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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 1, 11–16

Persistent URL: http://dml.cz/dmlcz/127697

Czechoslovak Mathematical Journal, 52 (127) (2002), 11–16

DOMINATION IN GENERALIZED PETERSEN GRAPHS

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(Received July 9, 1998)

Abstract. Generalized Petersen graphs are certain graphs consisting of one quadratic factor. For these graphs some numerical invariants concerning the domination are studied, namely the domatic number $d(G)$, the total domatic number $d_t(G)$ and the k-ply domatic number $d^k(G)$ for $k = 2$ and $k = 3$. Some exact values and some inequalities are stated.

Keywords: domatic number, total domatic number, k-ply domatic number, generalized Petersen graph

MSC 2000: 05C69, 05C38

In this paper we will study three numerical invariants of graphs which concern the domination, namely the domatic number $d(G)$, total domatic number $d_t(G)$ and k-ply domatic number $d^k(G)$ of a graph G. We will investigate them for generalized Petersen graphs. The vertex set of a graph G will be denoted by $V(G)$. For a vertex $v \in V(G)$ the symbol $N_G[v]$ denotes the closed neighbourhood of v in G, i.e. the set consisting of v and of all vertices adjacent to v in G.

A subset D of $V(G)$ is called dominating (or total dominating) in G, if for each $x \in V(G) \setminus D$ (or for each $x \in V(G)$ respectively) there exists a vertex $y \in D$ adjacent to x. The set D is called k-ply dominating for a positive integer k, if for each $x \in V(G) \setminus D$ there exist k distinct vertices $y_1, \ldots, y_k$ of D which are all adjacent to x.

A domatic (or total domatic, or k-ply domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or total dominating, or k-ply dominating respectively) sets in G. The maximum number of classes of a domatic (or total domatic, of k-ply domatic) partition of G is the domatic (or total domatic, or k-ply domatic respectively) number of G. The domatic number of G is denoted by $d(G)$, the total domatic number by $d_t(G)$, the k-ply domatic number by $d^k(G)$.

In this paper we will consider $d^k(G)$ for $k = 2$ and $k = 3$ and we will speak about the doubly domatic number and the triply domatic number.

The domatic number was introduced by E. J. Cockayne and S. T. Hedetniemi in [2], the total domatic number by the same authors and R. M. Dawes in [3], the k-ply domatic number by the author of this paper in [6].

Sometimes it is convenient to speak about the domatic colouring. The domatic number of G can be alternatively defined as the maximum number of colours by which the vertices of G can be coloured in such a way that each vertex is adjacent to vertices of all colours different from its own. Evidently this definition is equivalent to that written above. Similarly by means of colourings, also $d_t(G)$ and $d^k(G)$ may be defined.

As was mentioned, the number $d^k(G)$ will be used only for the concrete values $k = 2$ and $k = 3$. Thus in the sequel the symbol k will be used in another sense.

In the whole paper the symbols n, k will denote relatively prime positive integers such that $k < n$, $n \geqslant 3$. The generalized Petersen graph $\mathrm{GP}(n,k)$ is defined as follows. Let C_n, C'_n be two disjoint circuits of length n. Let the vertices of C_n be $u_1, \ldots, u_n$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, n-1$ and $u_n u_i$. Let the vertices of C'_n be $v_1, \ldots, v_n$ and edges $v_i v_{i+k}$ for $i = 1, \ldots, n$, the sum $i + k$ being taken modulo n. The graph $\mathrm{GP}(n,k)$ is obtained from the union of C_n and C'_n by adding the edges $u_i v_i$ for $i = 1, \ldots, n$.

The graph $\mathrm{GP}(5,2)$ is the well-known Petersen graph. The generalized Petersen graphs were studied e.g. in [1], [4], [5].

