal{H}_S(S)$ is stable under pointwise multiplication. It is natural to ask for a characterization of semigroups of Stieltjes moment functions, that is, families $(\varphi_t)_{t>0}$ such that $\varphi_t \in \mathcal{H}_S(S)$ for all t and $\varphi_{s+t} = \varphi_s \cdot \varphi_t$ for all $s, t > 0$. Restricting the problem a little bit, we ask: *What functions $\psi\colon S \to \mathbb{R}$ are such that $\mathrm{e}^{-t\psi} \in \mathcal{H}_S(S)$ for all $t > 0$?*

A *convolution semigroup* in $F_+(S_+^*)$ is a family $(\mu_t)_{t>0}$ such that $\mu_t \in F_+(S_+^*)$ for all t and $\mu_{s+t} = \mu_s * \mu_t$ for all $s, t > 0$. We ask: *Which convolution semigroups in $F_+(S_+^*)$ are continuous in the $\mathcal{L}$-topology?* From [7], Proposition 3.4, it follows that a convolution semigroup (μ_t) in $F_+(S_+^*)$ is continuous in the $\mathcal{L}$-topology if and only if there is some $\psi \in \mathcal{N}_c(S)$ such that $\mathcal{L}\mu_t = \mathrm{e}^{-t\psi}$ for all $t > 0$. So the question is: *For what functions $\psi\colon S \to \mathbb{R}$ does there exist a convolution semigroup (μ_t) in $F_+(S_+^*)$ such that $\mathcal{L}\mu_t = \mathrm{e}^{-t\psi}$ for all t?*

Define an ideal $\mathbb{R}[S]_0$ of $\mathbb{R}[S]$ by

$$\mathbb{R}[S]_0 = \{a \in \mathbb{R}[S] \mid \langle a, 1\rangle = 0\}$$

where 1 is the constant character. The square $\mathbb{R}[S]_0^2$ of this ideal is, by definition, the real linear span of the set of all elements of the form $a * b$ with $a, b \in \mathbb{R}[S]_0$. Define

$$(\mathbb{R}[S]_0^2)_{++} = \mathbb{R}[S]_0^2 \cap \mathbb{R}[S]_{++}.$$

An *additive function* on S is a homomorphism of S into the group $(\mathbb{R}, +)$. A *quadratic form* on S is a function $q\colon S \to \mathbb{R}$ satisfying $q(2s) = 4q(s)$ for all $s \in S$ and $\langle a * b * c, q\rangle = 0$ for all $a, b, c \in \mathbb{R}[S]_0$.

Proposition 14. *Real constants, additive functions, and negative definite quadratic forms are completely negative definite.*

Proof. For constants and additive functions, this is easy to see. Suppose q is a negative definite quadratic form on S. By [7], Proposition 4.1, there exist an inner product space $(X, \langle\cdot,\cdot\rangle)$ and an additive mapping $\pi\colon S \to X$ such that $q(s) = -\langle\pi(s), \pi(s)\rangle$ for $s \in S$. Now if $r \in S$, $s_1, \dots, s_n \in S$, and $c_1, \dots, c_n \in \mathbb{R}$ are such that $\sum\limits_{j=1}^{n} c_j = 0$ then an easy computation shows

$$\sum_{j,k=1}^{n} c_j c_k E_r q(s_j + s_k) = -2\Big\langle \sum_{j=1}^{n} c_j \pi(s_j), \sum_{j=1}^{n} c_j \pi(s_j) \Big\rangle \leqslant 0.$$

Thus q is completely negative definite. □

Define $\mathcal{A}(S^* \setminus \{1\}) = \{A \in \mathcal{A}(S^*) \mid 1 \notin A\}$ and $\mathcal{A}(S^*_+ \setminus \{1\}) = \{A \in \mathcal{A}(S^*_+) \mid 1 \notin A\}$. A *complex Lévy function* for S is a function $H\colon S \times S^* \to \mathbb{R}$ satisfying the following three conditions:

(i) $H(\cdot, \sigma)$ is additive for each $\sigma \in S^*$;

(ii) $H(s, \cdot)$ is $\mathcal{A}(S^*)$-measurable for each $s \in S$;

(iii) if μ is a measure on $\mathcal{A}(S^* \setminus \{1\})$ such that $\int (1 - \sigma(s))^2 \,\mathrm{d}\mu(\sigma) < \infty$ for all $s \in S$ then $\int |1 - \sigma(s) + H(s, \sigma)| \,\mathrm{d}\mu(\sigma) < \infty$ for all $s \in S$.

For every abelian semigroup there is a complex Lévy function ([7], Proposition 5.1).

Theorem 9. *Suppose S is an abelian semigroup with a complex Lévy function H. For a function $\psi\colon S \to \mathbb{R}$, the following four conditions are equivalent:*

(i) *There is a convolution semigroup $(\mu_t)_{t>0}$ in $F_+(S^*_+)$ such that $\mathcal{L}\mu_t = \mathrm{e}^{-t\psi}$ for all $t > 0$;*

(ii) $\mathrm{e}^{-t\psi} \in \mathcal{H}_S(S)$ *for all $t > 0$;*

(iii) $-\psi \in (\mathbb{R}[S]_0^2)^{\perp}_{++}$;

(iv) *there exist $a \in \mathbb{R}$, an additive function h on S, a negative definite quadratic form q on S, and a measure μ on $\mathcal{A}(S^*_+ \setminus \{1\})$, integrating $\sigma \mapsto (1 - \sigma(s))^2$ for all $s \in S$, such that*

$$\psi(s) = a + h(s) + q(s) + \int_{S^*_+ \setminus \{1\}} \big(1 - \sigma(s) + H(s, \sigma)\big) \,\mathrm{d}\mu(\sigma)$$

for all $s \in S$.

The convolution semigroups occurring in (i) *are all continuous in the $\mathcal{L}$-topology. There is a natural one-to-one correspondence between the convolution semigroups occurring in* (i) *and the measures μ occurring in* (iv), *each set being in a one-to-one* *correspondence with the set of those $\Psi \in \mathcal{N}_c(U)$ such that $\psi = \Psi \circ f$, where U and f are as in* [7], *Proposition* 7.1.

P r o o f. As the proof of [7], Theorem 7.1. □

We denote by $\mathcal{N}_S(S)$ the set of all functions $\psi\colon S \to \mathbb{R}$ satisfying the equivalent conditions of Theorem 9.

Proposition 15. *For every abelian semigroup S, $\mathcal{N}_S(S) \subset \mathcal{S}(S)$.*

P r o o f. As (i) $\Rightarrow$ (ii) in [4], 7.3.9, using condition (ii) of Theorem 9 and Proposition 12. □

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