For integers n, k fulfilling the above stated conditions we define the numbers $f(n,k)$, $g(n,k)$. They are positive integers such that $f(n,k) \leqslant n-1$, $g(n,k) \leqslant n-1$, $kf(n,k) \equiv 1 \pmod{n}$, $kg(n,k) \equiv -1 \pmod{n}$. It is easy to see that

$$f(n,k) + g(n,k) = n,$$
$$\mathrm{GP}(n,k) \cong \mathrm{GP}(n,n-k) \cong \mathrm{GP}(n,f(n,k)) \cong \mathrm{GP}(n,g(n,k)).$$

Theorem 1. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d(\mathrm{GP}(n,k)) = 4$$

if and only if $n \equiv 0 \pmod{4}$.

P r o o f. According to [2], $d(G) \leqslant \delta(G)+1$, where $\delta(G)$ is the minimum degree of a vertex in G. Every graph $\mathrm{GP}(n,k)$ is regular of degree 3, therefore $d(\mathrm{GP}(n,k)) \leqslant 4$. Suppose that $n \equiv 0 \pmod{4}$. We construct a domatic colouring c such that c: $V(\mathrm{GP}(n,k)) \to \{1,2,3,4\}$. For $i = 1, \ldots, n$ we define c by $c(u_i) \equiv i \pmod{4}$,

$c(v_i) \equiv i+2 \pmod 4$) The reader may verify himself that c is a domatic colouring of $\mathrm{GP}(n,k)$ by four colours and therefore $d(\mathrm{GP}(n,k)) = 4$.

On the other hand, suppose that $d(\mathrm{GP}(n,k)) = 4$. Let $\mathscr{D} = \{D_1, D_2, D_3, D_4\}$ be a domatic partition of $\mathrm{GP}(n,k)$. Evidently for any $i \in \{1,2,3,4\}$ no two vertices of D_i are adjacent and each vertex not belonging to D_i is adjacent to exactly one vertex of D_i. We will say that x dominates y, if either $y = x$, or y is adjacent to x. Let $a = |D_1 \cap V(C_n)|$, $b = |D_1 \cap V(C'_n)|$. Each vertex of $D_1 \cap V(C_n)$ dominates three vertices of C_n and one vertex of C'_n, while each vertex of $D_1 \cap V(C'_n)$ dominates three vertices of C'_n and one vertex of C_n. Therefore $3a + b = n$, $a + 3b = n$. These two equations imply $a = b = n/4$ and therefore $n \equiv 0 \pmod 4$. □

Remark. Let $n \equiv 0 \pmod 3$, let $\mathrm{GP}(n,k)$ be a generalized Petersen graph. Since it is easy to construct a domatic colouring of $\mathrm{GP}(n,k)$ by three colours, we have $d(\mathrm{GP}(n,k)) \geqslant 3$.

Theorem 2. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. If* $n \not\equiv 0 \pmod 3$ *and either* $k \equiv f(n,k) \equiv 0 \pmod 3$, *or* $k \equiv f(n,k) \equiv n \pmod 3$, *then the inequality* $d(\mathrm{GP}(n,k)) \geqslant 3$ *holds.*

P r o o f. First let $n \equiv 1 \pmod 3$, $k \equiv 1 \pmod 3$, $f(n,k) \equiv 1 \pmod 3$. Consider the Hamiltonian path P in $\mathrm{GP}(n,k)$ having subsequent vertices $u_1, u_2, \ldots, u_n, v_n, v_{n+k}, \ldots, v_{n-k}$, where the subscripts are taken modulo n. We colour its vertices subsequently by $1, 2, 3, 1, 2, 3, \ldots$ The last vertex $v_{(n-1)k} = v_{n-k}$ is coloured by 2 and is adjacent to $v_{(n-2)k}$ coloured by 1 and to u_{n-1} coloured by 3. The first vertex u_1 is coloured by 1 and is adjacent to u_n coloured by 2 and to v_1 coloured by 3. For any other vertex it is evident that it is adjacent to vertices of all colours different from its own. Therefore the described colouring is a domatic colouring of $\mathrm{GP}(n,k)$ by three colours.

Now let $n \equiv 2 \pmod 3$, $k \equiv 0 \pmod 3$, $f(n,k) \equiv 0 \pmod 3$. We construct the domatic colouring of $\mathrm{GP}(n,k)$ in the same way. The last vertex v_{n-k} is coloured by 1 and is adjacent to v_n coloured by 3 and to u_{n-k} coloured by 2. The first vertex u_1 is coloured by 1 and is adjacent to u_n coloured by 2 and to v_1 coloured by 3. Again the described colouring is domatic.

If $n \equiv 1 \pmod 3$, $k \equiv 0 \pmod 3$, $f(n,k) \equiv 0 \pmod 3$, then $n - k \equiv 1 \pmod 3$, $f(n, n-k) = g(n,k) = n - f(n,k) \equiv 1 \pmod 3$ and $\mathrm{GP}(n, n-k) \cong \mathrm{GP}(n,k)$; therefore the assertion also holds. Similarly if $n \equiv 2 \pmod 3$, $k \equiv 2 \pmod 3$, $f(n,k) \equiv 2 \pmod 3$, then $n - k \equiv 0 \pmod 3$, $f(n, n-k) \equiv 0 \pmod 3$ and the assertion holds. □

The following theorem concerns the graphs $\mathrm{GP}(n,1)$, i.e., graphs of n-side prisms.

Theorem 3. *For any integer* $n \geqslant 3$ *the inequality* $d(\mathrm{GP}(n,1)) \geqslant 3$ *holds.*

P r o o f. If $n \equiv 0 \pmod 3$, the assertion follows from Remark. If $n \equiv 1 \pmod 3$, then it follows from Theorem 2, because $f(n,1) = 1$. If $n \equiv 2 \pmod 3$, we define the colouring of vertices of $\mathrm{GP}(n,1)$ as follows. If $t \leqslant n-2$, then $c(u_t) \equiv t \pmod 3$, $c(v_t) \equiv 1 - t \pmod 3$. Then we put $c(u_{n-1}) = 2$, $c(u_n) = 1$, $c(v_{n-1}) = 2$, $c(v_n) = 2$. The colouring by 3 colours obtained is this way is domatic and $d(\mathrm{GP}(n,1)) \geqslant 3$. □

Example. The domatic number of the original Petersen graph $\mathrm{GP}(5,2)$ is 2.

P r o o f. The domatic number of a graph without isolated vertices is always at least 2. Suppose that there exists a domatic partition $\mathscr{D} = \{D_1, D_2, D_3\}$ of $\mathrm{GP}(5,2)$ with three classes. As the graph has ten vertices and no dominating set with less than three vertices, at least two classes of $\mathscr{D}$ must consist of three vertices. Without loss of generality let $|D_1| = 3$. It is easy to verify that then there exists a vertex v such that D_1 is its open neighbourhood. Without loss of generality suppose $v \in D_2$. Then $v \notin D_3$ and v is adjacent to no vertex of D_3, therefore D_3 is not dominating in $\mathrm{GP}(5,2)$, which is a contradiction. Therefore $d(\mathrm{GP}(5,2)) = 2$. □

Now we shall study total domatic numbers. According to [3] we have $d_t(G) \leqslant \delta(G)$. As $\mathrm{GP}(n,k)$ is regular of degree 3, we have always $d_t(\mathrm{GP}(n,k)) \leqslant 3$.

Theorem 4. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d_t(\mathrm{GP}(n,k)) = 3$$

if and only if $n \equiv 0 \pmod 3$.

P r o o f. Suppose that $d(\mathrm{GP}(n,k)) = 3$ and let $\{D_1, D_2, D_3\}$ be the corresponding total domatic partition. Evidently no vertex is adjacent to exactly one vertex of any class of this partition. Let u, v be two adjacent vertices from D_1. Then $M(u,v) = N_G[u] \cup N_G[v]$ has six elements. The sets $M(u,v)$ for different pairs $\{u,v\}$ of adjacent vertices from D_1 must be disjoint and therefore they form a partition of $V(\mathrm{GP}(n,k))$. This implies that the number $2n$ of vertices of $\mathrm{GP}(n,k)$ is divisible by 6 and therefore $n \equiv 0 \pmod 3$.

Now suppose that $n \equiv 0 \pmod 3$. For each vertex x of $\mathrm{GP}(n,k)$ we determine its colour $c(x) \in \{1,2,3\}$ in such a way that $c(u_i) = c(v_i) \equiv i \pmod 3$ for $i = 1, \ldots, n$. As k is relatively prime with n, it is also non-divisible by 3 and the colouring thus defined is total domatic. This implies $d(\mathrm{GP}(n,k)) = 3$. □

Theorem 5. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then the inequality* $d_t(\mathrm{GP}(n,k)) \geqslant 2$ *holds.*

P r o o f. The partition $\{V(C_n), V(C'_n)\}$ is evidently a total domatic partition of $\mathrm{GP}(n,k)$. □

At the end we turn to k-ply domatic numbers for $k=2$ and $k=3$. In [6] the inequality $d^k(G) \leqslant \lfloor \delta(G)/k \rfloor + 1$ is found, where again $\delta(G)$ is the minimum degree of a vertex in G. This implies $d^2(\mathrm{GP}(n,k)) \leqslant 2$, $d^3(\mathrm{GP}(n,k)) \leqslant 2$, $d^m(\mathrm{GP}(n,k)) = 1$ for $m \geqslant 4$. We prove two theorems.

Theorem 6. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d^3(\mathrm{GP}(n,k)) = 2$$

if and only if n *is even.*

Remark. As n,k must be relatively prime, in this case k is odd.

P r o o f. If and only if n is even, the graph $\mathrm{GP}(n,k)$ contains no circuit of odd length and thus it is a bipartite graph. Its bipartition classes are classes of a triply domatic partition and the assertion holds. On the other hand, if $\{D_1, D_2\}$ is a triply domatic partition of $\mathrm{GP}(n,k)$, then each edge joins a vertex of D_1 with a vertex of D_2, the graph is bipartite and n is even, because otherwise the graph $\mathrm{GP}(n,k)$ would contain circuits C_n, C'_n of odd lengths. Thus the assertion is proved. □

Theorem 7. *Let* $\mathrm{GP}(n,k)$ *be a generalized Petersen graph. Then*

$$d^2(\mathrm{GP}(n,k)) = 2.$$

P r o o f. If n is even, then by Theorem 7 there exists a triply domatic partition of $\mathrm{GP}(n,k)$ with two classes. This partition is also doubly domatic and thus $d^2(\mathrm{GP}(n,k)) = 2$. Suppose that n is odd. As $\mathrm{GP}(n,k) \cong \mathrm{GP}(n,n-k)$, we may suppose that $k \leqslant (n-1)/2$. We put $c(u_i) = 1$ for i odd and $c(u_i) = 2$ for i even. Further, $c(v_1) = c(v_n) = 2$. The circuit C'_n consists of two paths, both with the end vertices v_1, v_n. One of them has an odd length and the other has an even length; let the former be R_1 and the latter R_2. The vertices of R_2 can be coloured alternately by 1 and 2, starting in v_1 of colour 2 and ending in v_n of colour 2. If R_1 contains the edge $v_n v_k$, then it contains also the edge $v_k v_{2k}$. We put $c(v_k) = c(v_{2k}) = 1$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_{2k} of colour 1 and ending in v_1 of colour 2. If R_1 does not contain $v_n v_k$, it contains the edge $v_{n-k} v_n$. We put $c(v_{n-k}) = 2$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_1 of colour 2 and ending in v_{n-k} of colour 2. Now suppose that k is odd. If R_1 contains the edge $v_n v_k$, we put $c(v_k) = 2$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_k of colour 2 and ending in v_1 of colour 2. If R_1 does not contain $v_n v_k$, then it contains the edge $v_{n-k} v_k$ and the edge

$v_{n-2k}v_{n-k}$. We put $c(v_{n-k}) = c(v_{n-2k}) = 1$ and colour the vertices of the rest of R_1 alternately by 1 and 2, starting in v_1 of colour 2 and ending in v_{n-2k} of colour 1. In all the cases we obtain a doubly domatic colouring of $\mathrm{GP}(n,k)$, which proves the assertion. $\square$

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